ON THE BOUNDEDNESS OF INTEGRABLE AUTOMORPHIC FORMS

BY

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1. Let G be a fuchsian group acting on the unit disk $U : |z| < 1$. Let $q \geq 1$, and, for the moment, an integer. We say the function f, holomorphic in U, is an automorphic form of weight q with respect to G, if

$$(1.1) \qquad f(Az)(A'(z))^q = f(z) \quad \text{for all} \quad A \in G \quad \text{and} \quad z \in U.$$

The form f is *integrable* and we write $f \in A_q(G)$, if

$$\| f \|_1 \equiv \iint_R (1 - |z|^2)^{q-2} |f(z)| \, dx \, dy < \infty, \quad z = x + iy$$

where R is a fundamental region for G; f is *bounded*, and we write $f \in B_q(G)$ if

$$\| f \|_\infty \equiv \sup_{z \in U} (1 - |z|^2)^q |f(z)| < \infty.$$

The spaces $A_q(G)$, $B_q(G)$, with the indicated norms, are Banach spaces and were introduced by Bers.

It was conjectured some years ago that $A_q(G) \subset B_q(G)$, and that the injection is continuous. This has been proved for finitely generated G by a number of writers; see the bibliography in [4], where still another proof is given. In [5] we defined a class of infinitely generated groups for which the conjecture holds.

The purpose of this paper is to improve the result of [5]. Our main result is

THEOREM 1. *Let G be a fuchsian group satisfying the following condition:*

$$(1.2) \qquad |\operatorname{trace} A| - 2 \geq m > 0 \quad \textit{for all hyperbolic} \quad A \in G,$$

where m depends only on G. Then $A_q(G) \subset B_q(G)$ and the inclusion map is continuous.

The proof proceeds in two stages. First, we make no assumption about G. At each cusp p of G we erect a distinguished horocycle Π_p and show that in $\Pi = \bigcup_p \Pi_p$, $\phi(z) \equiv (1 - |z|^2)^q |f(z)|$ is bounded. This is done by utilizing the Fourier expansion of f at p. Next, about each elliptic vertex ω of G we describe a distinguished disk Λ_ω and prove that in $\Lambda = \bigcup_\omega \Lambda_\omega$, ϕ is likewise bounded. Here we use the Taylor series of f in a special form.

In the second stage of the proof, we consider the complementary region $\Sigma = (\Pi \cup \Lambda)' = \Pi' \cap \Lambda'$. It is here that the recent remarkable results of A. Marden ([7]) are needed. These have the effect of localizing the action of

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the group G in the neighborhood of a point ζ to one element of the group, which can be handled when ζ is in Σ, provided one assumes the trace condition (1.2).

One can also attempt to define a distinguished neighborhood of a pair of hyperbolic fixed points and estimate ϕ by expansion in an appropriate Fourier series. The coefficients d_k, $k \neq 0$, are easily estimated. But it has not so far proved possible to treat d_0 in the same way. Thus the method does not succeed in removing the hypothesis (1.2) from Theorem 1.

The trace condition has a natural geometric interpretation and has been used by other writers, for example in [1]–[3].

2. Let G be an arbitrary fuchsian group acting on $U : |z| < 1$; it need not satisfy the trace condition (1.2). Let $q \geq 1$. In what follows we shall assume q is an integer; see, however, Section 5. We denote by m a general positive constant depending at most on G and q. For $S \subset U$ we write $G(S)$ to denote $\{gz : g \in G, z \in S\}$; G_x is the stabilizer of x in G; $H = \{z : \operatorname{Im} z > 0\}$ is the upper half-plane; $\langle A, B, \cdots \rangle$ is the group generated by $A, B, \cdots$.

