NON-CONGRUENCE SUBGROUPS FOR THE HECKE GROUPS G($\sqrt{2}$) AND G($\sqrt{3}$)

BY

L. ALAYNE PARSON

1. Introduction

In [4], Newman exhibited a family of noncongruence subgroups for the modular group. In this note we generalize his construction to the Hecke groups $G(\sqrt{2})$ and $G(\sqrt{3})$.

In [1], Hecke introduced the discontinuous groups $G(\lambda_q)$ which are generated by the two linear fractional transformations $T(z) = -1/z$ and $S^{\lambda_q}(z) = z + \lambda_q$. Here $\lambda_q = 2\cos(\pi/q)$, where q is an integer, $q \geq 3$. When $q = 3$, we have the modular group; when $q = 4$ or 6, the resulting groups are $G(\sqrt{2})$ and $G(\sqrt{3})$.

For notational convenience let m stand for 2 or 3 throughout the remainder of this paper. By identifying the transformation $z' = (\alpha z + \beta)/(\gamma z + \delta)$ with the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$G(\sqrt{m})$ may be regarded as a multiplicative group of 2×2 matrices in which a matrix is identified with its negative. It is known [2], [6], that $G(\sqrt{m})$ consists of the entirety of elements of the following two forms:

$$\text{(i)} \qquad \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}, \quad a, b, c, d \in Z,\ ad - bcm = 1,$$

$$\text{(ii)} \qquad \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}, \quad a, b, c, d \in Z,\ mad - bc = 1.$$

Those of type (i) are called even whereas those of type (ii) are called odd.

For N a positive integer, we define the principal congruence subgroup of level N by

$$\Gamma_m(N) = \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(\sqrt{m}) : M \equiv \pm I \pmod{N} \right\}$$

where the congruence is elementwise and takes place in $Z[\sqrt{m}]$. It is easy to verify [5] that the above definition is equivalent to

$$\Gamma_m(N) = \left\{ M = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} : a \equiv d \equiv \pm 1 \pmod{N} \right.$$
$$\left. \text{and} \quad b \equiv c \equiv 0 \pmod{N} \right\}. \qquad (1)$$

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A subgroup Γ of $G(\sqrt{m})$ is called a congruence subgroup of level N if $\Gamma_m(N) \subset \Gamma$ and N is minimal with respect to this property.

2. Definition of the groups Γ_{4mn}

Let $G = \langle x_1, \ldots, x_k \rangle$ be the free group of rank k freely generated by $x_1, \ldots, x_k$. For every positive integer n, let G_n be the subgroup of G consisting of all words of G for which the sum of the exponents of x_1 is divisible by n. G_n is a normal subgroup of G of index n in G. Since G is of rank k, the rank of G_n is $1 + n(k - 1)$ by the Reidemeister-Schreier formula. It is easy to verify that

$$x_1^n,\ x_1^r x_j x_1^{-r}, \quad j = 2, \ldots, k,\ r = 0, \ldots, n - 1 \tag{2}$$

are generators of G_n. Since there are exactly $1 + n(k - 1)$ of them, they are the free generators of G_n.

We wish to apply the preceding remarks to the group $\Gamma_m(2)$. First, however, it is necessary to show that $\Gamma_m(2)$ is free of rank $2m - 1$. Since $\Gamma_m(2)$ contains only even elements, the map f defined by

$$f\left(\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ mc & d \end{pmatrix}$$

is a homomorphism from $\Gamma_m(2)$ into the modular group. From (1) we see that

$$f(\Gamma_2(2)) = \Gamma(2) \cap \Gamma_0(4) \quad \text{and} \quad f(\Gamma_3(2)) = \Gamma(2) \cap \Gamma_0(3).$$

Since $\Gamma(2)$ is a free group of rank 2 and since

$$|\Gamma(2) : f(\Gamma_2(2))| = 2 \quad \text{and} \quad |\Gamma(2) : f(\Gamma_3(2))| = 4,$$

