FINITE GROUPS WITH A QUASISIMPLE COMPONENT OF TYPE $PSU(3, 2^n)$ ON ELEMENTARY ABELIAN FORM

BY

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It is a quite common phenomenon among sporadic simple groups that some involution has a centralizer with a quasisimple component of even characteristic which is on elementary abelian form. By this we mean that the centralizer of the component has an elementary abelian Sylow 2-subgroup. (For definition of component, quasisimple etc., we refer the reader to for example D. Gorenstein's survey article on finite simple groups.) Examples of such sporadic simple groups are: Janko's first group J_1 ($Z_2 \times PSL(2, 4)$), the Mathieu group M_{12} ($Z_2 \times S_5$), the Hall-Janko group J_2 ($Z_2 \times Z_2 \times PSL(2, 4)$), the sporadic Suzuki group Su ($Z_2 \times Z_2 \times PSL(3, 4)$), Held's group He (a central extension of $PSL(3, 4)$ by $Z_2 \times Z_2$), Rudvalis' group Ru ($Z_2 \times Z_2 \times Sz(8)$), Conway's group Co_1 ($Z_2 \times Z_2 \times G_2(4)$) and Fischer's new simple group $F_2(?)$ ($Z_2 \times Z_2 \times F_4(2)$).

This gives rise to several classification problems, among which is the following natural one.

Classify finite (in particular simple) groups with an involution whose centralizer C is isomorphic to the direct product of an elementary abelian 2-group E and a group B containing a normal subgroup B_0 which is quasisimple of Bender-type such that $C_B(B_0) = Z(B_0)$.

However, to deal with this problem we need an additional assumption on the involutions of E. A natural one, at least when B_0 is of Bender-type, seems to be that C is the centralizer of all the involutions in E (trivially satisfied when $|E| = 2$.) This is a type of problem which for instance occurs in a recent work by D. Mason, in which he considers finite simple groups all of whose components are of Bender-type (and the centralizer of some involution not 2-constrained of course). Furthermore, J_2 and Ru satisfy this assumption.

Exactly this problem has been considered in the following cases when B_0 is isomorphic to one of the simple groups $PSL(2, q)$ or $Sz(q)$, $B = B_0$ and G is simple: $E \simeq Z_2$ and $B \simeq PSL(2, 2^n)$, by Z. Janko, $B \simeq PSL(2, 2^n)$, by F. L. Smith, $B \simeq Sz(q)$, by U. Dempwolff, and some as special cases in related problems which have been dealt with by M. Aschbacher and K. Harada.

Here we shall answer the question completely for all groups with B_0 quasisimple of $PSU(3, 2^n)$-type, the third class of groups of Bender-type.

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THEOREM 1. *Let G be a finite group with an involution whose centralizer C satisfies:*

(*) *$C = E \times U$, where $E \simeq E_{2^m}$ and U contains a normal subgroup U_0 which is quasisimple of $PSU(3, 2^n)$-type such that $C_U(U_0) = Z(U_0)$. Furthermore, C is the centralizer of every involution in E.*

Then either G contains a strongly closed elementary abelian 2-group or E is of order 2 and has a complement in G.

In particular, by D. Goldschmidt's classification of groups with a strongly closed abelian 2-subgroup, G is not simple. In case G does not contain such a strongly closed elementary abelian 2-group, let H be a complement in G of E. Now, an obvious question is whether H may be simple. To answer this we first recall that the unitary groups are groups of so-called "twisted" type due to the fact that they may be defined as the fixpoint-group of an automorphism of order two, namely, the product of a graph and a field automorphism, of a simple group of Chevalley-type, in this case the projective special linear groups. Thus $H \simeq PSL(3, 2^{2n})$ is a possibility. Our next theorem states that these are the only simple groups with that property.

THEOREM 2. *Let G be a simple group admitting an automorphism ρ of order 2, whose centralizer C in* Aut (G) *satisfies* (*). *Then ρ is an outer automorphism, $C \simeq PSU(3, 2^n)$ and $G \simeq PSL(3, 2^{2n})$.*

We shall obtain this by showing that a Sylow 2-subgroup of G is isomorphic to that of $PSL(3, 2^{2n})$ and then quote a classification theorem due to M. Collins, which may be found in [1].

Finally, Theorems 1 and 2 together with the theorem by D. Goldschmidt referred to above (see [2]) give the following.

MAIN THEOREM. *Let G be a finite group with an involution whose centralizer C satisfies* (*). *Then $G/O(G)$ contains a normal subgroup isomorphic to one of the following:*

(i) $PSU(3, 2^n)$,

(ii) $PSU(3, 2^n) \times PSU(3, 2^n)$,

(iii) $PSL(3, 2^{2n})$.

Furthermore, $O(G)$ is abelian and equal to $Z(U_0)$ if $|E| > 2$.

The most interesting fact about the proof of Theorem 1 and 2 is that except for the application of a few "classical" results (Sylow's Theorem, Grün's First Theorem and some transfer lemmas) and a result on the automorphism group of a special class of 2-groups, it is completely self-contained.

In Section 1 we describe those properties of $SU(3, 2^n)$ that we need and develop a very short method by which to determine the automorphism group of a special class of 2-groups, the so-called Suzuki 2-groups. More specifically, we find the automorphism group of the Sylow 2-subgroups of $Sz(q)$ and $PSU(3, 2^n)$.

Section 2 is a characterization of the Sylow 2-subgroups of $PSU(3, 2^n)$ and $PSL(3, 2^n)$ by a certain property of their automorphism group. The situation we consider seems to appear in several classification problems which is the main reason why we have stated the result in a special section.

In Section 3 we prove two elementary lemmas. The first gives those natural bounds that may be put on the elementary abelian component in general directly from the basic assumptions. The second is a straightforward application of Grün's First Theorem to a configuration that occurs many times whenever $|U/U_0|$ is even.

The last section consists of the proof of our theorems. Our method is merely to build up the possible structure of Sylow 2-subgroups of groups satisfying our assumption. The first step, namely when our involution is central, is easily reduced to the consideration of finite groups with a Sylow 2-subgroup isomorphic to a 2-subgroup of U containing a Sylow 2-subgroup of $PSU(3, 2^n)$. The idea in this proof will be used several times in what follows. Now, if $S \in \mathrm{Syl}_2(U)$ and $S_0 = S \cap U_0$, let $W \in \mathrm{Syl}_2(N_G(E \times S_0))$. Then S is the semidirect product $S_0 . \langle \eta \rangle$ of S_0 and a cyclic group. Our next step is to see that W contains a normal subgroup W_0 containing $E \times S_0$, which is a complement in W to $\langle \eta \rangle$ such that $W_0/E \times S_0 \simeq E_{2^n}$. Moreover, $W_0 = S_0 . C_{W_0}(S_0)$, and E has a complement in W_0 which is a central extension of S_0 by a homocyclic group F of exponent 2 or 4 such that $F \cap S_0$ is equal to $Z(S_0)$. If F is of exponent 4, we easily reduce to the case $|E| = 2$. However, it now takes a rather involved series of arguments to show that $Z(S_0)$ is strongly closed in a Sylow 2-subgroup P containing it. Anyway, we may in the following assume that F is elementary abelian. Now, a short argument allows us furthermore to assume that $F \cap E^g = \langle 1 \rangle$ for all $g \in G$, and also that $P > W$. We proceed to build up $V = N_P(W_0)$. Not surprisingly we obtain that V contains a normal subgroup $V_0 \geq W_0$ which is a complement to $\langle \eta \rangle$ such that $V_0/W_0 \simeq E_{2^{2n}}$. Moreover, $V_0 = C_{V_0}(F) . E$, and $C_{V_0}(F)/F \simeq E_{2^{4n}}$. Now two different cases occur, depending on whether $\Omega_1(C_{V_0}(F))$ equals F or not. In the former case we prove that F is strongly closed, in the latter that $C_{V_0}(F)$ is isomorphic to a Sylow 2-subgroup of $PSL(3, 2^{2n})$, using the above characterization of that. We finish by proving that G contains a normal subgroup L with $C_{V_0}(F)$ as Sylow 2-subgroup. The case where F is strongly closed of course corresponds to the case when $G/O(G)$ contains a normal subgroup isomorphic to the direct product of two copies of $PSU(3, 2^n)$ interchanged by the involution in E.

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1. Properties of $SU(3, 2^n)$ and Suzuki 2-groups

The unitary group $SU(3, q)$, $q = 2^n$, is defined over the field of q^2 elements. Its outer automorphism group is formed by the cyclic group of order $2n$ consisting of field automorphisms and a diagonal automorphism of order 3 when $(3, q + 1) = 3$. Furthermore,

$$PSU(3, q) = \frac{SU(3, q)}{Z(SU(3, q))},$$

and $Z(SU(3, q))$ has order 1 or 3 depending on whether 3 divides $q + 1$ or not. On the other hand, $SU(3, q)$ is the only nontrivial perfect central extension of $PSU(3, q)$. Thus, if U_0 is a quasisimple group of $PSU(3, q)$-type, U_0 is isomorphic to either $SU(3, q)$ or $PSU(3, q)$.