Let p be a parabolic cusp of G, $|p| = 1$. Define

$$(2.1) \quad w = T_p z \equiv Tz = -i(z + p)/(z - p), \quad z = x + iy, \quad w = u + iv.$$

Then T maps U on H, carrying p to $i\infty$. The mapping is isometric if we define the line element in H as $|dw|/2v$, the area element as $du\,dv/4v^2$. Let $G_1 = TGT^{-1}$; G_1 acts on H. If $G_p = \langle P \rangle$, $(G_1)_\infty = \langle P_1 \rangle$, where $P_1 : w \to w + \lambda$. Here $\lambda \equiv \lambda_p = 2\,|c(p)|$, and $c(p)$ is defined by

$$z' = Pz, \quad (z' - p)^{-1} = (z - p)^{-1} + c(p).$$

Let $f \in A_q(G)$; then $f_1(w) \equiv f(z)(dz/dw)^q \in A_q(G_1)$; moreover

$$(2.2) \qquad (1 - |z|^2)^q\,|f(z)| = (2v)^q\,|f_1(w)|, \quad \|f\|_1 = \|f_1\|_1,$$

the last symbol being the $A_q(G_1)$ norm of f_1.

The Fourier series of f_1 is

$$(2.3) \qquad \begin{aligned} f_1(w) &= \textstyle\sum_{k=1}^{\infty} a_k\, e^{2\pi i k w/\lambda}, \\ \lambda a_k &= \int_{-\lambda/2}^{\lambda/2} f_1(u + iv)e^{-2\pi i k w/\lambda}\,du; \end{aligned}$$

it is well known that $a_k = 0$ for $k \leq 0$. We get

$$(2.4) \qquad \begin{aligned} \lambda\,|a_k| \int_{v_0}^{\infty} & v^{q-2} e^{-2\pi k v/\lambda}\,dv \\ &\leq \int_{-\lambda/2}^{\lambda/2}\int_{v_0}^{\infty} v^{q-2}\,|f_1(w)|\,du\,dv \\ &= \iint_{R_1} n(S(v_0), w) v^{q-2}\,|f_1(w)|\,du\,dv, \qquad v_0 > 0, \end{aligned}$$

where R_1 is a fundamental region for G_1, $S(t) = \{w : |u| < \lambda/2, v > t\}$, and

$$n(\Omega, w) = \text{card}\{G_1 w \cap \Omega\}, \quad \Omega \subset H.$$

The last equality in (2.4) follows from the G-invariance of $v^q |f_1(w)|$ and of the Poincaré metric $v^{-2}\, du\, dv$.

It is known that for any fuchsian group K acting on H and containing translations, $c_0(K) > 0$, where

$$(2.5) \qquad c_0(K) = \min\{|c| \neq 0 : (ab : cd) \in K\}.$$

We obtain a lower bound for $c_0(G_1)$.

LEMMA 1. $c_0(G_1) \geq 1/\lambda$.

Proof. Let $(ab : c_0\, d) \in G_1$, then

$$\begin{pmatrix} a & b \\ c_0 & d \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c_0 & a \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ -c_0^2 \lambda & \cdot \end{pmatrix} \in G_1.$$

Hence $c_0^2 \lambda \geq c_0$.

We now let R_1 be the Ford fundamental region for G_1 situated in the strip $|u| \leq \lambda/2$, and we let $v_0 = \lambda$. Because of Lemma 1 every isometric circle $|cw + d| = 1$ lies below the line $v = \lambda$; hence $R_1 \supset S(\lambda)$. Thus $n(S(\lambda), w) \leq 1$. Then (2.4) yields

$$(2.6) \qquad \lambda |a_k| I_k \leq \|f_1\|_1, \quad I_k = \int_\lambda^\infty v^{q-2} e^{-2\pi k v/\lambda}\, dv.$$

We have

$$I_k = \lambda^{q-1} \int_1^\infty v^{q-2} e^{-2\pi k v}\, dv.$$

An elementary discussion shows that for all values of k and q,

$$I_k > m\lambda^{q-1} k^{2-q} e^{-4\pi k}.$$

We then obtain

$$(2.7) \qquad |a_k| < m\lambda^{-q} k^{q-2} e^{4\pi k} \|f_1\|_1.$$

As an estimate for the Fourier coefficients this is ridiculously large, but curiously enough it suffices for our purpose and the proof is easy.

