PRODUCTS OF WITT GROUPS

BY

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Introduction

We study products of Witt groups and answer the question of whether a direct factor of a product of Witt groups is also a product of Witt groups affirmatively.

The proof proceeds after an examination of the extendability of endomorphisms of products of Witt groups, and uses the cohomological study of algebraic group extensions given by Serre in *Groupes Algébriques et Corps de Classes.*

The category is that of affine algebraic groups.

0. Facts from Serre (Chapter 7)

Let k be an algebraically closed field.

(1) For commutative connected algebraic groups A and B, the group $\mathrm{Ext}\,(A, B)$ of classes of (commutative) extensions of A by B is the group $H^2_{\mathrm{reg}}\,(A, B)_S$ of classes of symmetric regular factor sets $f\colon A \times A \to B$ [Serre, Proposition 7, Number 6]. We will abbreviate this group to $H^2(A, B)$.

Let k have characteristic $p \neq 0$.

(2) For W_n the n-dimensional Witt group ($W_1 = G_a$, the additive group) and A_n the ring of endomorphisms of W_n, $H^2(W_n, G_a)$ is a right A_n-module and a left A_1-module under $\{f\} \cdot x = \{f \circ (x, x)\}$ and $y \cdot \{f\} = \{y \circ f\}$, for $\{f\}$ the element of $H^2(W_n, G_a)$ represented by the factor set f, $x \in A_n$ and $y \in A_1$.

(3) Let $F_n\colon W_n \times W_n \to G_a$ be a factor set with $F_n(0, 0) = 0$ which gives the extension

$$G_a \xrightarrow{V_n} W_{n+1} \xrightarrow{R} W_n,$$

$V^n(a) = (0, \ldots, 0, a)$ and $R(a_0, \ldots, a_n) = (a_0, \ldots, a_{n-1})$. $H^2(W_n, G_a)$ is a free A_1-module on base $\{F_n\}$ [Serre, Lemma 4, No. 9].

(4) $H^2(\prod_j W_{n_j+1}, \prod_i {}_iG_a) = \prod_{i,j} H^2(W_{n_j+1}, {}_iG_a)$ [Serre, (10), No. 1].

Here each ${}_iG_a$ is the additive group. The connection is given by

$$\{F\} \mapsto (\{F_{ij}\})_{ij},$$

where

$$F_{ij}\colon W_{n_j+1} \times W_{n_j+1} \xrightarrow{\text{inc}} \prod_k W_{n_k+1} \times \prod_k W_{n_k+1} \xrightarrow{F} \prod_k {}_kG_a \xrightarrow{\text{proj}} {}_iG_a$$

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(5) There is a sequence of subgroups of W_n:

$$G_a \xrightarrow{V} W_2 \xrightarrow{V} \cdots \xrightarrow{V} W_{n-1} \xrightarrow{V} W_n$$

where

$$W_i \xrightarrow{V} W_{i+1} \text{ is } V(a_0, \ldots, a_{i-1}) = (0, a_0, \ldots, a_{i-1}).$$

W_i is the unique connected dimension i subgroup of W_n.

(6) The elements of $\prod W_{n_i+1}$ of order p form the subgroup $\prod {}_iG_a$ (${}_iG_a \subset W_{n_i+1}$, the additive group) [Serre, No. 8].

(7) A commutative connected unipotent group of period p is a product of copies of the additive group [Serre, Proposition 11, No. 11].

1. Endomorphisms of products of Witt groups

For $x \in A_n$, there is a unique element $y \in A_1$ with $\{F_n\} \cdot x = y \cdot \{F_n\}$, by (3). The factor set $F_n \circ (x, x) - y \circ F_n$, which thus represents the trivial class in $H^2(W_n, G_a)$, must therefore arise out of a polynomial map T: $W_n \to G_a$ as

$$F_n \circ (x, x)(a, a') - y \circ F_n(a, a') = T(a + a') - T(a) - T(a')$$

for $a, a' \in W_n$.

Using y and T, the element $x \in A_n$ can be lifted to an element u of A_{n+1}. In fact, define

$$u(a_0, \ldots, a_n) = (x(a_0, \ldots, a_{n-1}), y(a_n) + T(a_0, \ldots, a_{n-1})).$$

One checks that u is indeed an element of A_{n+1}, and that

$$\begin{tikzcd} W_{n+1} \arrow[r, "R"] \arrow[d, "u"] & W_n \arrow[d, "x"] \\ W_{n+1} \arrow[r, "R"'] & W_n \end{tikzcd}$$

commutes.