A Sylow 2-subgroup $S_B = S_B(q)$ of $SU(3, q)$ is of Suzuki B-type: $|S_B| = q^3$, $|Z(S_B)| = q$, and $S_B' = Z(S_B) = \Phi(S_B)$, which of course also equals $\Omega_1(S_B)$ since S_B is a Suzuki 2-group. Let U be any group such that $U_0 \leq U$ is quasisimple of $PSU(3, q)$-type and $C_U(U_0) = Z(U_0)$. Let $|U/U_0|$ equal $n_1 n_2$, where n_1 is odd and n_2 is the 2-part. Then a Sylow 2-subgroup of U is isomorphic to the semidirect product of S_B and a cyclic group of order n_2.

Our first result gives the structure of the automorphism group A_B of S_B. We shall not use the specific structure of S_B to find A_B but the important property that it has a cyclic group of order $q^2 - 1$ acting on it (sitting inside Aut $(SU(3, q))$ and inside $SU(3, q)$ for $(3, q + 1) = 1$), such that the subgroup of order $q - 1$ acts trivially on the involutions. This is exactly what makes it of Suzuki B-type.

THEOREM 1.1. *The automorphism group A_B of S_B has the following structure: $O_2(A_B)$ is elementary abelian of order 2^{2n^2}. $A_B/O_2(A_B)$ has order $2n(q^2 - 1)$ and is isomorphic to the normalizer of a Singer-cycle in $GL(2n, 2)$.*

Proof. Let $B_B \leq A_B$ consist of those automorphisms acting trivially on $S_B/Z(S_B)$ and $C_B \leq A_B$ of those acting trivially on $Z(S_B)$. Clearly, $B_B \trianglelefteq A_B$ and $C_B \trianglelefteq A_B$. Moreover $B_B \leq C_B$ and as $\Phi(S_B) = Z(S_B)$, B_B is a 2-group. Since $|S_B/Z(S_B)| = 2^{2n}$ and $|Z(S_B)| = 2^n$,

$$B_B \simeq \text{Hom}\,(Z_2^1 \times Z_2^2 \times \cdots \times Z_2^{2n}, Z_2^1 \times Z_2^2 \times \cdots \times Z_2^n) \tag{1}$$

where $Z_2^k \simeq Z_2$ for all k. Hence B_B is elementary abelian of order 2^{2n^2}. Now A_B/B_B is isomorphic to a subgroup of $GL(2n, 2)$. We know it contains a subgroup of order $2^{2n} - 1$ acting irreducibly on $S_B/Z(S_B)$. Hence $O_2(A_B/B_B) = \langle 1 \rangle$ and $B_B = O_2(A_B)$. By a result of T. O. Hawkes [5], C_B/B_B is isomorphic to a subgroup of $D_{2q_1} \times \cdots \times D_{2q_k}$, where D_{2q_i} is a dihedral group of order $2q_i$, q_i an odd prime power. We know that A_B/B_B contains a subgroup D_B of order $2n(2^{2n} - 1)$ isomorphic to the normalizer of a Singer-cycle in $GL(2n, 2)$. Now D_B contains a dihedral subgroup $D_{2(q+1)}$ of order $2(q + 1)$, which lies inside C_B/B_B. Since the element of order $q + 1$ acts irreducibly on $S_B/Z(S_B)$, the normalizer in A_B/B_B of the subgroup Q of order $q + 1$ in $D_{2(q+1)}$ is equal to D_B as well. Hence Q is equal to its centralizer in C_B/B_B, so $C_B/B_B = D_{2(q+1)}$. But then $D_{2(q+1)}$ is normal in A_B/B_B, so $A_B/B_B = D_B$, and we are done.

COROLLARY. *Let R be any 2-group containing some S_B as a subgroup of index 2. Then $Z(R) \geq Z(S_B)$.*

Remark. This technique may easily be applied to find the automorphism group of other types of 2-groups, in particular other Suzuki 2-subgroups. Among these the most interesting are those of A-type, to which class belong the Sylow 2-subgroup $S_A = S_A(q)$ of the simple Suzuki groups $Sz(q)$, $q = 2^s$. Analogously we obtain the following (known) structure.

THEOREM 1.2. *The automorphism group A_A of S_A has the following structure: $O_2(A_A)$ is elementary abelian of order 2^{n^2}. $A_A/O_2(A_A)$ has order $n(q - 1)$ and is isomorphic to the normalizer of a Singer-cycle in $GL(n, 2)$.*

We will list those properties of $SU(3, q)$ we are going to use. Of course we are mostly interested in 2-elements.

$S_B(q)$ can be described in the following way:

$$S_B(q) \simeq \left\{\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix} : a, b \in GF(q^2), \quad b + b^q + a^{1+q} = 0 .\right\} \tag{2}$$

The cyclic group of order $q^2 - 1$, which is the complement of $S_B(q)$ in its normalizer in $SU(3, q)$ is generated by

$$\sigma_{q^2-1} = \begin{pmatrix} \varepsilon^{-q} & 0 & 0 \\ 0 & \varepsilon^{q-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} \tag{3}$$

where ε is a primitive $(q^2 - 1)$-th root of unity. Let furthermore $\sigma_{q-1} = (\sigma_{q^2-1})^{q+1}$ and $\sigma_{q+1} = (\sigma_{q^2-1})^{q-1}$. Unless 3 divides $q + 1$, $SU(3, q)$ is simple as mentioned earlier. If 3 does divide $q + 1$, $Z(SU(3, q))$ has order 3 and is contained in $\langle \sigma_{q+1} \rangle$. In this case the complement in the normalizer of $S_B(q)$ in $PSU(3, q)$ has order $(q^2 - 1)3^{-1}$. We will use the above notation for the elements of the complement independently of whether we deal with $SU(3, q)$ or $PSU(3, q)$. In the latter case, σ_{q^2-1} and σ_{q+1} have orders $(q^2 - 1)3^{-1}$ and $(q + 1)3^{-1}$ respectively.

Denote by (a, b) the element

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^q \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

in $S_B(q)$. Then

$$(a, b)(c, d) = (a + c, d + ac^q + b) \tag{5}$$

and in particular

$$(a, b)^2 = (0, a^{1+q}), \quad (a, b)^{-1} = (a, b^q). \tag{6}$$

$S_B(q)$ has one conjugacy class of involutions and one of elements of order 4 under the action of the group of order $q^2 - 1$ in its automorphism group. Let (a, b), $(1, c) \in S_B(q)$. Then

$$(a, b)^{-1}(1, c)(a, b) = (1, c + a + a^q) = (1, c) \tag{7}$$

if and only if $a^{q-1} = 1$, an equation with $q - 1$ solutions. Of course we do not get any bound on b since $Z(S_B(q))$ consists of elements of the form $(0, d)$ by (6). Hence it follows that the centralizer of an element of order 4 is of order q^2. It is easy to check that any such group M is normalized by σ_{q-1}, and σ_{q-1} acts irreducibly on $M/Z(S_B(q))$. Hence

$$M \simeq Z_4^1 \times \cdots \times Z_4^n \tag{8}$$

where $Z_4^k \simeq Z_4$ for all k. $S_B(q)$ has $q + 1$ groups of this type, $M_1, M_2, \ldots, M_q, M_0$, conjugate under the action of the element of order $q + 1$ in the automorphism group. Denote in the following $S_B(q)$ by S_0. For any k_1, k_2,

$$\frac{S_0}{Z(S_0)} = \frac{M_{k_1}}{Z(S_0)} \oplus \frac{M_{k_2}}{Z(S_0)}. \tag{9}$$

Let ξ be the field automorphism of order 2. We note that ξ acts trivially on $Z(S_0)$. Since S_0 contains $q + 1$ maximal abelian subgroups, ξ normalizes at least one of them, say M_0. However, (9) shows that it does not normalize any other, since otherwise it would act trivially on $S_0/Z(S_0)$, which is not the case. It is easy to check that M_0 is inverted by ξ. Finally, ξ centralizes σ_{q-1} and inverts σ_{q+1}, and the centralizer of ξ in $PSU(3, q)$ is isomorphic to $PSL(2, q)$.

2. A characterization of the Sylow 2-subgroups of $PSU(3, 2^n)$ and $PSL(3, 2^n)$

The following situation seems to occur in many classification problems, including the present one.