We now estimate f_1. In the region $v \geq 3\lambda$, we get, using $k^{q-2} \leq k^q$,

$$\|f_1\|_1^{-1} v^q |f_1(w)| \leq m\lambda^{-q} v^q \sum_1^\infty k^q e^{-2\pi k(v/\lambda - 2)},$$

and

$$\sum_1^\infty < \int_0^\infty t^q e^{-2\pi t(v/\lambda - 2)}\, dt + 2 \max\{t^q e^{-2\pi t(v/\lambda - 2)}\}$$

$$< m(v/\lambda - 2)^{-q-1} + m(v/\lambda - 2)^{-q},$$

giving,

$$(2.8) \qquad \|f_1\|_1^{-1} v^q |f_1(w)| \leq m(v/\lambda - 2)^{-1} + m \leq m, \quad v \geq 3\lambda.$$

Define $\Pi_p = T^{-1}\{v \geq 3\lambda\}$, a horocycle at p in U. Then by (2.2),

$$(2.9) \qquad (1 - |z|^2)^q |f(z)| = (2v)^q |f_1(w)| \leq m \| f_1 \|_1 = m \| f \|_1.$$

Setting $\Pi = \bigcup_p \Pi_p$, the union being over all cusps p, we get

LEMMA 2. *If G is any fuchsian group, $f \in A_q(G)$, then*

$$(1 - |z|^2)^q |f(z)| \leq m \| f \|_1, \quad z \in \Pi.$$

3. In this section we treat the elliptic vertices.

Let ω be an elliptic vertex of G of order $l_\omega \equiv l$, $|\omega| < 1$, $l \geq 2$. Define

$$(3.1) \qquad w = T_\omega z \equiv Tz = (z - \omega)/(\bar{\omega} z - 1).$$

T is an isometric mapping of U onto U. Let

$$G_1 = TGT^{-1}, \quad f_1(w) \equiv f(z)(dz/dw)^q \in A_q(G_1);$$

then

$$(3.2) \qquad (1 - |z|^2)^q |f(z)| = (1 - |w|^2)^q |f_1(w)|, \quad \| f \|_1 = \| f_1 \|_1.$$

(T, G_1, f_1 are of course not the same as in Section 2.) Write

$$G_\omega = \langle E \rangle, \quad (G_1)_0 = \langle E_1 \rangle, \quad E_1 : z \to \varepsilon^2 z, \quad \varepsilon = e^{\pi i/l}.$$

Because $w^q f_1(w)$ is invariant under E_1 we have an expansion of the form

$$(3.3) \qquad w^q f_1(w) = \sum_1^\infty b_k w^{kl}, \qquad b_k = \frac{1}{2\pi i} \int \frac{w^q f_1(w)}{w^{kl+1}}\, dw.$$

Hence

$$(3.4) \qquad \begin{aligned} |b_k| \rho^{lk+1-q} &\leq m \int_0^{2\pi} |f_1(\rho e^{i\theta})| \rho \, d\theta, \quad w = \rho e^{i\theta} \\ |b_k| J_k &\leq m \iint_C (1 - |w|^2)^{q-2} |f_1(w)| \, du \, dv, \end{aligned}$$

where

$$C = \{w : |w| < \tau\}, \quad J_k = \int_0^\tau \rho^{lk+1-q}(1 - \rho)^{q-2} \, d\rho.$$

Recall that $(G_1)_0$ is generated by an element of order l.

