LOCALIZATION OF MAPPING SPACES

BY
MARTIN BENDERSKY

1. Introduction

Let $\mathcal{C}$ be the category of spaces with the homotopy type of a c.w. complex, and continuous maps. In Anderson [1] a functor $R : \mathcal{C} \to \mathcal{C}$ is introduced which has the effect of localizing the homotopy groups of a 1-connected space. In this paper we study R for more general classes of spaces in $\mathcal{C}$.

In §2 we define R and prove the Universality Property (2.7). While much of this is in [1], a careful proof of (2.7) is necessary for §3.

§3 is devoted to generalizing the localizing properties of R to nilpotent spaces. Here we find that R localizes the homotopy groups above dimension 2, and kills the M-torsion of $\pi_1(X)$. While the effect on π_1 is not as pleasant as Hilton, Mislin and Roitberg's localizing functor $(\ )_{(P)}$ [6], R does have the advantages of being functorial on $\mathcal{C}$ (as opposed to the associated homotopy category), and being applicable to any space in $\mathcal{C}$ ($X_{(P)}$ is not defined if X is not nilpotent). It is also more conceptual than Bousfield-Kan's functor [2].

In §4 we show that mapping spaces X^Y are completely localized by $R(X)^Y$ if X is nilpotent, and Y is finite path connected. In §5 we modify R slightly to obtain a functor R_0. We show that X^Y is localized by the mapping space $R_0(X)^Y$, where now, Y is simply connected and finite, but X has no conditions on its fundamental group (5.7). It should be remarked that $(X^Y)_{(P)}$ localizes, but $(X_{(P)})^Y$ is not even defined.

Unless otherwise indicated all spaces belong to $\mathcal{C}$, have finitely generated homotopy groups, and are path connected.

2. The functor R

Let P be a set of primes. Let $Z_{(P)}$ be the integers localized at P. $M \subset Z$ shall denote the set of integers which are invertible in $Z_{(P)}$. A group is M-torsion or P-torsion if its elements are all torsion of order belonging to M, or P respectively.

For $a \in Z$, the Moore space, $M(a)$, is defined to be the cofibre of a map of degree a from S^1 to S^1.

DEFINITION 2.1. $\pi_n(X; Z/a) = [S^{n-1}M(a), X]$ where $[\ ,\ ]$ is homotopy classes of base point preserving maps. For $n \geq 2$ this is a group.

From the Puppe sequence

$$S^1 \xrightarrow{a} S^1 \to M(a) \to S^2 \to S^2 \to SM(a) \to \cdots$$

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we obtain a long exact sequence

$$(2.2) \qquad \pi_1(X) \xleftarrow{\;a\;} \pi_1(X) \leftarrow \pi_1(X; Z/a) \leftarrow \pi_2(X) \xleftarrow{\;a\;} \cdots.$$

DEFINITION 2.3. X is a $Z_{(P)}$-space if $\pi_i(X; Z/m) = 0$ for all $i \geq 1$, $m \in M$. For $i = 1$ this means $\pi_1(X; Z/m)$ has only one element.

A group G is local if the map $G \to G$ defined by $g \to g^m$ is a bijection for $m \in M$. A space is local if all its homotopy groups are local.

From 2.2 it follows that a local space is a $Z_{(P)}$-space. The converse is almost true:

(2.4) If X is a $Z_{(P)}$-space $\pi_i(X)$ is a local group for $i \geq 2$, and $\pi_1(X)$ has no M-torsion.

Let $R^1(X)$ be obtained from X by attaching a cone on each map of any $S^kM(a)$, for $a \in M$. Inductively define $R^{i+1}(X) = R^1(R^i(X))$.

DEFINITION 2.5. $R(X) = \bigcup_i R^i(X)$. Clearly R is a functor.

PROPOSITION 2.6 (Anderson [1]). *$R(X)$ is a $Z_{(P)}$-space. The inclusion $e: X \to R(X)$ induces a P-bijection in reduced, integral homology, and a surjection in π_1.*

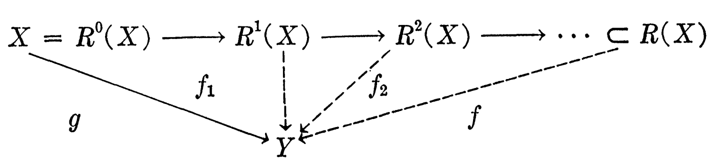
(A homomorphism $g: G_1 \to G_2$, between arbitrary groups, G_1, and G_2 is a P-bijection if ker g is M-torsion, and for $x \in G_2$, there is an $m \in M$ such that x^m is in the image of g.)