(*) Q is a 2-group admitting an automorphism α of order 2 and an automorphism ρ of order $2^n - 1$ such that

(i) α and ρ commute with each other under the action on Q,

(ii) $C_Q(\alpha) \simeq E_{2^n}$,

(iii) ρ acts transitively on $C_Q(\alpha)^\#$.

The purpose of this section is to prove the following

THEOREM 2.1. *Let Q be a (nonabelian) 2-group satisfying* (*). *Then Q is isomorphic to a Sylow 2-subgroup of $PSU(3, 2^n)$ or $PSL(3, 2^n)$.*

Remark. We note that is easily verified that in case Q is abelian, then Q is either homocyclic of rank n or elementary abelian of order 2^{2n}.

The first step towards a characterization of such 2-groups has also been obtained by G. N. Thwaites in [7] as a corollary to a general result on p-groups:

LEMMA 2.2. *Let Q be a 2-group satisfying (*). Then Q contains a homocyclic subgroup Q_0 of rank n such that*

(i) *α inverts Q_0,*

(ii) *ρ acts transitively on $Q_0/\Phi(Q_0)$,*

(iii) *$Q/\Phi(Q) \simeq E_{2^{2n}}$ and $\Phi(Q) = \Phi(Q_0)$.*

Proof. See Lemma 2.5 in [7].

We now consider the semidirect product of Q and $\langle\alpha\rangle \times \langle\rho\rangle$. Let $Q_0 = \langle r_1, \ldots, r_n\rangle$ and choose notation such that $r_i = r_1^{\rho^{i-1}}$ for $i = 2, \ldots, n$. Let $q_1 \in Q\backslash Q_0$, and set $q_i = q_1^{\rho^{i-1}}$, $i = 2, \ldots, n$. Then

$$Q = \langle r_1, \ldots, r_n, q_1, \ldots, q_n\rangle. \tag{10}$$

Since $\alpha q_1^{-1}\alpha q_1 \in Q_0\backslash\Phi(Q)$, we may as well assume that $[\alpha, q_1] = r_1$. In particular, it follows that the map $r_i \to q_i$ induces an isomorphism between $Q_0/\Phi(Q)$ and $\langle q_1, \ldots, q_n, \Phi(Q)\rangle/\Phi(Q)$ which commutes with ρ. Thus we have

LEMMA 2.3. *$Q/\Phi(Q)$ is the direct sum of two isomorphic ρ-modules.*

The next lemma is due to R. Solomon and occurs in another context. We include the proof.

LEMMA 2.4. *Let Q be a nonabelian group satisfying* (*). *Then Q is of class* 2 *and exponent* 4.

Proof. To see this we will consider the associated Lie ring. Let $Q, Q_1, Q_2, Q_3, \ldots$ be the lower central series of Q and set $L = Q/Q_1$, $L^+ = Q_0/Q_1$ (Q_0 defined as above) and $L = Q_i/Q_{i+1}$ for $i = 1, 2, \ldots$. Let L^- be a complement in L under the action of ρ. Now, by Lemmas 2.2 and 2.3, $Q_i/Q_{i+1} \simeq E_{2^n}$ for all $i \geq 0$, and $L^+, L^-, L_1, L_2, \ldots$ are all vector spaces of dimension n over Z_2 and isomorphic as ρ-modules. Thus there exists a primitive $(2^n - 1)$-th root of unity λ such that $\lambda, \lambda^2, \lambda^{2^2}, \ldots, \lambda^{2^n}$ are the eigenvalues of ρ on $L_K^- = L^- \otimes_{Z_2} K$, where $K = Z_2(\lambda)$. Let $L_K = L \otimes_{Z_2} K$, $L_K^+ = L^+ \otimes_{Z_2} K$ and $L_{iK} = L_i \otimes_{Z_2} K$ for $i = 1, 2, \ldots$. Let $u_0, \ldots, u_{n-1}$ be eigenvectors of ρ in L_K^- with corresponding eigenvalue λ^{2^i}. It easily follows that $u_0 + u_0^{\alpha_1}, \ldots, u_{n-1} + u_{n-1}^{\alpha_1}$ form a basis for L_K^+ and corresponding eigenvalues are λ^{2^i}. Next, we want a basis of eigenvectors for L_1. Clearly L_1 is generated by vectors of the form $[u_i, u_j^{\alpha_1}]$ or $[u_i, u_j]$, each of which is either 0 or an eigenvector of σ_{q-1} with corresponding eigenvalue $\lambda^{2^i+2^j}$. Hence $[u_i, u_j] = 0$ for all i, j, and $[u_i, u_j^{\alpha_1}] \neq$

0 if and only if $i = j$, so $[u_0, u_0^{\alpha_1}], \ldots, [u_{n-1}, u_{n-1}^{\alpha_1}]$ form a basis for L_1. Finally, consider L_2. A similar calculation shows that L_2 is generated by vectors of the form $[[u_i, u_i^{\alpha_1}], u_j]$. However, by Jacobi's identity,

$$[[u_i, u_i^{\alpha_1}], u_j] = [[u_i^{\alpha_1}, u_j], u_i] + [[u_j, u_i], u_i^{\alpha_1}] \tag{11}$$

so by our calculations above $[[u_i, u_i^{\alpha_1}], u_j] = 0$ unless $i = j$. But now, as $\lambda^{3 \cdot 2^i}$ is never an eigenvalue, $[[u_i, u_i^{\alpha_1}], u_i] = 0$ as well, i.e. $L_2 = 0$. Hence Q is of class at most 2.

LEMMA 2.5. *Let P be a group of order 2^{3n}, class 2 and exponent 4 admitting an automorphism ρ of order $2^n - 1$ such that*

(i) *$Z(P) \geq Z \simeq E_{2^n}$, and ρ acts transitively on $Z^\#$,*

(ii) *$P/Z \simeq E_{2^{2n}}$,*

(iii) *P/Z is the direct sum of two irreducible ρ-modules, each of which is isomorphic to Z as a ρ-module.*

Then P is isomorphic to the Sylow 2-subgroup of $PSU(3, 2^n)$ or $PSL(3, 2^n)$.

Proof. Identity P and $\langle \rho \rangle$ with the corresponding subgroups of $P \,.\, \langle \rho \rangle$. Let $p \in P \backslash Z$, and let $R = \langle p, \rho \rangle \cap P$. Then $RZ/Z \simeq E_{2^n}$ by (iii). Moreover, $RZ/Z \simeq Z$ as a ρ-module. On the other hand, as R is either abelian or a Suzuki 2-group of A-type, it follows from [6] that R is abelian.

Let h_0 be an element of order 4 and set $H = \langle h_0, \rho \rangle \cap P$. As P is nonabelian, $\Omega_1(H) = Z$. Next we claim that $C_P(h) = H$ for all $h \in H \backslash Z$. Suppose $p \in P \backslash H$ centralizes h and consider $R = \langle ph^\rho, \rho \rangle \cap P$. As R is abelian, $[ph^\rho, p^\rho h^{\rho^2}] = 1$. However, since $[p, p^\rho] = 1$ as well, this implies that $[p, h^{\rho^2}] = [p^\rho, h^\rho] = 1$. Thus p centralizes h^{ρ^2} as well, and it follows by induction that p centralizes H, a contradiction since P is nonabelian.

It now follows that every $h \in H \backslash Z(P)$ is inverted by exactly one element p in P modulo H. In particular, if $\Omega_1(P) > Z(P)$, P contains exactly 2 maximal elementary abelian subgroups of order 2^{2n}. Anyway, P is generated by the subgroups $P_1 = \langle p, \rho \rangle \cap P$ and $P_2 = \langle ph, \rho \rangle \cap P$. Now, since $[p, p^{\rho^k} h^{\rho^k}] = [p, h^{\rho^k}]$, all commutators are uniquely determined from commutators of type $[p, h^{\rho^k}]$, $1 \leq k \leq 2^n - 1$. However, as $\langle ph^\rho, \rho \rangle \cap P$ is abelian as we have seen above,

$$[ph^\rho, p^\rho h^{\rho^2}] = [p, h^{\rho^2}][p^\rho, h^\rho] = 1. \tag{12}$$

Thus $[p, h^{\rho^2}] = [p, h]^\rho = (h^2)^\rho$, so it follows by induction that all commutators are uniquely determined. Thus there exists at most one such group of a given order with P_1 elementary abelian and at most one with P_2 homocyclic of exponent 4. As both the Sylow 2-subgroup of $PSU(3, 2^n)$ and that of $PSL(3, 2^n)$ satisfy the assumption of the lemma, we are done.

Theorem 2.1 is an immediate consequence of these lemmas.