LEMMA 3. *If c_0 has the meaning of (2.5), then $c_0(G_1) \geq \alpha l$ if $l \geq 7$, $\alpha =$ abs. const.*

Proof. Let $(a\bar{c} : c\bar{a}) \in G_1$ with $|c| = c_0$. Then

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} \bar{a} & -\bar{c} \\ -c & a \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \bar{a}c(\varepsilon - \varepsilon^{-1}) & \cdot \end{pmatrix} \in G_1,$$

which implies that

$$|a| c_0 \cdot 2 \sin \pi/l \geq c_0, \quad (1 + c_0^2)^{1/2} = |a| \geq 2^{-1} \csc \pi/l \geq l/2\pi > 1,$$

or

$$c_0^2 \geq l^2/4\pi^2 - l^2/49 = \alpha^2 l^2.$$

We now assume $l \geq 7 + 3/\alpha$. We construct the Ford fundamental region R_1 for G_1, using for the purpose the sector $\{| \arg w | < \pi/l\}$ as a fundamental region for $(G_1)_0 = \langle E_1 \rangle$. Lemma 3 implies that

$$(3.5) \qquad K = \{w : | \arg w | < \pi/l, | w | < 1 - 1/\alpha l\} \subset R_1.$$

Choose $\tau = 1 - 1/\alpha l$, so that $C = \bigcup_{k=0}^{l-1} \varepsilon^{2k} K$. Then $n(C, w) \leq l$ for w in U. It follows that the right member of (3.4) is less than

$$m \iint_{R_1} n(C, w)(1 - | w |^2)^{q-2} | f_1(w) | \, du \, dv \leq ml \| f_1 \|_1 .$$

Also

$$J_k \geq \int_x^y \rho^{lk+1-q}(1 - \rho)^{q-2} \, d\rho, \quad x = 1 - 2/\alpha l, \quad y = 1 - 1/\alpha l.$$

Since from (3.3), $lk \geq q$,

$$J_k \geq (1 - 2/\alpha l)^{lk+1-q}(m/\alpha l)^{q-2} \cdot 1/\alpha l \geq m e^{-\beta k} l^{1-q}, \quad \beta = 2/\alpha,$$

which gives

$$(3.6) \qquad | b_k | \leq m l^q e^{\beta k} \| f_1 \|_1 .$$

At this point assume

$$(3.7) \qquad l \geq l_0 \equiv 3/\alpha + 7 + 2q,$$

which implies in particular

$$q \leq l_0/2 \leq l/2, \quad lk - q \geq l(k - 1) \quad \text{and} \quad l - q \geq l/2.$$

Also

$$-\log | w | > 1 - | w |, \quad 0 < | w | < 1.$$

From (3.3), (3.6),

$$\begin{aligned} \| f_1 \|_1^{-1}(1 - | w |^2)^q | f_1(w) | & \\ &\leq m(1 - | w |^2)^q l^q \{| w |^{l/2} + \textstyle\sum_{k=2}^{\infty} e^{\beta k} e^{-l(k-1)(1-|w|)}\} \\ &= m(l(1 - | w |^2)^q \{S_1 + S_2\}. \end{aligned}$$

In the region $| w | \leq 1 - (1 + \beta)/l$ we have $l(1 - | w |) - \beta \geq 1$, and so

$$S_2 = e^{\beta} \textstyle\sum_{k=1}^{\infty} e^{-k(l(1-|w|)-\beta)} = e^{\beta}(e^{l(1-|w|)-\beta} - 1)^{-1};$$

thus

$$(l(1 - | w |^2))^q S_2 \leq m \frac{(l(1 - | w |))^q}{e^{l(1-|w|)-\beta} - 1} \leq m.$$

And for S_1 we have

$$l^q S_1 \leq \max_l l^q | w |^{l/2} \leq m(-\log | w |)^{-q} \leq m(1 - | w |)^{-q},$$

so that
$$(l(1 - |w|))^q S_1 \le m.$$

Putting these results together, we get
$$(1 - |w|^2)^q |f_1(w)| \le m \, \| f_1 \|_1, \quad w \in D_\omega,$$
where