$\Gamma_2(2)$ is a free group of rank 3 and $\Gamma_3(2)$ is a free group of rank 5. In [3] Knopp shows that $\Gamma_2(2)$ is generated by

$$X_1 = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} -3 & 4\sqrt{2} \\ -2\sqrt{2} & 5 \end{pmatrix}, \quad X_4 = \begin{pmatrix} -3 & 2\sqrt{2} \\ -4\sqrt{2} & 5 \end{pmatrix},$$

and that $\Gamma_3(2)$ is generated by

$$X_1 = \begin{pmatrix} 1 & 2\sqrt{3} \\ 0 & 1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 1 & 0 \\ 2\sqrt{3} & 1 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} -5 & 6\sqrt{3} \\ -2\sqrt{3} & 7 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} -5 & 2\sqrt{3} \\ -6\sqrt{3} & 7 \end{pmatrix}, \quad X_5 = \begin{pmatrix} -11 & 8\sqrt{3} \\ -6\sqrt{3} & 13 \end{pmatrix}, \quad X_6 = \begin{pmatrix} -11 & 6\sqrt{3} \\ -8\sqrt{3} & 13 \end{pmatrix}.$$

Since $X_1 X_2^{-1} X_4 X_3 = I$ in $\Gamma_2(2)$ and $X_5 X_3 X_1 X_2^{-1} X_4 X_6 = \mathrm{I}$ in $\Gamma_3(2)$, we may choose X_1, X_2, and X_3 as the free generators of $\Gamma_2(2)$ and X_1, X_2, X_3, X_4, and X_5 as the free generators of $\Gamma_3(2)$.

We now define the groups Γ_{4mn}. Γ_{8n} is the subgroup of $\Gamma_2(2)$ generated by the $1 + 2n$ elements

$$\begin{pmatrix} 1 & 2n\sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 + 8r & -16r^2\sqrt{2} \\ 2\sqrt{2} & 1 - 8r \end{pmatrix},$$
$$\begin{pmatrix} -3 - 8r & (4 + 16r + 16r^2)\sqrt{2} \\ -2\sqrt{2} & 8r + 5 \end{pmatrix}, \quad r = 0, \ldots, n - 1. \tag{3}$$

Γ_{12n} is the subgroup of $\Gamma_3(2)$ generated by the $1 + 4n$ elements

$$\begin{pmatrix} 1 & 2n\sqrt{3} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 + 12r & -24r^2\sqrt{3} \\ 2\sqrt{3} & 1 - 12r \end{pmatrix},$$
$$\begin{pmatrix} -5 - 12r & (6 + 24r + 24r^2)\sqrt{3} \\ -2\sqrt{3} & 12r + 7 \end{pmatrix}, \quad \begin{pmatrix} -5 - 36r & (2 + 24r + 72r^2)\sqrt{3} \\ -6\sqrt{3} & 7 + 36r \end{pmatrix} \tag{4}$$
$$\begin{pmatrix} -11 - 36r & (8 + 48r + 72r^2)\sqrt{3} \\ -6\sqrt{3} & 13 + 36r \end{pmatrix}, \quad r = 0, \ldots, n - 1.$$

If we make the correspondence $X_i \leftrightarrow x_i$, $i = 1, \ldots, 2m - 1$, (2) and our earlier comments on free groups give:

THEOREM 1. *The group Γ_{4mn} is a normal subgroup of $\Gamma_m(2)$ of index n in $\Gamma_m(2)$. It is the free group on the $1 + 2(m - 1)n$ generators* (3) *when $m = 2$ or* (4) *when $m = 3$. In addition, Γ_{4mn} consists of those elements of $\Gamma_m(2)$ for which the sum of the exponents of*

$$S^{2\sqrt{m}} = \begin{pmatrix} 1 & 2\sqrt{m} \\ 0 & 1 \end{pmatrix}$$

is divisible by n.