3. General results

LEMMA 3.1. *Let G be a finite group with an involution α_1 whose centralizer has the form $C_G(\alpha_1) = C = E \times H$ where $\alpha_1 \in E$, E is elementary abelian and H is any group. Assume furthermore that for any $\beta \in E^{\#}$, a Sylow 2-subgroup of $C_G(\beta)$ is isomorphic to that of C. Then one of the following occurs:*

(i) *α_1 is central,*

(ii) *$r(E) \leq r(\Omega_1(Z(S)))$, where $S \in \mathrm{Syl}_2(H)$*

Proof. Assume α_1 is not central and let $E \times S \leq P$, where $P \in \mathrm{Syl}_2(G)$. Furthermore, let $p \in N_P(E \times S)\backslash E \times S$ such that $p^2 \in E \times S$. Then p acts on $E \cap E^p$, so $E \cap E^p = \langle 1 \rangle$ by assumption. On the other hand,

$$p^{-1}Ep \leq \Omega_1(Z(E \times S)) = E \times \Omega_1(Z(S)). \tag{13}$$

Hence $|E|^2 \leq |E|\,|\Omega_1(Z(S))|$, which proves (ii).

Notation. If K is a group acting on the group H, let $H \,.\, K$ denote the semidirect product of H and K.

LEMMA 3.2. *Let G be a finite group, $P \in \mathrm{Syl}_2(G)$. Suppose P contains a normal subgroup P_0 with a complement $C = E \times \langle c \rangle$, where E is elementary abelian (or $\langle 1 \rangle$). Assume furthermore that* ord $(c) \geq \exp(P_0)$. *Then*

$$P \cap G' \leq P_0 \,.\, (E \times \langle c^{\mathrm{ord}(C)/\exp(P_0)} \rangle).$$

Proof. This is just a straightforward application of Grün's First Theorem (see [4, p. 252]):

$$P \cap G' = \langle P \cap N_G(P)', \bigcup_{x \in G} P \cap (P^x)' \rangle. \tag{14}$$

First consider $P \cap N_G(P)'$. Let N be a complement of P in $N_G(P)$. Let $n_i \in N$, $p_i \in P$, $i = 1, 2$. Then, independently of the present structure of P,

$$[n_1 p_1, n_2 p_2] = p_1^{-1} n_1^{-1} p_2^{-1} n_1 n_1^{-1} n_2^{-1} n_1 n_2 n_2^{-1} p_1 n_2 p_2 \tag{15}$$

belongs to P if and only if $n_1^{-1} n_2^{-1} n_1 n_2 = [n_1, n_2]$ does. Hence it suffices to consider elements $[p_1 n_1, p_2 n_2]$ where $[n_1, n_2] = 1$ in order to determine $P \cap N_G(P)'$, in which case

$$[n_1 p_1, n_2 p_2] = p_1^{-1} n_1^{-1} p_2^{-1} n_1 n_2^{-1} p_1 n_2 p_2. \tag{16}$$

Before we continue, we note the following elementary fact.

Let P be a p-group, P_0 a normal subgroup of P with a complement $C = E \times \langle c \rangle$ where E is elementary abelian (or $\langle 1 \rangle$) and ord $(c) > \exp(P_0/P')$. Let ρ be an automorphism of P and set $P_1 = P_0 \,.\, E$. Then

(a) $\langle c^\rho \rangle \cap P_1 = \langle 1 \rangle$,

(b) $c^\rho = p_\rho c^j$ for some $p_\rho \in P_1^\rho$ and $j \in \mathbf{N}$, $(j, p) = 1$.

This is easily verified in the following way: Let $\bar{E} = EP'/P'$ and $\langle \bar{c} \rangle = \langle c \rangle P'/P'$. Then

$$\frac{P}{P'} = \bar{E} \times \langle \bar{c} \rangle \times \bar{Q} \tag{17}$$

for some $Q \leq P_0$, $\bar{Q} = QP'/P'$, as $P' \leq P_0$ and (a) follows, since by assumption ord $(c) > \exp(P_0/P') = \exp(\bar{Q})$. Hence $P_1^{\varrho} \cap \langle c \rangle = \langle 1 \rangle$ as well, and (b) follows.

This has the following consequence. Let $p_i = p_{1i}c^{k_i}$, where $p_{1i} \in P_1$, $i = 1, 2$. Suppose ord $(c) > \exp(P_0/P')$. Then

$$n_2^{-1} p_1 n_2 = p'_{11} c^{k_1 k_1'}, \quad n_1^{-1} p_2 n_1 = p'_{12} c^{k_2 k_2'} \tag{18}$$

where $p'_{11} \in P_1^{n_2}$, $p'_{12} \in P_1^{n_1}$ and k'_i is odd, $i = 1, 2$, by (b). Thus

$$[n_1 p_1, n_2 p_2] = p' c^{k_1(1-k_1') + k_2(1-k_2')} \tag{19}$$

for some

$$p' \in \langle P_1^{n_1}, P_1^{n_2} \rangle \leq P_1 \cdot \langle c^{\mathrm{ord}(c)e^{-1}} \rangle \leq P_1 \langle c^2 \rangle,$$

where $e = \exp(P_0/P')$, by assumption. Since $1 - k'_i$, $i = 1, 2$, is even, $P \cap N_G(P)' \leq P_1 \cdot \langle c^2 \rangle$.

Next let $n_0 = \min(\mathrm{ord}(c), \exp(P'))$. Now the result above together with our assumption, namely that $P = P \cap G'$, implies that for some $x \in G$ there exists a $j \in \mathbf{N}$, j odd, $p_0 \in P_0$ and $\alpha \in E$ such that $p = p_0 \alpha c^j \in P \cap (P^x)' \leq P_1 c^{\mathrm{ord}(c) n_0^{-1}}$. But clearly, ord $(p) \geq$ ord (c). This proves the lemma.

The following result was first observed by K. Harada.

LEMMA 3.3. *Let G be a finite group, $P \in \mathrm{Syl}_2(G)$, and let P_0 be a maximal subgroup of P. Assume that $x \in P \backslash P_0$ belongs to the focal subgroup of P with respect to G. Then either x is conjugate to an element of P_0 or x^{2^r} is conjugate to an element of $P \backslash P_0$ for some $r \geq 1$.*

Proof. By transfer.

Finally we shall use a transfer lemma due to D. Goldschmidt, which extends the result of Lemma 3.3 in the special case when ord $(c) = 2$, namely the following:

Definition. Let G be a finite group, $x \in P \in \mathrm{Syl}_2(G)$ an involution. Then x is said to be extremal in P provided that $C_P(x) \in \mathrm{Syl}_2(C_G(x))$.

LEMMA 3.4. *Let G be a finite group, $P \in \mathrm{Syl}_2(G)$, and let $x \in P$ be an involution which belongs to the focal subgroup of P with respect to G. Assume x has a complement P_0 in P. Then x has an extremal conjugate in P_0.*

Proof. See [3].

4. The classification

Assumption. Let G be a finite group with an involution α_1 such that

$$C_G(\alpha_1) = C = E \times U \tag{*}$$

where E is elementary abelian and U contains a normal subgroup U_0 which is quasisimple of $PSU(3, q)$-type such that $C_U(U_0) = Z(U_0)$. Assume furthermore that $C_G(\alpha) = C_G(\alpha_1)$ for all $\alpha \in E^{\#}$ (a trivial assumption when $|E| = 2$).

Notation. If H and K are subgroups of the group G such that $[H, K] = 1$, we denote by $H \times_* K$ the central product of H and K w.r.t. $H \cap K$ in addition to the standard use.

If G is a finite group, we denote by G_p a p-group isomorphic to a Sylow p-subgroup of G. Similarly, we denote by $|G|_p$ the order of a Sylow p-subgroup of G.

Otherwise our notation will be standard as in [4].

$E = \langle \alpha_1, \ldots, \alpha_m \rangle$.

$S \in \mathrm{Syl}_2(U)$, $S_0 = U_0 \cap S \simeq S_B(q)$. $S = S_0 \cdot \langle \eta \rangle$, where $\mathrm{ord}(\eta) = |U/U_0|_2$. Let ξ be the involution in $\langle \eta \rangle$ (if $\eta \neq 1$). Furthermore, let $\eta_r = \eta^{2^{k-r}}$, where $2^k = \mathrm{ord}(\eta)$.

$Z(S_0) = \langle i_1, \ldots, i_n \rangle$.

$T_0 = E \times S_0$.

$T = E \times S$. We note that all maximal elementary abelian subgroups of T are conjugate to $E \times Z(S_0) \times \langle \xi \rangle$ inside of $C_T(E \times Z(S_0))$.

Let $W \in \mathrm{Syl}_2(N_G(T_0))$, $W_0 = C_W(S_0) \cdot S_0$, $V \in \mathrm{Syl}_2(N_G(W_0))$ and $V \leq P \in \mathrm{Syl}_2(G)$.