THEOREM 2.7. *Let $g: X \to Y$ be a map of X to a $Z_{(P)}$-space Y. Then there is up to homotopy a unique map, $f: R(X) \to Y$, such that*

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & R(X) \\ & {\scriptstyle g}\searrow & \downarrow {\scriptstyle f} \\ & & Y \end{array}$$

homotopy commutes.

Proof. Existence.



To construct f_i, we must extend f_{i-1} to a cone on any map

$$S: S^kM(a) \to R^{i-1}(X), \quad a \in M$$

Since Y is a $Z_{(P)}$-space

$$S^k M(a) \xrightarrow{S} R^{i-1}(X) \xrightarrow{f_{i-1}} Y$$

is null homotopic. The null homotopy gives us the required extension.

Uniqueness. Suppose we have

$$R(X) \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y,$$

f_1 and f_2 extensions of g. We denote by f_i^ε ($\varepsilon = 1, 2$), the restrictions to $R^i(X)$. Suppose we have found a homotopy

$$H_i : R^i(X) \times I \to Y$$

from f_i^1 to f_i^2. We wish to find an $H_i : R^{i+1}(X) \times I \to Y$ such that

$$\begin{array}{ccc} R^i(X) \times I & \longrightarrow & R^{i+1}(X) \times I \\ & {}_{H_i}\searrow \quad \swarrow_{H_{i+1}} & \\ & Y & \end{array}$$

communtes.

We will then be able to pass to the limit to obtain a homotopy H from f^1 to f^2.

For a map s: $S^k M(a) \to R^i(X)$, $a \in M$, D is defined to be the space

$$(S^k M(a) \times I) \cup Cone(S^k M(a) \times \{0\}) \cup Cone(S^k M(a) \times \{1\}).$$

We have a map $T : D \to Y$ defined by

$$(H_i \circ (s \times \text{identity})) \cup (f_{i+1}^1 \circ s) \cup (f_{i+1}^2 \circ s).$$

But D is the same homotopy type as $S(S^k M(a))$. Hence, since Y is a $Z_{(P)}$-space, T is null homotopic, and we obtain a map

$$T^1 : (D \times I)/(D \times \{1\}) \to Y.$$

$(D \times I)/(D \times \{1\})$ is homeomorphic to $\text{Cone}(S^k M(a)) \times I$, and we thus obtain a map

$$h : (R^i(X) \cup \text{Cone}(S^k M(a))) \times I \to Y.$$

Using the above procedure for each map, s, gives us the desired homotopy H_{i+1}.

(2.8) *Remark.* (i) From the proof, it is clear that the Universality Property is true for any space obtained from X by attaching cones on mappings of Moore spaces. Furthermore, if Y is path connected the mappings of Moore spaces need not be base point preserving.

(ii) In Anderson [1] the uniqueness is shown for maps which restrict to null homotopic maps on each $R^i(X)$.

3. Properties of R

A space, X, is said to be nilpotent if $\pi_1(X)$ is a nilpotent group, and acts nilpotently on the higher homotopy groups of X. See Bousfield-Kan [2] for details.

PROPOSITION 3.1. *If X and $R(X)$ are nilpotent spaces, then*

$$e_i\colon \pi_i(X) \to \pi_i(R(X))$$

localizes for $i \geq 2$, and e_1 is the quotient map

$$\pi_1(X) \to \pi_1(X)/(M\text{-Torsion}) = \pi_1(R(X)).$$

Proof. The first part of 3.1 follows from 2.6 and 3.2 below. The last part is Proposition 3.4.

LEMMA 3.2. *Let $f : X \to Y$ be a mapping between nilpotent spaces. Then $H_*(f)$ is a P-bijection if and only if $\pi_*(f)$ is a P-bijection.*

Proof. One may use the argument in Dror [3], modified for P-bijection or refer to Hilton, Mislin, and Roitberg [6].