Secondly, we lift an endomorphism x of $\prod_i W_{n_i}$ to an endomorphism u of $\prod_i W_{n_i+1}$. Denote by X_{jk} the homomorphism

$$W_{n_k} \xrightarrow{\text{inc}} \prod_i W_{n_i} \xrightarrow{\times} \prod_i W_{n_i} \xrightarrow{\text{proj}} W_{n_j};$$

since $X_{jk}(W_{n_k})$ is a connected subgroup of W_{n_j}, $X_{jk}(W_{n_k}) = W_n \subset W_{n_j}$ for some n, by (5). Furthermore, the connected component of the identity of $\ker X_{jk}$ must be $W_{n_k-n} \subset W_{n_k}$.

Therefore, X_{jk} may be factored as follows:

$$\begin{tikzcd} W_{n_k} \arrow[r, "R^{n_k-n}"] \arrow[d, "X_{jk}"'] & W_n \arrow[dl, dashed, "X'_{jk}"] \\ W_n & \\ W_{n_j} \arrow[u, phantom, "\cap"] & \end{tikzcd}$$

Let $u'_{jk} \in A_{n+1}$ lift $X'_{jk} \in A_n$; then

$$u_{jk}\colon W_{n_k+1} \xrightarrow{R^{n_k-n}} W_{n+1} \xrightarrow{u_{jk}'} W_{n+1} \xrightarrow{\text{inc}} W_{n_j+1}$$

lifts X_{jk}. The map $u\colon \prod W_{n_i+1} \to \prod W_{n_i}$, determined by the condition that

$$W_{n_k+1} \xrightarrow{\text{inc}} \prod W_{n_i+1} \xrightarrow{u} \prod W_{n_i+1} \xrightarrow{\text{proj}} W_{n_j+1}$$

is u_{jk}, lifts x.

Lastly, the lifting of an automorphism is an automorphism. In fact, let u' lift x^{-1}. $u' \circ u$ lifts the identity map:

$$\begin{array}{ccccc} \prod {}_iG_a & \longrightarrow & \prod W_{n_i+1} & \longrightarrow & \prod W_{n_i} \\ \downarrow{\scriptstyle u'\circ u|} & & \downarrow{\scriptstyle u'\circ u} & & \downarrow{\scriptstyle \text{id}} \\ \prod {}_iG_a & \longrightarrow & \prod W_{n_i+1} & \longrightarrow & \prod W_{n_i} \end{array} \qquad (*)$$

The extension $(*)$ can be described in two ways:

(1) $(\delta_{ij} \cdot \{F_{n_j}\})_{ij} \in \prod_{i,j} H^2(W_{n_j}, {}_iG_a)$

(2) as the image of $(*)$ under $u' \circ u|$; namely

$$(u' \circ u|_{ij} \cdot \{F_{n_j}\})_{ij} \in \prod_{i,j} H^2(W_{n_j}, {}_iG_a).$$

By (4) and (3), $u' \circ u|_{ij} = \delta_{ij}$; that is, $u' \circ u|$ is the identity. Therefore (cf. [Serre, p. 164, (3) and below]) $u' \circ u$ is an automorphism, since $G_a \subsetneq W_{n+1} \to W_n$ decomposes to $G_a \subsetneq W_n \times G_a \to W_n$ as varieties. Similarly, $u \circ u'$ is an automorphism. Therefore, u is an automorphism.

2. Decompositions of products of Witt groups

Let W be the product of Witt groups

$$W = \prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{r} W_{n_i+1}, \quad n_i \geq 1, \text{ all } i \geq 1.$$

Display W as the extension

$$\prod_{i=-u}^{r} {}_iG_a \to \prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{r} W_{n_i+1} \to \prod_{i=1}^{r} W_{n_i}, \qquad (+)$$

where ${}_iG_a \subset W_{n_i+1}$ is the additive group.

$\prod_{i=-u}^{r} {}_iG_a$ is the subgroup of W of elements of order p by (6).

THEOREM. *A direct factor G of a product W of Witt groups is a product of Witt groups.*

Proof. The theorem is true for groups W of dimension 1. Suppose that it is true for groups of dimension less than the dimension of W. We show inductively that the theorem is true for W.