Let M_0 denote the maximal abelian subgroup of S_0 which is inverted by ξ.

Finally, let σ_{q^2-1}, σ_{q+1} and σ_{q-1} denote the same elements of U_0 as in Section 1.

We note that the assumption on the centralizers of the involutions in E implies that E is a T.I.-set and that the automizer of E is of odd order.

Also, since every involution of S_0 is a square, no involution of E is conjugate to the involutions of S_0.

LEMMA 4.1. *Suppose $|E| > 2$. Let $\alpha \in E^{\#}$, $\gamma \in E$. Then α is not conjugate to $\xi\gamma$.*

Proof. Clearly $E \times \langle \xi \rangle \times C_{U_0}(\xi) \leq C_G(\xi\gamma)$, which is isomorphic to $E \times U$ if $\xi\gamma$ is conjugate to α. Since $C_{U_0}(\xi)$ is isomorphic to $PSL(2, q)$, the assumption $|E| > 2$ implies that

$$O_2(C_G(\xi\gamma)) \cap E \neq \langle 1 \rangle.$$

But this contradicts that E is a T.I.-set.

LEMMA 4.2. *Suppose the weak closure of* $E \times Z(S_0)$ *in* P *is contained in* T, *and assume that* $\xi\gamma$ *is conjugate to an involution of* $Z(S_0)$ *for some* $\gamma \in E$. *Then there exists for any* $\alpha \in E^{\#}$ *a* $\beta \in E^{\#}$ *such that* α *is conjugate to* $\beta\xi\gamma$. *In particular,* $|E| = 2$.

Proof. If the weak closure of $E \times Z(S_0)$ in P is contained in T, it follows that whenever $(E \times Z(S_0) \times \langle\xi\rangle)^g \leq P$ for some $g \in G$,

$$(E \times Z(S_0) \times \langle\xi\rangle)^g = E \times Z(S_0) \times \langle\xi^s\rangle \tag{20}$$

for some $s \in M_0$ by the remark above. Suppose $(\xi\gamma)^h = i \in Z(S_0)$ for some $h \in G$. As $Z(P) \leq Z(S_0)$, we may as well assume that $i \in Z(P)$. Then

$$(E \times Z(S_0) \times \langle\xi\rangle)^h \leq C_G(i),$$

so for some $c \in C_G(i)$ we have that $(E \times Z(S_0) \times \langle\xi\rangle)^{hc} \leq P$. Hence

$$(E \times Z(S_0) \times \langle\xi\rangle)^g = E \times Z(S_0) \times \langle\xi\rangle \tag{21}$$

and $(\xi\gamma)^g = i$ for some $s_0 \in M_0$ with $g = hcs_0$. But i and $\xi\gamma$ are not conjugate in $N_G(C)$, so $E \cap E^g = \langle 1\rangle$. Since all involutions of $Z(S_0) \times \langle\xi\gamma\rangle$ are conjugate by assumption, $g\alpha g^{-1}$ equals βj or $\xi\beta j$ for some $\beta \in E^{\#}$, $j \in Z(S_0)$. If $g\alpha g^{-1}$ equals $\xi\beta j$, we are done. If $g\alpha g^{-1}$ equals βj, replace g by $g_0 = g^\sigma$, where σ is a power of σ_{q-1} such that $j^\sigma = i$. Then $gg_0\alpha g_0^{-1}g^{-1} = g\beta g^{-1}\xi\gamma$. Now, if $g\beta g^{-1} \notin E \times Z(S_0)$, $\beta \neq \alpha$ and in particular $|E| > 2$. But then $g\beta g^{-1} \in E \times Z(S_0)$ by Lemma 4.1, a contradiction. Hence $g\beta g^{-1} \in E \times Z(S_0)$, and we are done.

LEMMA 4.3. *Let* G *be a finite group with* S *(in the above notation) as Sylow 2-subgroup. Then we have the following constraints on* S.

(i) *Assume* $S_0 \leq S \cap G'$. *Then* $S \cap G' \leq S_0 . \langle\xi\rangle$.

(ii) *If furthermore* G *contains a subgroup isomorphic to* U *(in the above notation),* $S \cap G' = S_0$.

Proof. By Lemma 3.2, we may as well assume that $\eta^4 = 1$. Suppose $\langle\eta\rangle \leq S \cap G'$ and ord $(\eta) = 4$. Then, by Lemma 3.3, ξ is conjugate to some involution in $Z(S_0)$, say $\xi^g = i \in Z(S)$ for some $g \in G$. By Sylow's Theorem we may assume that

$$(Z(S_0) . \langle\eta\rangle)^g \leq S.$$

But then $\eta^g \in S_0$ as $(\eta^g)^2 = i$ and ord $(\eta) = 4$, so η^g acts trivially on $Z(S_0)$. Thus

$$|Z(S_0)\colon C_{Z(S_0)}(\eta)| = 2. \tag{22}$$

However, $|Z(S_0)| = |C_{Z(S_0)}(\eta)|^2$ as ord $(\eta) = 4$, and consequently $|Z(S_0)| = 4$. On the other hand, as $(Z(S_0) \times \langle\xi\rangle)^g \leq S$ we may as well assume that $(Z(S_0) \times \langle\xi\rangle)^g = Z(S_0) \times \langle\xi\rangle$. Now, as $\eta^g \in S_0$, $\xi\eta^g\xi = \eta^g \bmod (Z(S_0))$.

But then η^g is inverted by ξ, as we have seen in Section 1. This is a contradiction as η is not inverted by any element in $Z(S_0)$, and (i) follows.

Next assume that G contains a subgroup isomorphic to U. In order to prove (ii), we may assume that ord $(\eta) = 2$ by (i). If $S \cap G' = S_0 \,.\, \langle \xi \rangle$, $\xi^g = i \in Z(S_0)$ for some $g \in G$ by Lemma 3.4. Thus $C_G(i) = C_i$ contains a subgroup $H = H_0 \times \langle i \rangle$ such that $Z(S_0) \times \langle \xi \rangle$ is a Sylow 2-subgroup of $\langle i \rangle \times H_0$ and $\xi \in H_0$, where $H_0 \simeq PSL(2, q)$. Since $S \leq C_i$ as well, we obtain $O_2(C_i) = \langle i \rangle$. Now let $s \in S_0$ such that $\xi s \xi = s^{-1}$ and $s^2 = i$. For every element a (or subgroup A) of C_i, denote by $\bar{a}$ (resp. $\bar{A}$) the corresponding element (resp. subgroup) of $\bar{C}_i = C_i/\langle i \rangle$. Let $\xi^h = j \in Z(S_0) \cap H_0$ for some $h \in H_0$. Again we may assume that $(\langle \bar{s} \rangle \times Z(S_0)^- \times \langle \bar{\xi} \rangle)^h \leq \bar{S}$ by Sylow's Theorem. Hence

$$(\langle \bar{s} \rangle \times Z(S_0)^- \times \langle \bar{\xi} \rangle)^h = \langle \bar{s}^h \rangle \times Z(S_0)^- \times \langle \bar{\xi} \rangle \tag{23}$$

where $\xi^h = j$. Thus

$$((\langle s \rangle \times_* Z(S_0)) \,.\, \langle \xi \rangle)^h = (\langle s^h \rangle \times_* Z(S_0)) \,.\, \langle \xi \rangle. \tag{24}$$

But then $j = \xi^h$ centralizes s^h, a contradiction.

LEMMA 4.4. *Suppose α_1 is central. Then $Z(S_0)$ is strongly closed.*

Proof. Suppose α_1 is central. Then $E \times Z(S_0)$ is strongly closed if $|E| > 2$ by Lemmas 4.1 and 4.2. But clearly, no element of $E \times Z(S_0) \backslash Z(S_0)$ is conjugate to an involution of $Z(S_0)$. Hence $Z(S_0)$ is strongly closed if $|E| > 2$. Assume therefore that $E = \langle \alpha_1 \rangle$. By Lemma 3.4, α_1 is conjugate to some involution in $\Omega_1(S)$ if E does not have a complement in G. So in that case $|U/U_0|$ is even and $\langle \alpha_1 \rangle$ is conjugate to ξ. In particular, ξ is not a square. But then $S_0 \,.\, \langle \xi\alpha_1 \rangle$ is a complement in P to α_1, so α_1 is conjugate to $\xi\alpha_1$ as well, and again $Z(S_0)$ is strongly closed. Hence E is of order 2 and has a complement in G. But then by Lemma 4.3, we are done.,

COROLLARY. $m \leq n$.

Proof. By Lemma 3.1.

LEMMA 4.5. $\Omega_1(W) > \Omega_1(T)$.