3.1 becomes interesting in light of

THEOREM 3.3. *If X is nilpotent, then $R(X)$ is nilpotent.*

Proof. Let

$$R(X)^{\sim} \xrightarrow{\mathcal{P}} R(X)$$

be the Universal covering of $R(X)$. Define $R^i(X)^{\sim} = \mathcal{P}^{-1}(R^i(X))$. In particular

$$\mathcal{P}^{-1}(X) = \tilde{X} \xrightarrow{\mathcal{P}} X$$

is a fibration with discrete fibre. Hence $\tilde{X}$ is nilpotent. Let η be the cone on a map of a suspension of a Moore space into $R^i(X)$; $\tilde{\eta}$ denotes a lift of η to $R(X)^{\sim}$. $R^{i+1}(X)^{\sim}$ is obtained from $R^i(X)^{\sim}$ by attaching the cones $\omega\cdot\tilde{\eta}$, for all $\omega \in \pi_1(R(X))$, η as above. Note the attaching maps do not preserve base points. It follows from 2.8 (i) that $\tilde{e} : \tilde{X} \to R(X)^{\sim}$ satisfies the Universal Property. Since the 1-skeleton of $R(X)$ is contained in X, and X is path connected, $\pi_1(R(X), X)$ is the 1-point set. By the homotopy extension property, $\pi_1(R(X)^{\sim}, \tilde{X})$ is the 1-point set, and $\tilde{X}$ is path connected. Since $R(X)$ is a $Z_{(P)}$-space $R(X)^{\sim} = R(\tilde{X})$. It follows from 3.2 that $\tilde{e}$ localizes homotopy. Hence e localizes homotopy in dimension ≥ 2.

Suppose $\alpha \in \pi_1(R(X))$. Since $\pi_1(e)$ is onto, we may suppose $\alpha = \pi(e)(\alpha^1)$ for some $\alpha^1 \in \pi_1(X)$. Since $\pi_i(e)$, $i > 1$, is the localization map, it follows by a simple induction that $\Gamma_j(X) \to \Gamma_j(R(X))$ is P-surjective, where

$$\Gamma_1(Y) = \pi_*(Y) \supset \Gamma_2(Y) \supset \cdots$$

is the filtration of $\pi_*(Y)$ determined by the action of $\pi_1(Y)$.

Since X is a nilpotent space there is a k, such that $\Gamma_k(X) = 0$. Therefore every element in $\Gamma_k(R(X))$ is M-torsion. But $\pi_*(R(X))$ is local for $* \geq 2$, so $\Gamma_k(R(X))$ must be 1, and $R(X)$ is a nilpotent space.

PROPOSITION 3.4. *Suppose $\pi_1(X)$ is a nilpotent group. Then*

$$\pi_1(R(X)) = \pi_1(X)/(M\text{-torsion})$$

and $\pi_1(e)$ is the canonical quotient map.

Proof. Since X is path connected, and X contains the 1-skeleton of $R(X)$, $\pi_1(R(X), X) = 0$. We wish to show that $\pi_2(R(X), X)$ is M-torsion. 3.4 will then follow, since $\pi_1(R(X))$ has no M-torsion. To this end we state

LEMMA 3.5 (Hilton [4]). *There is a localization functor which assigns to each nilpotent group G, a nilpotent group $G_{(P)}$, and a map $e : G \to G_{(P)}$ such that*

(i) *$G_{(P)}$ is a local group and e is a P-bijection,*
(ii) *localization is an exact functor.*

We now consider the homotopy sequence of the pair $(R(X)^\sim, \bar{X})$ (the notation is that of 3.3). By the homotopy extension property we may identify $\pi_2(R(X), X)$ with $\pi_2(R(X)^\sim, \bar{X})$. We therefore obtain the sequence

$$\pi_2(R(X)) \xrightarrow{j} \pi_2(R(X), X) \to \pi_1(\bar{X}) \to 0.$$

For this situation there are two relevant observations:

(1) The image of j lies in the center of $\pi_2(R(X), X)$
(2) The subgroup, Γ, of $\pi_2(R(X), X)$ generated by elements of the form $(\omega \circ \alpha)\alpha^{-1}$, $\omega \in \pi_1(\bar{X})$, $\alpha \in \pi_2(R(X), X)$ coincides with the commutator subgroup (Spanier [7, pg. 385]).