Remarks. Since $\Gamma_m(2)$ is of index $4m$ in $G(\sqrt{m})$ [3], Γ_{4mn} is of index $4mn$ in $G(\sqrt{m})$. Also, for $n > 1$, Γ_{4mn} is not a normal subgroup $G(\sqrt{m})$. For example, $X_2 \in \Gamma_{4mn}$ but $TX_2T^{-1} = S^{-2\sqrt{m}}$ is not in Γ_{4mn} if $n > 1$.

3. Γ_{4mn} is not a congruence subgroup if $m = 2$ and $n \neq 1, 2, 4$ or if $m = 3$ and $n \neq 1, 2$

For technical reasons we now introduce the groups

$$\Gamma_m(R\sqrt{m}) = \{M \in G(\sqrt{m}) : M \equiv \pm I \pmod{R\sqrt{m}}\},$$

R a positive integer, where the congruence is elementwise and takes place in $Z[\sqrt{m}]$. It is easily verified [5] that

$$\Gamma_m(R\sqrt{m}) = \left\{ M = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in G(\sqrt{m}) : a \equiv d \equiv \pm 1 \pmod{Rm}, \right.$$
$$\left. b \equiv c \equiv 0 \pmod{R} \right\}.$$

LEMMA 2. *Assume that Γ_{4mn} is a congruence subgroup of $G(\sqrt{m})$. Then*

$$\Gamma_{4mn} \supset \Gamma_m(2n\sqrt{m}) \quad \text{if } n \not\equiv 0 \pmod{m}$$

and

$$\Gamma_{4mn} \supset \Gamma_m(2n) \quad \text{if } n \equiv 0 \pmod{m}.$$

Proof. Let l be the level of Γ_{4mn}. Then $\Gamma_m(l) \subset \Gamma_{4mn}$. Since $S^{l\sqrt{m}} \in \Gamma_m(l)$, $2n \mid l$.

Consider first the case when $n \not\equiv 0 \pmod{m}$. Take

$$M = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$$

in $\Gamma_m(2n\sqrt{m})$. Without loss of generality we may assume that $a \equiv d \equiv 1 \pmod{2mn}$. We must show that $M \in \Gamma_{4mn}$.

We may assume that $(d, l) = 1$; for, if not, $bc \neq 0$ and, since $ad - mbc = 1$, $(d, 2nmc) = 1$. Therefore, there exists an integer f such that $(d + 2nmfc, l) = 1$. Now consider

$$MS^{2fn\sqrt{m}} = \begin{pmatrix} a & (2afn + b)\sqrt{m} \\ c\sqrt{m} & 2fcnm + d \end{pmatrix}.$$

$MS^{2fn\sqrt{m}} \in \Gamma_m(2n\sqrt{m})$ and $MS^{2fn\sqrt{m}} \in \Gamma_{4mn}$ iff $M \in \Gamma_{4mn}$. Therefore, we may replace M by $MS^{2fn\sqrt{m}}$.

We may also assume that $b \equiv 0 \pmod{l}$. If not, consider

$$S^{2gn\sqrt{m}}M = \begin{pmatrix} a + 2gcnm & (b + 2gdn)\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$$

for any integer g. Now choose g so that $b + 2gdn \equiv 0 \pmod{l}$. This is possible since $(2dn, l) = 2n$ divides b.

A similar argument shows that we may also assume that $c \equiv 0 \pmod{l}$.

The above conditions on M imply that $ad \equiv 1 \pmod{ml}$ and that

$$M \equiv \begin{pmatrix} a & (1 - ad)\sqrt{m} \\ \dfrac{ad - 1}{m}\sqrt{m} & d(2 - ad) \end{pmatrix} \pmod{l}.$$

Thus there exists $V \in \Gamma_m(l)$ such that $M = M'V$ where

$$M' = \begin{pmatrix} a & (1 - ad)\sqrt{m} \\ \dfrac{ad - 1}{m}\sqrt{m} & d(2 - ad) \end{pmatrix}$$

and it suffices to show that $M' \in \Gamma_{4mn}$. However,

$$M' = \begin{pmatrix} 1 & 0 \\ \dfrac{d - 1}{m}\sqrt{m} & 1 \end{pmatrix}\begin{pmatrix} a & (1 - a)\sqrt{m} \\ \dfrac{a - 1}{m}\sqrt{m} & 2 - a \end{pmatrix}\begin{pmatrix} 1 & (1 - d)\sqrt{m} \\ 0 & 1 \end{pmatrix}$$

$$= X_2^{(d-1)/2m} X_3^{(1-a)/2m} X_1^{(1-d)/2};$$

and $M' \in \Gamma_{4mn}$ because of the congruence conditions on a and d.