Proof. Suppose not. Let $p \in N_P(W)$. Then $p \in N_P(Z(\Omega_1(W)))$. But $Z(\Omega_1(W)) = E \times Z(S_0)$, so p normalizes

$$C_W(E \times Z(S_0)) = E \times (S_0 \,.\, \langle \xi \rangle) \tag{25}$$

and hence also $E \times S_0$. Thus $p \in W$ by definition, so $P = W$.

By Lemma 4.2 $Z(S_0)$ is strongly closed if $|E| > 2$. So assume $E = \langle \alpha_1 \rangle$. By Lemma 4.4, α_1 is not central, so T is a proper subgroup of W. Hence α_1 is conjugate to $\alpha_1 i$ for all $i \in Z(S_0)$ by the action of σ_{q-1} on $w^{-1}\alpha_1 w$, where $w \in W \backslash T$. If $Z(S_0)$ is not strongly closed, the involutions of $Z(S_0)$ are conjugate to $\gamma\xi$ for some $\gamma \in E$. Hence α_1 is conjugate to $\gamma\xi\alpha_1$ by Lemma 4.2. Now, as in the proof of Lemma 4.2, there exists an $h \in G$ such that $i^h = \gamma\xi$ and

$$(\langle\alpha_1\rangle \times Z(S_0) \times \langle\xi\rangle)^h = \langle\alpha_1\rangle \times Z(S_0) \times \langle\xi\rangle. \tag{26}$$

On the other hand,

$$h^{-1}ih\sigma_{q+1}h^{-1}ih = \xi\gamma\sigma_{q+1}\xi\gamma = \sigma_{q+1}^{-1} \tag{27}$$

Hence $ih\sigma_{q+1}h^{-1}i = h\sigma_{q+1}^{-1}h^{-1}$. By the structure of U, $h\sigma_{q+1}h^{-1}$ does not belong to U_0 then, since $h\sigma_{q+1}h^{-1}$ is not real in U_0. Thus $h\sigma_{q+1}h^{-1} \notin C$. Consequently, $\alpha_1^h \notin E \times Z(S_0)$, so $\alpha_1^h = \xi\gamma\alpha_1 j$ for some $j \in Z(S_0)$. Let $s \in M_0$ such that $s^{-1}\xi s$ equals ξj. Then $\alpha_1^{hs} = \xi\gamma\alpha_1$ and $i^{hs} = \xi\gamma j$. Now let c be an arbitrary element in $C_{U_0}(\xi)$. Then

$$\alpha_1 hscs^{-1}h^{-1}\alpha_1 = hs\xi\gamma\alpha_1 c\xi\gamma\alpha_1 s^{-1}h^{-1} = hscs^{-1}h^{-1} \tag{28}$$

so $hsC_{U_0}(\xi)s^{-1}h^{-1} \leq C$. As $C_{U_0}(\xi) \simeq PSL(2, q)$ we deduce that

$$hsC_{U_0}(\xi)s^{-1}h^{-1} \leq U_0.$$

Thus

$$Z(S_0)^{s^{-1}h^{-1}} \leq (\langle\alpha_1\rangle \times Z(S_0) \times \langle\xi\rangle) \cap U_0 = Z(S_0), \tag{29}$$

a contradiction since $i^{hs} = \xi\gamma j$.

COROLLARY 1. $\Omega_1(W) \leq N_P(S_0)$.

Proof. Let $t \in \Omega_1(W)\backslash T$ be an involution. Then t acts on

$$T_1 = O_{2,2'}(C_G(E \times Z(S_0))) = E \times (S_0 \cdot \langle\sigma_{q+1}\rangle) \tag{30}$$

so t acts on $T'_1 = S_0$.

COROLLARY 2. $\Omega_1(W) \leq C_P(Z(S_0))$.

Proof. By Corollary 1 and the corollary of Theorem 1.1.

LEMMA 4.6. (i) $\sigma_{q-1} \in N_G(\Omega_1(W))$.

(ii) *W contains a normal subgroup $W_0 > T_0$, which is a complement to $\langle\eta\rangle$ such that $W_0/T_0 \simeq E_{2^n}$. Moreover σ_{q-1} acts faithfully and irreducibly on W_0/T_0, and $W_0/Z(S_0)$ is elementary abelian.*

Proof. Let $\tau \in \Omega_1(W)\backslash\Omega_1(T)$ be an involution and $\alpha \in E^{\#}$. Then τ acts trivially on $Z(S_0)$ and on $E \times Z(S_0)/Z(S_0)$ as well, as E is a T.I.-set and $|N_G(E): C|$ is odd. Thus $\tau\alpha\tau = \alpha i$ for some $i \in Z(S_0)$. So for any k there exists an r such that $\tau\tau^{\rho^k} = \tau^{\rho^r}$ mod (C), where $\rho = \sigma_{q-1}$ and r is determined by $ii^{\rho^k} = i^{\rho^r}$. Now let $\tau\tau^{\rho^k} = \tau^{\rho^r}a$, where $a \in C$. As τ, τ^{ρ^k}, τ^{ρ^r} belong to $C_G(Z(S_0))$,

$$a \in C_G(Z(S_0)) \cap C_G(\alpha) = E \times (S_0 \cdot (\langle\sigma_{q+1}\rangle \cdot \langle\xi\rangle)). \tag{31}$$

By Corollary 1 of Lemma 4.5, τ acts on S_0. Suppose $|U/U_0|$ is odd. Then

$$a \in E \times (S_0 \,.\, \langle \sigma_{q+1} \rangle).$$

But $\tau\tau^{\rho^k}$ acts trivially on $S_0/Z(S_0)$ by Theorem 1.1 as σ_{q-1} centralizes ξ. Thus, as $\tau^{\rho^r} = \tau\tau^{\rho^k}a^{-1}$, τ^{ρ^r} acts trivially on $S_0/Z(S_0)$, so does τ.

If τ acts trivially on $S_0/Z(S_0)$, independently of whether $|U/U_0|$ is odd or not, a does as well, and it follows that $a \in T_0$.

If τ acts nontrivially on $S_0/Z(S_0)$, $|U/U_0|$ is even by the remark above. In particular, as $\tau\xi \in W$, $\tau\xi$ must act trivially on $S_0/Z(S_0)$, again by the structure of A_B. Consequently $(\tau\xi)^2 \in Z(T_0)$. Hence $(\tau\xi)^2$ actually belongs to $Z(S_0)$, since no element of $Z(T_0)\backslash Z(S_0)$ is a square. Let $t = \tau\xi$. It now follows that

$$a = (t\xi)^{\rho^{-r}}(t\xi)(t\xi)^{\rho^k} = t^{\rho^{-r}}t^{-1}t^{\rho^k}\xi \tag{32}$$

is a 2-element, and (i) follows.

To prove (ii), we define W_0 as follows. If τ acts trivially on $S_0/Z(S_0)$, we let $W_0 = \langle T_0, \tau^{\rho}, \ldots, \tau^{\rho^{q-1}} \rangle$. If τ acts nontrivially on $S_0/Z(S_0)$, we let $W_0 = \langle T_0, t^{\rho}, \ldots, t^{\rho^{q-1}} \rangle$. For any $w \in W$, clearly $w = \tau^{\rho^z}t_0$ for some $z \in \mathbf{N}$, $t_0 \in T$, so $W = \langle W_0, \eta \rangle$. As $S_0 \trianglelefteq W$, it follows from the structure of A_B that $W_0 \trianglelefteq W$. Also, W_0/T_0 acts trivially on $T_0/Z(S_0)$. Now, let $w \in W_0$. Then $w^2 \in T_0$. On the other hand, as w acts trivially on $S_0/Z(S_0)$, $w^2 \in C_{W_0}(S_0)$. Hence $w^2 \in E \times Z(S_0)$ and it follows that $W_0/Z(S_0)$ is elementary abelian.

We can now determine the structure of W_0 completely.

LEMMA 4.3.7. *$W_0 = S_0 \,.\, C_{W_0}(S_0)$ and $C_{W_0}(S_0) = F \,.\, E$, where $F \cap T_0 = Z(S_0)$ and $F/Z(S_0) \simeq E_{2^n}$. Moreover, $F \trianglelefteq W$.*

Proof. It is not difficult to see that σ_{q+1} acts on W_0. But $\overline{W}_0 = W_0/E \times Z(S_0)$ is elementary abelian, from which it follows that $\bar{S}_0 = S_0 \times E/E \times Z(S_0)$ has a complement in $\overline{W}_0$ under the action of σ_{q+1}, $\bar{F}_0 = F_0/E \times Z(S_0)$. Once again, $E \times Z(S_0)/Z(S_0)$ has a complement in $F_0/Z(S_0)$, say $\bar{F} = F/Z(S_0)$, as $F_0/Z(S_0)$ is elementary abelian by Lemma 4.6(ii). Since σ_{q+1} acts trivially on $Z(S_0)$ and $F/Z(S_0) \simeq Z(S_0)$, σ_{q+1} actually centralizes F. Let $f \in F$ be any element outside $Z(S_0)$. Then $|S_0 \colon C_{S_0}(f)| \leq 2^n$. As f centralizes σ_{q+1} and $\langle S_0, \sigma_{q+1} \rangle' = S_0$, we deduce immediately that f centralizes S_0, and the first part of the lemma follows.