(3.8) $$D_\omega = \{w : |w| < 1 - (1 + \beta)/l\}, \quad l \ge l_0, \quad \beta = \text{abs. const.}$$

Let
$$\Lambda_\omega = T_\omega^{-1} D_\omega, \quad \Lambda = \bigcup_\omega \Lambda_\omega$$
the sum being over all elliptic vertices of G. The preceding results together with (3.2) establish

(3.9) $$(1 - |z|^2)^q |f_1(z)| \le m \, \| f \|_1, \quad z \in \Lambda_\omega, \quad l \ge l_0.$$

We must now consider the possibility $l < l_0$. Here we define D_ω to be the empty set:

(3.10) $$D_\omega = \emptyset, \quad l_\omega = l < l_0.$$

Then $\Lambda_\omega = \emptyset$ and (3.9) is fulfilled vacuously. This proves

LEMMA 4. *If G is any fuchsian group, $f \in A_q(G)$, then*
$$(1 - |z|^2)^q |f(z)| \le m \, \| f \|_1, \quad z \in \Lambda.$$

4. We must now estimate f in the complementary region
$$\Sigma = (\Pi \cup \Lambda)' = \Pi' \cap \Lambda',$$
where the dash means the complement in U. It is here that we need the trace condition (1.2) as well as A. Marden's results in [7], which we now describe.

Write $D(z, t)$ for the H-disk of center z and radius t. Let
$$\mathcal{J}(z, t) = \{A \in G : A(D(z, t)) \cap D(z, t) \ne \emptyset\}$$
and let $\mathcal{G}(z, t)$ be the subgroup of G generated by $\mathcal{J}(z, t)$.

THEOREM A (Marden). *There exists a universal constant $r > 0$ with the following property. Given any point $z \in U$ and any fuchsian group G, either $\mathcal{G}(z, r)$ is cyclic, or there exist $E, F \in \mathcal{G}(z, r)$ such that*
$$\mathcal{G}(z, r) = \langle E, F : E^2 = F^2 = 1 \rangle.$$

LEMMA 5. *If $z \in D(\zeta, t)$, $|\zeta| < 1$, $t > 0$, then*
$$n(D(\zeta, t), z) = \text{card}\, \{\mathcal{G}(\zeta, t) z \cap D(\zeta, t)\}.$$

Proof. We have to show that $Gz \cap D = \mathcal{G}z \cap D$, where $D = D(\zeta, t)$, $\mathcal{G} = \mathcal{G}(\zeta, t)$. If gz is in the left member, then $gz \in D$ but also $gz \in gD$. Thus $gD \cap D$ contains gz, so $g \in \mathcal{J} \subset \mathcal{G}$.

For $\zeta \in U$ and $0 < c < 1$ define
$$\Delta_c(\zeta) = \{z : |z - \zeta| < c(1 - |\zeta|)\}.$$

By calculation we find the H-radius of $\Delta_c(\zeta)$ is less than

$$(\tfrac{1}{2}) \log((1 + c)/(1 - c))$$

for all ζ in U; hence we can choose an absolute constant $c = c_0$ so that the radius of $\Delta_{c_0}(\zeta)$ is less than r, with the r of Theorem A. Thus to each $\zeta \in U$ there is a unique point $s = s(\zeta)$ lying on the line $\arg z = \arg \zeta$ such that the disk $D(s(\zeta), r)$ contains $\Delta_{c_0}(\zeta)$ as an internal tangent at the point nearest $|z| = 1$. We shall abbreviate $\Delta_{c_0}(\zeta)$ to $\Delta(\zeta)$, $D(s(\zeta), r)$ to $D(\zeta)$, and $\mathcal{G}(s(\zeta), r)$ to $\mathcal{G}(\zeta)$.