Now take $n \equiv 0 \pmod m$. Let

$$M = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$$

be in $\Gamma_m(2n)$ and assume that $a \equiv d \equiv 1 \pmod{2n}$. As above, to show that $M \in \Gamma_{4mn}$ it suffices to show that

$$M' = X_2^{(d-1)/2m} X_3^{(1-a)/2m} X_1^{(1-d)/2} \in \Gamma_{4mn}.$$

However, since $n \equiv 0 \pmod m$, $a \equiv d \equiv 1 \pmod{2m}$; and $M' \in \Gamma_{4mn}$. This completes the proof of the lemma.

THEOREM 3. *Γ_{4mn} is not a congruence subgroup if $m = 2$ and $n \neq 1, 2, 4$ or if $m = 3$ and $n \neq 1, 2$.*

Proof. By Lemma 2 it suffices to exhibit an element of $\Gamma_m(2n\sqrt{m})$ or $\Gamma_m(2n)$ which is not in Γ_{4mn}.

Consider the element

$$M = X_2^x X_3^y X_1^z = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$$

where

$$a = 1 - 2ym, \quad b = 2z - 4zym + 2ym, \quad c = 2x - 4xym - 2y,$$

and

$$d = 4xzm - 8xyzm^2 - 4yzm + 4xym^2 + 2ym + 1.$$

Suppose that x, y, and z have been chosen so that $d \equiv 1 \pmod{2mn}$ if $n \not\equiv 0 \pmod m$ and $d \equiv 1 \pmod{2n}$ if $n \equiv 0 \pmod m$; that is, so that

$$2xz - 4xyzm - 2yz + 2xym + y \equiv 0 \quad \left(\operatorname{mod} \frac{n}{(n, m)}\right). \tag{5}$$

Then

$$M \equiv \begin{pmatrix} 1 + mbc & b\sqrt{m} \\ c\sqrt{m} & 1 \end{pmatrix} \equiv X_1^{b/2} X_2^{c/2} \quad \left(\operatorname{mod} \frac{2mn}{(m, n)}\right)$$

and $V = X_2^{-c/2} X_1^{-b/2} M \in \Gamma_m(2mn/(m, n))$. Since the sum of the exponents of X_1 in V is $ym(2z - 1)$, V will belong to $\Gamma_m(2mn/(m, n))$ but not to Γ_{4mn} if (5) is satisfied and, in addition,

$$ym(2z - 1) \not\equiv 0 \pmod n. \tag{6}$$

Suppose first that $n > 1$, $(n, 2m) = 1$. Set $z = 0$ and $y = 1$. Then (5) becomes $2xm + 1 \equiv 0 \pmod n$ which has a solution since $(2m, n) = 1$. (6) becomes $-m \not\equiv 0 \pmod n$ which is satisfied since $(n, m) = 1$, $n > 1$; and the theorem is proved in this case. However, since $\Gamma_{4md} \supset \Gamma_{4mn}$ whenever $d \mid n$, the theorem is proved for any n with a positive divisor d such that $(d, 2m) = 1$. When $m = 2$, it remains to consider $n = 2^k$, $k \geq 3$. If we take $z = 1$, $x = 2^{k-3}$, and $y = 2^{k-2}$, both (5) and (6) are satisfied. When $m = 3$ and

$n = 2^k$, $k \geq 2$, (5) and (6) are both satisfied if we choose $z = 1$, $x = 2^{k-2}$, and $y = 2^{k-1}$. Finally, if $m = 3$ and $n = 3^k$, $k \geq 1$, we note that $X_1^{2 \cdot 3^{k-1}} X_4^{3^{k-1}}$ belongs to $\Gamma_3(2 \cdot 3^k)$ but not to $\Gamma_{12 \cdot 3^k}$. This completes the proof of the theorem.