It follows from (1) that $\pi_2(R(X), X) = \pi$ is nilpotent. Suppose π contains an element, α, which is not M-torsion. The exact sequence

$$0 \to [\pi, \pi] \to \pi \to \pi/[\pi, \pi] \to 0$$

induces, by 3.5 (ii), an exact sequence

$$0 \to [\pi, \pi]_{(P)} \to \pi_{(P)} \to (\pi/[\pi, \pi])_{(P)} \to 0.$$

But the relative Hurewicz theorem implies $(\pi/[\pi, \pi])_{(P)} = 0$. Furthermore $[\pi_{(P)}, \pi_{(P)}] = [\pi, \pi]_{(P)}$ (Hilton [5]). Since $\pi_{(P)}$ is nilpotent, we must have $\pi_{(P)} = 1$. However by 3.5 (i), $e(\alpha) \neq 1$. We conclude that π is M-torsion, proving 3.4.

In order to obtain information about $H_*(R(X); Z)$ we proceed as follows.

Let Y be a $Z_{(P)}$-space. Then if Y is 1-connected, $H_*(Y; Z/m) = 0$,

$* \geq 1$ (Sullivan [8]). For Y arbitrary, we have the fibration

$$\tilde{Y} \to Y \to K(\pi_1(Y), 1).$$

From the above remark, this fibration is orientable with Z/m coefficients, and we obtain the following from the Serre spectral sequence.

PROPOSITION 3.6. *Let* $\pi_i(Y; Z/m) = 0 \; i \geq 1, \pi = \pi_1(Y)$. *Then*

$$H_*(Y; Z/m) = H_*(\pi; Z/m),$$

i.e., the obstructions to $H_*(Y; Z)$ *being local, are the groups* $H_*(\pi; Z/m)$.

COROLARY 3.7. *If* π *is nilpotent and local then* $H_*(Y; Z)$ *is local.*

For, by Hilton [4], $\tilde{H}_*(\pi)$ is local, $* \geq 0$. Note that no assumption on the action of π on the higher homotopy groups is necessary.

COROLLARY 3.8. *Suppose* $\pi_1(X)$ *is abelian with* k *free summands. Then*

$$H_*(X) \xrightarrow{e} H_*(R(X))$$

is the localization map for $* > k$.

4. Applications to mapping spaces

We now consider the problem of the localizing the space X^Y. The base point shall be the constant map.

LEMMA 4.1. $R(X)^Y$ *is a* $Z_{(P)}$*-space for* X, Y *arbitrary c.w. complexes.*

Proof. $\pi_*(R(X)^{S^k M(m)}) = 0$ for $* = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots, m \in M$. Therefore $R(X)^{S^k M(m)}$ has the homotopy type of a point. It follows that $[Y, R(X)^{S^k M(m)}] = 0$. By adjunction $[S^k M(m), R(X)^Y] = 0$, $k \geq 0$, $m \in M$. Hence $R(X)^Y$ is a $Z_{(P)}$-space.

LEMMA 4.2. *Let* Y *be a finite c.w. complex,* X *a nilpotent path connected space, and* F *the fibre of the map* $e : X \to R(X)$. *Then* $\pi_n(F^Y)$ *is an* M*-torsion group,* $n \geq 1$.

Proof. From the long exact sequence of the fibration $F \to X \to R(X)$ it follows that $\pi_*(F)$ is M-torsion, $* \geq 1$, and F is connected. Assume the result is true for a complex Y with k cells. There is the Puppe sequence

$$Y \to Y \cup D^r \to S^r,$$

and the induced exact sequence

$$[S^{*+r}, F] \to [S^*(Y \cup D^r), F] \to [S^*Y, F],$$

from which it follows that $\pi_*(F^{Y \cup D^r})$ is M-torsion, $* \geq 1$ proving 4.2.

Let $X^Y \to R(X)^Y$ be composition with $e : X \to R(X)$.

PROPOSITION 4.3. *With* X, Y *as in* 4.2,

(i) $\pi_*(e) : \pi_*(X^Y) \to \pi_*(R(X)^Y)$ *localizes for* $* \geq 2$,
(ii) $\ker \pi_1(e)$ *is the M-torsion of* $\pi_1(X^Y)$.

