A TRANSFER THEOREM FOR FINITE GROUPS WITH SYLOW p-SUBGROUPS OF MAXIMAL CLASS

BY

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In [1], Cline proved a transfer theorem in response to a question of Glauberman [2, Problem 8] for the primes $p = 3, 5$ under the assumption that the Sylow p-subgroups of a finite group G have maximal class. We prove a similar result for the primes $p \geq 7$. The proof uses recent work of Shepherd [5] on p-groups of maximal class.

Our notations are standard [4]. In particular, if P is a p-group of nilpotence class $n - 1$, then $P = K_1(P), K_2(P), \cdots, K_n(P) = 1$ is the lower central series for P.

THEOREM 1. *Let G be a finite group with Sylow p-subgroup P of maximal class with p an odd prime. Let $N = N_G(C_P(K_2(P)/K_4(P)))$. Then*

$$G/O^p(G) \cong N/O^p(N).$$

To prove this, we use the Hall-Wielandt Theorem [3, 14.4.2] and the following theorem of Shepherd.

THEOREM 2 [5]. *Let P be a p-group of maximal class, with $p \geq 7$. Then the nilpotence class of $C_P(K_2(P)/K_4(P))$ does not exceed $\frac{1}{2}(p + 1)$.*

Let $P_1 = C_P(K_2(P)/K_4(P))$ and $P_i = K_i(P)$, $i > 1$. In view of the Hall-Wielandt Theorem, for the case $p \geq 7$ it is sufficient to prove that P_1 is weakly closed in P.

In [4, III.14], the following properties of P_1 are established:

LEMMA 1. *Let P be a p-group of maximal class with $|P| = p^n$, and $n \geq 5$ and $P_1 = C_P(P_2/P_4)$. Then:*

(a) *P_1 is a maximal subgroup of P.*

(b) *$P_1 = C_P(P_i/P_{i+2})$ for $2 \leq i \leq n - 3$.*

We call P exceptional if $P_1 \neq C_P(P_{n-2}/P_n) = C_P(Z_2(P))$, where $Z_2(P)$ is the second center of P.

(c) *If P is not exceptional, then P_1 is the only maximal subgroup of P that does not have maximal class.*

(d) *P/P_{n-1} is not an exceptional group.*

We now establish the following.

LEMMA 2. *Let P be a p-group of maximal class with $|P| = p^n$, and $n \geq 5$. Then P_1 is not isomorphic to any other subgroup of P.*

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Proof. If P is not exceptional, then by (c) above, P_1 is not isomorphic to any other maximal subgroup of P.

If P is exceptional, then let $P^* = C_P(Z_2(P))$. Then P^* is a maximal subgroup of P. If M is a maximal subgroup of P and $M \neq P^*$, then

$$Z(M) = Z(P) = P_{n-1};$$

if we had $| Z(M) | > p$, then $Z_2(P) \leq Z(M)$, and $M \leq C_P(Z_2(P)) = P^*$. Thus P^* is the only maximal subgroup of P whose center has order larger than p, and so P_1 is not isomorphic to P^*.

Now let M be any maximal subgroup of P, $M \neq P_1, P^*$, and consider $\bar{M}$, its image in $\bar{P} = P/P_{n-1}$. Now $\bar{P}_1 = P_1/P_{n-1} = C_{\bar{P}}(\bar{P}_2/\bar{P}_4)$. For $n > 5$, since $\bar{P}$ is not exceptional, $\bar{P}_1$ is the unique maximal subgroup of $\bar{P}$ which does not have maximal class, by Lemma 1. For $n = 5$, it is easy to see that $\bar{P}_1$ is the unique maximal subgroup of $\bar{P}$ that does not have maximal class, since $\bar{P}$ has order p^4. Thus $\bar{M}$ does have maximal class, and since

$$\bar{M} = M/P_{n-1} = M/Z(M),$$

it follows that M has maximal class. Thus P_1 is not isomorphic to M, and so P_1 is not isomorphic to any other maximal subgroup of P, completing the proof of Lemma 2.

Thus if $| P | = p^n$, $n \geq 5$, P_1 is weakly closed in P. For $n < 4$, $P_1 = P$. For $n = 4$, it is easy to see that P_1 is the only maximal subgroup of P that does not have maximal class. Thus for all n, P_1 is weakly closed in P.

So for $| P | = p^n$, $p \geq 7$, Theorem 1 follows from Shepherd's Theorem and the Hall-Wielandt Theorem. For $p = 3, 5$ the theorem is essentially identical to Cline's result.

References

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