4. Γ_{4mn} is a congruence subgroup if $m = 2$ and $n = 1, 2. 4$ or if $m = 3$ and $n = 1, 2$

In this section we prove:

THEOREM 4. *$\Gamma_{8n} \supset \Gamma_2(2n)$ if $n = 1, 2, 4$; and $\Gamma_{12n} \supset \Gamma_3(2n\sqrt{3})$ if $n = 1, 2$.*

Proof. First take $m = 2$. By Lemma 2 it suffices to show that $\Gamma_{32} \supset \Gamma_2(8)$. Since $\Gamma_2(8) \subset \Gamma_2(2)$, any element of $\Gamma_2(8)$ may be written as

$$M_t = X_2^{b_1} X_3^{c_1} X_1^{a_1} X_2^{b_2} X_3^{c_2} X_1^{a_2} \cdots X_2^{b_t} X_3^{c_t} X_1^{a_t}$$

where $a_1, b_1, c_1, \ldots, a_t, b_t, c_t$ are integers. Set $\alpha_t = \sum_{i=1}^t a_i$ and $\gamma_t = \sum_{i=1}^t c_i$. To show that $M_t \in \Gamma_{32}$ we must prove that $\alpha_t \equiv 0 \pmod 4$. Since

$$X_2^b X_3^c X_1^a = \begin{pmatrix} 1 - 4c & 2(a - 4ac + 2c)\sqrt{2} \\ 2(b - 4bc - c)\sqrt{2} & 1 + 4(2ab - 8abc - 2ac + 4bc + c) \end{pmatrix},$$

an induction argument gives

$$M_t = \begin{pmatrix} 1 - 4A_t & 2B_t\sqrt{2} \\ 2C_t\sqrt{2} & 1 + 4D_t \end{pmatrix}.$$

In addition, since $M_t = M_{t-1} X_2^{b_t} X_3^{c_t} X_1^{a_t}$, the integers A_t and B_t satisfy the recursion congruences

$$A_t \equiv A_{t-1} + c_t \pmod 2 \tag{7}$$

$$B_t \equiv B_{t-1} + a_t + 2c_t \pmod 4 \tag{8}$$

with initial conditions $A_1 = c_1$, $B_1 \equiv a_1 + 2c_1 \pmod 4$. Summing both sides of (7) and then (8), we find that

$$A_t \equiv \gamma_t \pmod 2 \tag{9}$$

$$B_t \equiv \alpha_t + 2\gamma_t \pmod 4. \tag{10}$$

However, since $M_t \in \Gamma_2(8)$, $A_t \equiv 0 \pmod 2$ and $B_t \equiv 0 \pmod 4$; and (9) and (10) imply that $\alpha_t \equiv 0 \pmod 4$. This completes the proof of the theorem when $m = 2$.

Now let $m = 3$. We need only show that $\Gamma_{24} \supset \Gamma_3(4\sqrt{3})$. Since $\Gamma_3(4\sqrt{3}) \subset \Gamma_3(2)$, any element of $\Gamma_3(4\sqrt{3})$ may be written as

$$M_t = X_2^{b_1} X_3^{c_1} X_1^{a_1} X_4^{d_1} X_5^{e_1} \cdots X_2^{b_t} X_3^{c_t} X_1^{a_t} X_4^{d_t} X_5^{e_t}$$

where the a_i, b_i, c_i, d_i, e_i, $i = 1, \ldots, t$, are integers. Set $\alpha_t = \sum_{i=1}^t a_i$, $\gamma_t = \sum_{i=1}^t c_i$, and $\delta_t = \sum_{i=1}^t d_i$. We must prove that $\alpha_t \equiv 0 \pmod 2$. Since

$$X_2^b X_3^c X_1^a X_4^d X_5^e = \begin{pmatrix} 1 - 6A & 2B\sqrt{3} \\ 2C\sqrt{3} & 1 + 6D \end{pmatrix}$$

with $A \equiv c + d \pmod 2$ and $B \equiv a + c + d \pmod 2$, an induction argument gives that

$$M_t = \begin{pmatrix} 1 - 6A_t & 2B_t\sqrt{3} \\ 2C_t\sqrt{3} & 1 + 6D_t \end{pmatrix}.$$