Proof. Both parts follow from the exact sequence

$$[S^nY, F] \to [S^nY, X] \to [S^nY, R(X)] \to [S^{n-1}Y, F]$$

and the lemmas above.

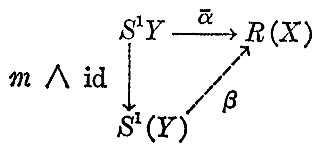
We now come to the main result of this section.

THEOREM 4.4. *Let X be a nilpotent space, Y finite and path connected. Then* $\pi_*(e)$ *localizes the homotopy groups of* X^Y *for* $* \geq 1$.

It remains to show that $\pi_1(e)$ is the localization map. We first prove a special case.

LEMMA 4.5. *Let X be simply connected, Y finite. Then* $\pi_*(e)$ *localizes the homotopy groups of* X^Y *for* $* \geq 1$.

Proof. Let $\alpha \in \pi_1(R(X)^Y)$. There is the diagram

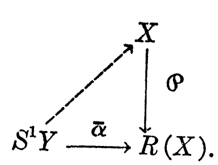


where $m : S^1 \to S^1$ is a map of degree m, and $\bar{\alpha}$ is the adjoint of α. If we can show that there is a $\beta : SY \to R(X)$, unique up to homotopy, making (4.5) commute, for $m \in M$, then α will have a unique *m-th* root. The mapping cone of $m \wedge \mathrm{id}$ is $M(m) \wedge Y$. The obstructions to unique extension lie in

$$\tilde{H}^*(M(m) \wedge Y; \pi_*(R(X)))$$

Since $\pi_*(R(X))$ is a local group, and $M(m)$ is a Z/m Moore space, it follows that all obstructions are 0, and $\pi_1(R(X)^Y)$ is local.

To show $\pi_1(e)$ is a P-bijection we consider the lifting problem



The obstructions $\mathcal{O}_*$ to lifting $\bar{\alpha}$ lie in $H^{*+1}(S^1 \wedge Y; \pi_*F)$ where F is the fibre of $\mathcal{P}$. $\pi_*(F)$ is M-torsion. Hence for each $*$, there is a $d_* \in M$ such that $\mathcal{O}_*$ is of order d_*. Since Y is finite there is a k, such that a lift to the *k-th* stage of the Moore-Postnikove resolution of $\mathcal{P}$ implies a lift to X. Using the Co-H space structure of S^1, we replace $\bar{\alpha}$, by $(d_1d_2 \cdots d_k)\bar{\alpha}$ which we denote $d\bar{\alpha}$. Since the group structures of $[SY, K(G, n)]$ given by the Co-H space structure of S^1 and by the H-space structure of $K(G, n)$ coincide, we see that the obstructions to lifting $d\alpha$ are zero, and $\pi_1(e)$ is P-surjective. The P-injectivity of $\pi_1(e)$ follows from 4.3 (ii). This proves 4.5.

To prove 4.4 in general we consider the diagram

$$\begin{array}{ccc} \pi & \longrightarrow & F \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & R(\tilde{X}) \\ \mathcal{P}\Big\downarrow & & \Big\downarrow R(\mathcal{P}) \\ X & \xrightarrow{\ e\ } & R(X) \end{array}$$

where

$$\tilde{X} \xrightarrow{\mathcal{P}} X$$

is the universal cover of X, $R(\mathcal{P})$ is given by universality, and F is the fibre of $R(\mathcal{P})$. e localizes higher homotopy groups, and $\pi_1(R(\tilde{X})) = 0$. It follows that $\pi_i(F) = 0$, $i \geq 1$. The Whitehead theorem implies the path component of the base point in F has the homotopy type of a point. Since Y is path connected, F^Y has the homotopy type of a point. It follows that $\pi_i(R(\tilde{X})^Y) \to \pi_i(R(X)^Y)$ is an isomorphism for $i \geq 1$. Similarly $\pi_i(\tilde{X}^Y) \to \pi_i(X^Y)$ is an isomorphism for $i \geq 1$. 4.4 now follows from 4.5.