In order to prove the last statement we note that $F \,.\, E$ is normal in W. Furthermore, $\langle \sigma_{q-1} \rangle$ is normalized by η. Clearly σ_{q-1} acts on $F \,.\, E$ and hence on $F \,.\, E/Z(S_0)$. Therefore $E \times Z(S_0)/Z(S_0)$ has a complement under the action of σ_{q-1}, which we may as well assume to be F itself. It now follows that

$$((F \times_* S_0) \,.\, \langle \sigma_{q-1} \rangle) \,.\, (E \times \langle \eta \rangle)' \leq (F \times_* S_0) \,.\, \langle \sigma_{q-1} \rangle \tag{33}$$

is normalized by η and hence that $F \trianglelefteq W$.

LEMMA 4.8. *F is homocyclic of exponent* 2 *or* 4.

Proof. We have seen that $F/Z(S_0)$ and $Z(S_0)$ are isomorphic as σ_{q-1}-modules. Also $Z(S_0) \leq Z(F)$. Hence all elements in a coset of $Z(S_0)$ in F have the same order. If F is not elementary abelian, $\Omega_1(F) = Z(S_0)$ and F is homocyclic of exponent 4 if abelian, otherwise of Suzuki A-type by definition. However, the last case is impossible, since, by [6], this would imply that $F/Z(S_0)$ and $Z(S_0)$ are not isomorphic as σ_{q-1}-modules.

LEMMA 4.9. *Suppose $P = W$. Then $Z(S_0)$ is strongly closed in W_0 with respect to G. In particular $Z(S_0)$ is strongly closed if $W_0 = W$.*

Proof. An involution of $W_0 \backslash E \times Z(S_0)$ is of the form vs where $v \in F . E$ and $s \in S_0$, and $C_{W_0}(vs) \geq C_{S_0}(s) \times \langle vs \rangle$. If $(vs)^g \in Z(S_0)$ for some $g \in G$, we may as well assume that W contains $(C_{W_0}(vs))^g$ by Sylow's Theorem. But $\Omega_1(C_{S_0}(s))$ is equal to $Z(S_0)$, and every involution in $Z(S_0)$ is a square in $C_{S_0}(s)$, which contains a maximal abelian subgroup M of S_0. As $W = W_0 . \langle \eta \rangle$, $M^g = M_1 \times \langle m \rangle$, where $M_1 \leq W_0$, for some $m \in M^g$. Furthermore,

$$\langle vs \rangle^g \times \Omega_1(M_1) = Z(S_0) \tag{34}$$

as $Z(S_0) = \mho^1(W_0)$. Since $M \times \langle vs \rangle$ is abelian,

$$M^g \leq C_W(Z(S_0)) = W_0 . \langle \xi \rangle. \tag{35}$$

As any square in $W_0 . \langle \xi \rangle$ lies in S_0 and m^2 is an involution, $m^2 \in Z(S_0)$, a contradiction.

LEMMA 4.10. *Suppose $P = \langle P \cap G', E \rangle$ and assume furthermore that $Z(S_0)$ is not strongly closed in P with respect to G.*

(i) *Suppose $|E| = 2$ and $P = W$. Then E has a complement in G.*

(ii) *Suppose $|E| = 2$. Then $P > W$.*

(iii) *Suppose $P = W$. Then* ord $(\eta) \leq 2$.

(iv) *The weak closure of E in P is not contained in $E \times Z(S_0)$.*

Proof. (i) Suppose $|E| = 2$ and $P = W$. As $F \times_* S_0/Z(S_0)$ is the direct sum of three isomorphic σ_{q-1}-modules, it follows that if w_0 is an involution in $F \times_* S_0$ then $C_{F \times_* S_0}(w_0)$ contains an elementary abelian group of order 2^{2n}. In particular, α_1 is not conjugate to any involution in $F \times_* S_0$. Now suppose E does not have a complement in G. Then $\alpha_1^g = \xi fs$ for some $f \in F \backslash Z(S_0)^\#$, $s \in S_0$ and $g \in G$ by Lemma 3.3 since $(F \times_* S_0) . \langle \eta \rangle$ is a complement in P to α_1. But then ξ inverts fs, so $\xi s \xi = s \bmod (Z(S_0))$. Thus ξ inverts s (see Section 1) and hence ξfs is conjugate to ξf. Suppose $f \neq 1$. If F is elementary abelian, $C_{W_0}(\xi f) \geq F$ in contradiction to the assumption that α_1 is conjugate to ξf. If F is of exponent 4, f centralizes the diagonal D of F and M_0, $D \simeq E_{2^n}$. But ξf is conjugate to $\xi f s_1$ where $s_1 \in M_0$ and $s_1^2 = f^2$. Hence $fs_1 \in D$ and a conjugate of α_1 centralizes $D \times Z(S_0)$, again a contradiction. Thus $f = 1$

and α_1 is conjugate to ξ. In particular, ord $(\eta) = 2$. By the same argument, α_1 is conjugate to $\alpha_1\xi$. It now easily follows that all involutions of $W\backslash W_0$ are conjugate to α_1. Hence $Z(S_0)$ is strongly closed by Lemma 4.9.

(ii) Let G_0 be a complement of E in G by (i). We may as well choose notation so, that a Sylow 2-subgroup of G_0 is of the form $(F \times_* S_0) \cdot \langle\eta\rangle$. Moreover, we may assume that ord $(\eta) \leq 4$ by Lemma 3.2, and if ord $(\eta) = 4$ that ξ is conjugate to an involution in $Z(S_0)$ by Lemma 4.9, again using the fact that if $m \in W_0 \cdot \langle\xi\rangle$ is of order 4, then $m^2 \in Z(S_0)$. If on the other hand ord $(\eta) = 2$ it follows immediately from Lemma 3.4 that ξ is conjugate to some involution fs in $F \times_* S_0$. But then $C_G(\xi)$ contains subgroups isomorphic to $F \times_* C_{S_0}(s)$ and $\langle\alpha_1\rangle \times Z(S_0) \times \langle\xi\rangle$, so ξ is not extremal in P. Furthermore, if $\xi^g \in P$ is extremal, ξ^g centralizes some conjugate of α_1 lying in P, which, by (i), must be of the form $\alpha_1 v$ for some $v \in (F \times_* S_0) \cdot \langle\xi\rangle$. It is now easy to see, using Lemma 4.9, that $\xi^g \in Z(S_0)$. Thus, in any case, ξ is conjugate to an involution of $Z(S_0)$. Now, let $i \in Z(P) \leq Z(S_0)$ and $C_i = C_G(i)$. Obviously, as ξ is conjugate to i, $O_2(C_i) = \langle i\rangle$, since $C_G(\xi)$ contains a subgroup isomorphic to $PSL(2, q)$. Thus we may use the idea in the proof of Lemma 4.3 (ii). For every element a (or subgroup A) of C_i, denote by $\bar{a}$ (resp. $\bar{A}$) the corresponding element (resp. subgroup) of $\bar{C}_i = C_i/\langle i\rangle$. Now, $(\xi)^h = j$ belongs to $Z(S_0)\backslash\langle i\rangle$ for some $h \in C_i$, where $\bar{j} \in Z(W)^-$. Let $s \in M_0, s^2 = i$. Then, as $\langle\bar{s}\rangle \times Z(S_0)^- \times \langle\bar{\xi}\rangle \leq C_{\bar{C}_i}(\bar{\xi})$, we may assume by Sylow's Theorem that $(\langle\bar{s}\rangle \times Z(S_0)^- \times \langle\bar{\xi}\rangle)^h \leq \bar{W}$. But then

$$((\langle s\rangle \times_* Z(S_0)) \cdot \langle\eta\rangle)^h \leq (F \times_* S_0) \cdot \langle\xi\rangle \tag{36}$$

so $j = (\xi)^h$, a contradiction as ξ inverts s. Now, by Lemma 4.9, we are done.