Suppose $W = G_1 \times G_2$. The subgroup of G_j of elements of order p is $G_j \cap \prod_{i=-u}^{r} {}_iG_a$; denote it by P_j. It is immediate that $P_1 \times P_2 = \prod_{i=-u}^{r} {}_iG_a$.

The sequence $(+)$ can be decomposed as

$$P_1 \times P_2 \to G_1 \times G_2 \to G_1/P_1 \times G_2/P_2,$$

the product of the sequences $\{P_i \to G_i \to G_i/P_i\}_{i=1,2}$. Since

$$\frac{G_1}{P_1} \times \frac{G_2}{P_2} = \prod_{i=1}^{r} W_{n_i},$$

by the inductive assumption the groups G_i/P_i are products of Witt groups. Using (6) and induction, one shows that $\prod_{i=1}^{r} W_{n_i} = \prod_{i=1}^{r'} W_{n_i'}$ implies that $r = r'$ and that there is a permutation w of $\{1, \ldots, r\}$ such that $n_{w(i)} = n_i'$. Thus one sees that the product $\prod_{i=1}^{r} W_{n_i}$ can be reordered so that

$$\pi_1 : \frac{G_1}{P_1} \cong \prod_{i=1}^{m} G_{n_i} \quad \text{and} \quad \pi_2 : \frac{G_2}{P_2} \cong \prod_{i=m+1}^{r} G_{n_i}$$

for some m.

The map

$$\frac{G_1}{P_1} \times \frac{G_2}{P_2} \xrightarrow{\pi_1 \times \pi_2} \prod_{i=1}^{m} W_{n_i} \times \prod_{i=m+1}^{r} W_{n_i}$$

is an automorphism of $\prod_{i=1}^{r} W_{n_i}$. Extend this automorphism to an automorphism of $\prod_{i=1}^{r} W_{n_i+1}$, and then complete the extension to an automorphism y of W by setting $y = \text{id}$ on $\prod_{i=-u}^{0} {}_iG_a$.

The exact sequence $(+)$ decomposes in two ways:

$$y(P_1) \times y(P_2) \to y(G_1) \times y(G_2) \to \prod_{i=1}^{m} W_{n_i} \times \prod_{i=m+1}^{r} W_{n_i} \tag{$*$}$$

$$\begin{aligned} \prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{m} {}_iG_a \times \prod_{i=m+1}^{r} {}_iG_a \to \prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{m} W_{n_i+1} \times \prod_{i=m+1}^{r} W_{n_i+1} \\ \to \prod_{i=1}^{m} W_{n_i} \times \prod_{i=m+1}^{r} W_{n_i} \end{aligned} \tag{$**$}$$

By (7), $y(P_1) = \prod_{i=1}^{s} {}^iG_a$ and $y(P_2) = \prod_{i=s+1}^{t} {}^iG_a$ for some s, where the ${}^iG_a$ are copies of the additive group and $t = r + u + 1$.

By (4) and (3), the extension $(+)$, as represented by $(*)$, is described by

$$(g_{ij} \cdot \{F_{n_j}\})_{ij} \in \prod_{i,j=1,1}^{t,r} H^2(W_{n_j}, {}^iG_a)$$

for a unique collection (g_{ij}) of elements of A_1. As represented by $(**)$, the extension $(+)$ is described by

$$(\delta_{ij} \cdot \{F_{n_j}\})_{ij} \in \prod_{i,j=-u,1}^{r,r} H^2(W_{n_j}, {}_iG_a).$$

Write the coordinates of $a \in {}_kG_a$ in term of the product $\prod_{i=1}^{t} {}^iG_a$ as $\{x_{ik}(a)\}_i$, where $x_{ik}(a) \in {}^iG_a$ and $x_{ik} \in A_1$. As indicated in (4), the connection between

$(\delta_{ij}\cdot\{F_{n_j}\})$ and $(g_{ij}\cdot\{F_{n_j}\})$ is given, via the x_{ik}, as $(x_{ij}\cdot\{F_{n_j}\}) = (g_{ij}\cdot\{F_{n_j}\})$. Hence, by (4) and (3), $x_{ij} = g_{ij}$.