5. Localizing X^Y, Y simply connected

Let X be a space. $R_0^1(X)$ is defined to be the space obtained from X by attaching a cone on all maps $S^k M(m) \to X$, $m \in M$, $k \geq 1$. Inductively define $R_0^{i+1}(X) = R_0^1(R_0^i(X))$.

DEFINITION 5.1. $R_0(X) = \bigcup_i R_0^i(X)$.

Clearly R_0 is a functor. The proof of the following proposition is similar to the proof of the corresponding statements for R.

PROPOSITION 5.2. (i) $\pi_i(R_0(X); Z/m) = 0$, $m \in M$, $i \geq 2$.

(ii) *The natural map* $e_0 : X \to R_0(X)$ *is universal with respect to maps* $X \to Y$, *where* Y *is a space such that* $\pi_i(Y; Z/m) = 0$, $i \geq 2$, *all* $m \in M$.

In the following lemma $X(n)$ denotes the *n-th* connected cover of X.

LEMMA 5.3. *If* Y *is* $(n - 1)$*-connected the natural homomorphism*

$$\pi_i(X(n)^Y) \to \pi_i(X^Y)$$

is an isomorphism for $i \geq 1$.

Proof. In the proof of 4.4 it is shown that $X(1)^Y \to X^Y$ induces an isomorphism for π_i, $i \geq 1$, if Y is path connected. There is the fibration

(5.4) $\qquad F = K(\pi_k(X), k - 1) \to X(k) \to X(k - 1).$

Furthermore $\pi_i(F^Y) = [S^i Y, F] = H^{k-i-1}(Y; \pi_k(X))$.

Since Y is $(n - 1)$-connected and $\pi_k(X)$ is finitely generated, the Universal coefficient theorem implies

$$H^{k-i-1}(Y; \pi_k(X)) = 0, \quad i \geq 0, k \leq n.$$

5.3 now follows from the long exact homotopy sequence associated to (5.4).

PROPOSITION 5.5. *If X is path connected then $R_0(X(n)) = R_0(X)(n)$.*

Proof. There is the diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\tilde{e}_0} & R_0(X)^\sim \\ \big\downarrow \bar{\mathcal{P}} & & \big\downarrow \mathcal{P} \\ X & \xrightarrow{e_0} & R_0(X) \end{array}$$

where $R_0(X)^\sim$ is the universal cover of $R_0(X)$, and $\bar{X} = \mathcal{P}^{-1}(X)$. As in the proof of 3.3, $R_0(X)^\sim = R_0(\bar{X})$. Since R_0 is obtained from X by attaching cones on suspensions of Moore spaces, $\pi_1(R_0(\bar{X}), \bar{X}) = \pi_2(R_0(\bar{X}), \bar{X}) = 0$. It follows that $\bar{X} = \tilde{X}$. 5.5 is therefore true for $n = 1$. Since $H_*(e_0)$ is a P-bijection, $\pi_*(\tilde{e}_0)$ is the localization map for $* > 2$, and $\pi_2(e_0)$ is the canonical map

$$\pi_2(X) \to \pi_2(X)/(M\text{-torsion}).$$

Since $\pi_*(e_0)$ is the localization homomorphism for $* \geq 3$, 5.5 follows for $n \geq 2$.

COROLLARY 5.6. *Suppose Y is a 1-connected, finite c.w. complex. Then*

$$\lambda : X^Y \to R_0(X)^Y$$

localizes all homotopy groups.

Proof. 5.6 follows from the diagram below with "=" meaning "induces an isomorphism for π_*, $* \geq 1$".

$$\begin{array}{ccccccc} X(2)^Y & & & \overset{f}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=} & & & X^Y \\ \big\downarrow \lambda_1 & & & & & & \big\downarrow \lambda \\ R(X(2))^Y & \underset{f_1}{=} & R_0(X(2))^Y & \underset{f_2}{=} & R_0(X)(2)^Y & \underset{f_3}{=} & R_0(X)^Y \end{array}$$

f and f_3 are equivalences by 5.3. f_2 is an equivalence by 5.5. f_1 is an equivalence since $R = R_0$ for 2-connected spaces. λ_1 induces the localization homomorphism in π_*, $* \geq 1$ by 4.5.

COROLLARY 5.7. *R_0 localizes π_* $* \geq 3$.*

Proof. Take $Y = S^2$ in 5.6.

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University of Washington
Seattle, Washington