COROLLARY 1. *For $z \in U$ we have*

$$n(\Delta(\zeta), z) \leq \operatorname{card}\{\mathcal{G}(\zeta)z \cap D(\zeta)\}.$$

If z has no G-images in $\Delta(\zeta)$, $n(\Delta, z) = 0$ and the result is proved. Otherwise $z = Bz'$ with $B \in G$, $z' \in \Delta(\zeta)$, and then $n(\Delta(\zeta), z) = n(\Delta(\zeta), z')$. We may thus assume $z \in \Delta(\zeta)$, so $z \in D(\zeta)$. By Lemma 5,

$$n(\Delta(\zeta), z) \leq n(D(\zeta), z) = \operatorname{card}\{\mathcal{G}(\zeta)z \cap D(\zeta)\}.$$

LEMMA 6. *Let H be the group*

$$H = \langle M, N : M^2 = N^2 = 1\rangle$$

where $M, N \in SL(2, R)$. Then H is conjugate over $GL(2, R)$ to the group

$$H_\rho = \left\langle A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix} \right\rangle$$

for exactly one $\rho > 1$. Let K be an H-disk in H of radius r. Then for $w \in H$, $\rho \geq 1 + m$, we have

$$\operatorname{card}\{H_\rho w \cap K\} \leq m.$$

Proof. The first statement is proved in [6, Th. 1]. Next, the elements of H_ρ are seen to be

$$\pm\begin{pmatrix} \rho^{-k} & 0 \\ 0 & \rho^k \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -\rho^{-k} \\ \rho^k & 0 \end{pmatrix}, \quad k \in \mathbf{Z}.$$

Let $w_k' = \rho^{-2k} w$ and let h, k be the smallest and largest integers, respectively, for which w_h', w_k' lie in K. Then

$$d(w_h', w_k') \geq d(i\,|w_h'|, i\,|w_k'|) = 2^{-1} \log \rho^{2(k-h)}.$$

Hence $(k - h) \log \rho \leq 2r$, or

$$0 \leq k - h \leq 2r/\log \rho \leq m.$$

It follows that

$$\operatorname{card}\left\{\left\langle\begin{pmatrix} \rho^{-k} & 0 \\ 0 & \rho^k \end{pmatrix}\right\rangle w \cap K\right\} \leq 1 + m = m.$$

With a similar result for the transformations $\{(0, -\rho^{-k} : \rho^k, 0)\}$, we get the conclusion of the lemma.

Let $\zeta \in \Pi_p'$. Consider $D(s(\zeta), r) = D(\zeta)$, as defined after Lemma 5. The H-diameter of $D(\zeta)$ is $2r$. When we map into H by the T_p of (2.1), $D(\zeta)$ goes into a disk D^* whose lowest point $u_0 + iv_0$ satisfies $v_0 < 3\lambda$. The euclidean diameter of D^* is therefore $v_0(e^{4r} - 1) < m\lambda$. Hence card $\{\langle P_1\rangle w \cap D^*\} < m$, where $\langle P_1\rangle = (G_1)_\infty$ and P_1 is $w \to w + \lambda$. Mapping back to U we get

$$\text{card}\,\{\langle P\rangle z \cap D(\zeta)\} < m, \quad \zeta \in \Pi_p', \quad z \in U, \quad \langle P\rangle = G_p.$$

Since this is true for all p,

$$\text{card}\,\{\langle V\rangle z \cap D(\zeta)\} < m, \ V \text{ parabolic}, \quad \zeta \in \Pi', \quad z \in U. \tag{4.1}$$