Since $(*)$ decomposes the extension $(+)$ as a product, $g_{ij} = 0$ for $j \leq m$ and $i > s$ or $j > m$ and $i \leqq s$. So $x_{ij} = 0$ for $j \leq m$ and $i > s$ or $j > m$ and $i \leqq s$; in other words,

$$\prod_{i=1}^{m} {}_iG_a \subset \prod_{i=1}^{s} {}^iG_a \quad\text{and}\quad \prod_{i=m+1}^{r} {}_iG_a \subset \prod_{i=s+1}^{t} {}^iG_a.$$

Since $\prod_{i=1}^{m} {}_iG_a$ and $\prod_{i=m+1}^{n} {}_iG_a$ are direct factors of $\prod_{i=-u}^{r} {}_iG_a$, they are also direct factors (resp.) of $\prod_{i=1}^{s} {}^iG_a$ and $\prod_{i=s+1}^{t} {}^iG_a$. Let R_1 and R_2 be complementary subgroups; these are products of copies of the additive group by (7).

The expression $(*)$ can then be expanded as

$$\left(\prod_{i=1}^{m} {}_iG_a \times R_1\right) \times \left(\prod_{i=m+1}^{r} {}_iG_a \times R_2\right) \to y(G_1) \times y(G_2)$$
$$\to \prod_{i=1}^{m} W_{n_i} \times \prod_{i=m+1}^{r} W_{n_i}. \qquad (*')$$

Moreover, $R_1 \times R_2$ is a direct factor of W complementary to $\prod_{i=1}^{r} W_{n_i+1}$. To see this, take the surjective map

$$v\colon \prod_{i=-u}^{0} {}_iG_a \xrightarrow{\text{inc}} \prod_{i=1}^{m} {}_iG_a \times R_1 \times \prod_{i=m+1}^{r} {}_iG_a \times R_2 \xrightarrow{\text{proj}} R_1 \times R_2;$$

from the fact that

$$v = \text{inc} - \left(\text{proj to} \prod_{i=1}^{m} {}_iG_a \times \prod_{i=m+1}^{r} {}_iG_a\right) \circ \text{inc},$$

it is immediate that

$$\prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{r} W_{n_i+1} \xrightarrow{(v,\,\text{id})} \prod_{i=-u}^{0} {}_iG_a \times \prod_{i=1}^{r} W_{n_i+1}$$

is an automorphism. Thus the image of the direct factor $\prod_{i=-u}^{0} {}_iG_a$, $R_1 \times R_2$, is a direct factor complementary to $\prod_{i=1}^{r} W_{n_i+1}$.

Finally, we consider $(*')$ and

$$(R_1 \times R_2) \times \prod_{i=1}^{m} {}_iG_a \times \prod_{i=m+1}^{r} {}_iG_a \to R_1 \times R_2 \times \prod_{i=1}^{m} W_{n_i+1} \times \prod_{i=m+1}^{r} W_{n_i+1}$$
$$\to \prod_{i=1}^{m} W_{n_i} \times \prod_{i=m+1}^{r} W_{n_i}. \qquad (**')$$

Under the map

$$x\colon R_1 \times \prod_{i=1}^{m} W_{n_i+1} \xrightarrow{\text{inc}} R_1 \times R_2 \times \prod_{i=1}^{m} W_{n_i+1} \times \prod_{i=m+1}^{r} W_{n_i+1}$$
$$= y(G_1) \times y(G_2) \xrightarrow{\text{proj}} y(G_1),$$

the diagram

$$\begin{tikzcd}
R_1 \times \prod_{i=1}^{m} {}_iG_a \arrow[r] & y(G_1) \arrow[r] & \prod_{i=1}^{m} W_{n_i} \\
R_1 \times \prod_{i=1}^{m} {}_iG_a \arrow[r] \arrow[u, "\mathrm{id}"] & R_1 \times \prod_{i=1}^{m} W_{n_i+1} \arrow[r] \arrow[u, "x"] & \prod_{i=1}^{m} W_{n_i} \arrow[u, "\mathrm{id}"]
\end{tikzcd}$$

commutes. Therefore (cf. [Serre, p. 164, (3) and below]) x is an isomorphism and so $y(G_1)$ is a product of Witt groups. So G_1 is.

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Reference

J-P. Serre, *Groupes Algébriques et Corps de Classes*, Hermann, Paris, 1962.

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