Moreover, since $M_t = M_{t-1} X_2^{b_t} X_3^{c_t} X_1^{a_t} X_4^{d_t} X_5^{e_t}$, the integers A_t and B_t satisfy the recursion congruences

$$A_t \equiv A_{t-1} + c_t + d_t \pmod 2 \tag{11}$$

$$B_t \equiv B_{t-1} + a_t + c_t + d_t \pmod 2 \tag{12}$$

with the initial conditions $A_1 \equiv c_1 + d_1 \pmod 2$ and $B_1 \equiv a_1 + c_1 + d_1 \pmod 2$. Summing both sides of (11) and then (12), we find that

$$A_t \equiv \gamma_t + \delta_t \pmod 2 \tag{13}$$

$$B_t \equiv \alpha_t + \gamma_t + \delta_t \pmod 2. \tag{14}$$

However, since $M_t \in \Gamma_3(4\sqrt{3})$, $A_t \equiv 0 \pmod 2$ and $B_t \equiv 0 \pmod 2$. It follows from (13) and (14) that $\alpha_t = 0 \pmod 2$; and the proof of the theorem is complete.

5. Conclusion

The groups Γ_{4mn} are an important example of a family of noncongruence subgroups of $G(\sqrt{m})$ of strictly increasing index but of genus 0. In addition, they are related to λ_m, the invariant of $\Gamma_m(2)$, which is given in [6] when $m = 2$ and [2] when $m = 3$ as a quotient of theta-null series. When $m = 2$, Γ_{8n} is the invariance group of $\lambda_2^{1/n}$. When $m = 3$, the invariance group of $\lambda_3^{1/n}$ is the subgroup Γ of Γ_{12n} consisting of all words for which the sum of the exponents of X_3 and the sum of the exponents of X_4 are both divisible by n. Γ is of index n^2 in Γ_{12n} and, by Theorem 3, is not a congruence subgroup of $G(\sqrt{3})$ whenever $n \geq 3$. When $n = 2$, the argument in Lemma 2 shows that Γ is a congruence subgroup iff $\Gamma \supset \Gamma_3(4\sqrt{3})$. However, since

$$|G(\sqrt{3})\colon \Gamma_3(4\sqrt{3})| = |G(\sqrt{3})\colon \Gamma| = 96 \qquad [5],$$

Γ is a congruence subgroup iff $\Gamma = \Gamma_3(4\sqrt{3})$. Since $X_2 \in \Gamma$ but $X_2 \notin \Gamma_3(4\sqrt{3})$, Γ is not a congruence subgroup; and the invariance group of $\lambda_3^{1/n}$ is a congruence subgroup only when $n = 1$.

References

1. E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, Math. Ann., vol. 112 (1936), pp. 664–699.
2. J. I. Hutchinson, *On a class of automorphic functions*, Trans. Amer. Math. Soc., vol. 3 (1902), pp. 1–11.
3. M. I. Knopp, *Determination of certain roots of unity in the theory of automorphic forms of dimension zero*, Duke Math. J., vol. 32 (1960), pp. 497–506.
4. M. Newman, *On a problem of G. Sansone*, Ann. Mat. Pura Appl. (4), vol. 65 (1964), pp. 27–33.
5. L. A. Parson, *Generalized Kloosterman sums and the Fourier coefficients of cusp forms*, Doctoral dissertation, University of Illinois, 1973.
6. J. Young, *On the group belonging to the sign* $(0, 3; 2, 4, \infty)$ *and the functions belonging to it*, Trans. Amer. Math. Soc., vol. 5 (1904), pp. 81–104.

Ohio State University
Columbus, Ohio