Next, let $\zeta \in \Lambda_\omega'$. We again assume $l \geq l_0$ but now require

$$l_0 \geq 2(1 + \beta)e^{4r} \tag{4.2}$$

in addition to the previous restrictions of (3.7); β appears in (3.8). When we map by the T_ω of (3.1), $D(\zeta)$ goes into a disk D^* that lies partly or wholly in D_ω' (see (3.8)). Let $w \in D^*$. The points w, $E_1 w$ lie on a circle about 0 and subtend an arc of euclidean length $2\pi \mid w \mid /l$. If δ is the euclidean diameter of D^*,

$$\text{card}\,\{\langle E_1\rangle w \cap D^*\} \leq m\delta/2\pi \mid w \mid l^{-1}.$$

We first show that 0 lies outside D^*. In fact, the H-radius of D_ω is

$$\tfrac{1}{2} \log(2l - (1 + \beta))/(1 + \beta),$$

and this exceeds $2r$ because of (4.2). Next, let

$$\min \mid w \mid = x > 0, \quad \max \mid w \mid = y > 1 - (1 + \beta)/l, \quad w \in D^*.$$

We have

$$\frac{1 + y}{1 - y} \cdot \frac{1 - x}{1 + x} = e^{4r},$$

$$\frac{1 - x}{1 + y} < \frac{1 - x}{1 + x} = \frac{1 - y}{1 + y} e^{4r},$$

which gives

$$x > 1 - (1 - y)e^{4r} > 1 - e^{4r}(1 + \beta)/l \geq \tfrac{1}{2},$$

the last inequality by (4.2). Hence

$$\delta = y - x < 1 - (1 - e^{4r}(1 + \beta)/l) = m/l,$$

so

$$\begin{aligned} \text{card}\,\{\langle E\rangle z \cap D(\zeta)\} &= \text{card}\,\{\langle E_1\rangle w \cap D^*\} \\ &< ml^{-1}/2\pi x l^{-1} \qquad (4.3) \\ &< m, \quad z \in D(\zeta). \end{aligned}$$

This estimate holds for $z \in U$, by standard reasoning.

On the other hand, if $l < l_0$ we obviously have

$$\text{card}\,\{\langle E\rangle z \cap D(\zeta)\} \leq \text{card}\,\{\langle E\rangle z \cap U\} = l < l_0 = m, \quad z \in U.$$

This is (4.3), which is now true for all ω; hence

$$(4.4) \qquad \operatorname{card}\{\langle V\rangle z \cap D(\zeta)\} < m, \quad V \text{ elliptic}, \zeta \in \Lambda', \quad z \in U.$$

Let $\mathcal{G}(\zeta)$ be as defined above. We now make use of Marden's Theorem A. If $\mathcal{G}(\zeta)$ is an elliptic or parabolic cyclic group, (4.4) and (4.1) give

$$\operatorname{card}\{\mathcal{G}(\zeta)z \cap D(\zeta)\} \leq m, \quad \zeta \in \Sigma, \quad z \in U.$$

Suppose $\mathcal{G}(\zeta)$ is hyperbolic cyclic, say $\langle L\rangle$. The images of z under $\langle L\rangle$ lie on a circle through the fixed points of L. Map the figure into H by a Moebius transformation W, carrying the fixed points to $0, i\infty$; L becomes $L_1 : w \to \kappa w$. Let w and $\kappa^h w$ be the extreme images lying in $D^* = WD(\zeta)$. Then

$$2r \geq d(w, \kappa^h w) \geq d(i\,|\,w\,|, i\kappa^h\,|\,w\,|) = 2^{-1}h \log \kappa.$$

Because of the trace condition (1.2), $\kappa > 1 + m$, so we find

$$h = \operatorname{card}\{\langle L_1\rangle w \cap D^*\} = \operatorname{card}\{\langle L\rangle z \cap D(\zeta)\} \leq m.$$

This shows that

$$(4.5) \qquad \operatorname{card}\{\mathcal{G}(\zeta)z \cap D(\zeta)\} \leq m \quad \text{if } \mathcal{G} \text{ is cyclic}, \quad \zeta \in \Sigma.$$

Finally, suppose $\mathcal{G}(\zeta)$ is a 2-generator group:

$$\mathcal{G}(\zeta) = \langle E, F : E^2 = F^2 = 1\rangle.$$

Map U to H and then apply the first part of Lemma 6: $\mathcal{G}$ is conjugate under $T \in GL(2, \mathbf{C})$ to H_ρ for some $\rho > 1$. Now

$$|\operatorname{tr} EF\,| = |\operatorname{tr} AB\,| = \rho + \rho^{-1} > 2;$$

hence EF is hyperbolic and by the trace assumption (1.2), $\rho + \rho^{-1} \geq 2 + m$, or $\rho \geq 1 + m$. We are now in the situation of the last part of Lemma 6 and conclude that

$$(4.6) \quad \operatorname{card}\{\mathcal{G}(\zeta)z \cap D(\zeta)\} = \operatorname{card}\{H_\rho Tz \cap TD(\zeta)\} \leq m, \quad \zeta \in \Sigma, \quad z \in U.$$

This estimate, therefore, holds in every case.

By Corollary 1,

$$(4.7) \qquad n(\Delta(\zeta), z) \leq m, \quad \zeta \in \Sigma, \quad z \in U.$$

Now

$$f(\zeta) = \frac{1}{\pi a^2}\iint_{\Delta(\zeta)} f(w)\, du\, dv, \qquad w = u + iv,$$

where $a = c_0(1 - |\,\zeta\,|)$ is the Euclidean radius of $\Delta(\zeta)$, as explained in the lines after Lemma 5. Since $c_0 = m$, we get

$$(1 - |\zeta|^2)^q |f(\zeta)| \leq m \iint_{\Delta(\zeta)} (1 - |w|^2)^{q-2} |f(w)|\, du\, dv \cdot \sup\left\{\left(\frac{1 - |\zeta|^2}{1 - |w|^2}\right)^{q-2}\right\}.$$

It is easily seen that the sup $\leq m$; hence

$$(1 - |\zeta|^2)^q |f(\zeta)| \leq m \iint_R n(\Delta(\zeta), w)(1 - |w|^2)^{q-2} |f(w)| \, du \, dv,$$

and this with (4.7) gives

$$(1 - |\zeta|^2)^q |f(\zeta)| \leq m \|f\|_1, \quad \zeta \in \Sigma. \tag{4.8}$$

Now Lemmas 2 and 4, and (4.8) yield Theorem 1.

5. We have assumed q is an integer. If this is not the case, it is necessary to introduce a multiplier system $\{v(A) : A \in G, |v| = 1\}$ and modify the functional equation for an automorphic form. However, multiplier systems for an arbitrary group G and nonintegral q have not been shown to exist. In any event, the treatment of arbitrary multiplier systems is routine, as has been demonstrated in the literature, and can be carried out without difficulty if the need should arise.

References

1. L. Bers, *A remark on Mumford's compactness theorem*, Israel J. Math., vol. 12 (1972), pp. 400–407.
2. G. Kiremidjian, *Complex structures on Riemann surfaces*, Trans. Amer. Math. Soc., vol. 169 (1972), pp. 317–336.
3. D. Mumford, *A remark on Mahler's compactness theorem*, Proc. Amer. Math. Soc., vol. 28 (1971), pp. 289–294.
4. J. Lehner, *On the $A_q(\Gamma) \subset B_q(\Gamma)$ conjecture*, Springer Lecture Notes, no 320 (1973), pp. 189–194.
5. ———, *On the $A_q(G) \subset B_q(G)$ conjecture for infinitely generated groups*, Annals of Mathematics Studies, no. 79, 1974.
6. J. Lehner and M. Newman, *Real two-dimensional representations of the free product of two finite cyclic groups*, Proc. Cambridge Philos. Soc., vol. 62 (1966), pp. 135–141.
7. A. Marden, *Universal properties of fuchsian groups in the Poincaré metric*, Annals of Mathematics Studies, no. 79, 1974.

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