(iii) By Lemma 4.7, $W = (F \times_* S_0) \cdot (E \times \langle\eta\rangle)$. Hence we may assume, by Lemma 3.2, that ord $(\eta) \leq 4$. Furthermore, if ord $(\eta) = 4$, then, by Lemma 3.3, ξ is conjugate to a square in $W_0 \cdot \langle\xi\rangle$, i.e. to an involution $i \in Z(S_0)$, say $\xi^g = i \in Z(W)$ for some $g \in G$. Now we use the idea of the proof of Lemma 4.3 (i). By Lemma 4.9, $Z(S_0)$ is strongly closed in W_0 w.r.t. G. Thus we may assume by Sylow's Theorem that

$$(Z(S_0) \times \langle\xi\rangle)^g = Z(S_0) \times \langle w_0\xi\rangle \tag{37}$$

for some $w_0 \in W_0$. Furthermore, since $(\eta^g)^2 = i$, $\eta^g \in W_0 \cdot \langle\xi\rangle$. Hence η^g centralizes $Z(S_0)$ and thus $n = 2$. Now, as E is a T.I.-set and ord $(\eta) = 4$ by assumption, $|E| = 2$, and, by (ii), we are done.

(iv) If the weak closure of E in W w.r.t. G is contained in $E \times Z(S_0)$, then $P = W$. Hence we may assume, by (iii) and Lemma 4.9, that ord $(\eta) = 2$ and after possibly change of notation, by Lemma 3.4, that ξ is conjugate to some involution of W_0 say $\xi^g = vs$, $v \in F \cdot E$, $s \in S_0$, for some $g \in G$. But then $C_G(\xi)$ contains subgroups isomorphic to $E \times Z(S_0) \times \langle\xi\rangle$ and $F \times_* C_{S_0}(s)$. This, together with the assumption that the weak closure of E is contained in

$E \times Z(S_0)$, again implies that ξ is conjugate to an involution of $Z(S_0)$. This is a contradiction by Lemma 4.2.

LEMMA 4.11. *Suppose F is homocyclic of exponent 4. Then either $Z(S_0)$ is strongly closed or $|E| = 2$ and α_1 inverts F.*

Proof. Assume $Z(S_0)$ is not strongly closed in P w.r.t. G. Then, by Lemma 4.10 (iv), $E \times Z(S_0)$ is not weakly closed in W. Suppose some $\alpha \in E^{\#}$ does not invert F. This will occur if $|E| > 2$. Let $F = \langle f_1, \ldots, f_n \rangle$ and $f \in F \backslash Z(S_0)$. If $\alpha f \alpha = ff^2 i$, where $i \in Z(S_0)^{\#}$, and $s \in S_0$ such that s^2 equals i, αfs is an involution. Assume therefore that αfs is conjugate to some involution in E. As $C_{S_0}(\alpha fs) = C_{S_0}(s)$ equals some maximal abelian subgroup $M_1 = \langle s_{11}, \ldots, s_{1n} \rangle$ in S_0 we may choose notation such that

$$C_W(\alpha fs) \geq E_\alpha = \langle \alpha_1 f_1 s_{11}, \ldots, \alpha_m f_m s_{1m} \rangle \simeq E_{2^m} \tag{38}$$

as E is a T.I.-group and σ_{q-1} centralizes $F \cdot E$. Let $M_2 = \langle s_{21}, \ldots, s_{2n} \rangle$ be another maximal abelian subgroup of S_0. Here we choose notation such that $\alpha\alpha^{f_i} = [s_{1i}, s_{2i}]$. Then

$$M_3 = \langle f_1 s_{21}, \ldots, f_n s_{2n} \rangle \leq C_W(\alpha fs).$$

Now fix s_{2k_0} for some $k_0 \in \mathbf{N}$. For every $j \in Z(S_0)^{\#}$ there exists an $s_j \in M_1$ such that $(s_{2k_0} s_j)^2 = j$, in particular if $j = j_{k_0} = f_{k_0}^2$. Hence $f_{k_0} s_{2k_0} s_j$ is an involution, so $\Omega_1(\langle M_1, M_3 \rangle) > Z(S_0)$ and thus $\langle M_1, M_3 \rangle$ is not a Suzuki 2-group. Thus, by Lemma 4.10(iv), an involution of the form fs, $f \in F \backslash Z(S_0)$, $s \in S_0 \backslash Z(S_0)$ is conjugate to an element of E. As every noncentral involution in $F \times_* S_0$ belongs to an elementary abelian subgroup of order 2^{2n} then $|E| = 2^n$ and we may reverse the above process. If $s_0 \in S_0 \backslash C_{S_0}(s)$, there exists an $\alpha_0 \in E$ such that $[\alpha_0, f] = [s_0, s]$. Now let $f_0 \in F$ such that $(\alpha_0 f_0)^2 = s_0^2$. Then $\alpha_0 f_0 s_0$ is an involution centralizing fs, a contradiction. Since any involution of $W_0 \backslash E \times Z(S_0)$ is of the form $\alpha_0 f_0 s_0$ or $f_0 s_0$ for suitable $\alpha_0 \in E^{\#}$, $f_0 \in F \backslash Z(S_0)$, we have reached a final contradiction.

LEMMA 4.12. *Suppose F is homocyclic of exponent 4. Then $Z(S_0)$ is strongly closed.*

Proof. We will prove this in a series of steps by way of contradiction. So assume $Z(S_0)$ is not strongly closed. By the previous lemma, $|E| = 2$ and α_1 inverts F. Moreover, by Lemma 4.10(ii), $P > W$.

As W_0 char W, $N_G(W) \leq N_G(W_0)$ and in particular $V > W$. Now let $v \in V \backslash W$ such that $v^2 \in W$. Then, by Lemma 4.11, $v^{-1} \alpha_1 v = \alpha_1 f$ for some $f \in F$. Furthermore, $v^{-2} \alpha_1 v^2 = \alpha_1 i$ for some $i \in Z(S_0)$, so v acts on

$$C_{W_0}(\alpha_1) \cap C_{W_0}(\alpha_1^v) = S_0. \tag{39}$$

Hence v acts on $C_{W_0}(S_0) = F.\langle\alpha_1\rangle$ as well, so v acts on F. By counting conjugates of α_1 we obtain

$$|\langle W, v, \sigma_{q-1}\rangle : \langle W, \sigma_{q-1}\rangle| = q. \tag{40}$$

Clearly $v \notin N_G(\langle W, \sigma_{q-1}\rangle)$.

(1) Suppose $W_0 < W$. Then $r(\Omega_1(V/W_0)) > 1$.

Proof. Suppose not. Then the above remarks and Theorem 1.1 imply that V/W_0 is cyclic. But then $\langle W, v, \sigma_{q-1}\rangle/W$ has a cyclic Sylow 2-subgroup, a contradiction since this forces v to lie in $N_G(\langle W, \sigma_{q-1}\rangle)$.

(2) V contains a normal subgroup $V_0 > W_0$ which is a complement to $\langle\eta\rangle$. Moreover $V_0/W_0 \simeq E_{2^n}$.

Proof. Let v be as above. By (1) we may replace v by v_0 such that in addition we have $v_0^2 \in W_0$. We now use the idea in the proof of Lemma 4.6. v_0 acts trivially on $W_0/F \times_* S_0$, and E has q conjugates in W_0 under the action of $\langle F, v_0, \sigma_{q-1}\rangle$. Moreover we have seen in (39) that v_0 acts on S_0. As in the proof of Lemma 4.6 we find that v_0 acts trivially on $S_0/Z(S_0)$ if $\eta = 1$. If $\eta \neq 1$, either v_0 or $v_0\xi$ acts trivially on $S_0/Z(S_0)$ by Theorem 1.1. Thus we may as well assume that v_0 acts trivially on $S_0/Z(S_0)$. Therefore

$$|\langle W_0, v_0, \sigma_{q-1}\rangle : \langle W_0, v_0\rangle| = q \tag{41}$$

and $\langle W_0, v_0, \sigma_{q-1}\rangle$ has a normal Sylow 2-subgroup V_0. Now (2) follows easily.

(3) $V_0 = (R \times_* S_0).\langle\alpha_1\rangle$ where $F \leq R \leq C_{V_0}(S_0)$ and R/F is isomorphic to E_{2^n}.

Proof. We first observe that σ_{q+1} acts on V_0 as

$$F.\langle\alpha_1\rangle \leq C_G(\sigma_{q+1}).$$

Moreover, $V_0/F.\langle\alpha_1\rangle$ is elementary abelian since $v_0^2 \in C_{V_0}(S_0)$. Hence $\bar{V}_0 = V_0/F$ is elementary abelian, so

$$(S_0 \times \langle\alpha_1\rangle)^- = S_0 \times \frac{\langle\alpha_1\rangle . F}{F} \tag{42}$$

has a complement $\bar{R} = R/F$ under the action of σ_{q+1}. As $\bar{R}$ is isomorphic to E_{2^n}, σ_{q+1} centralizes R. Now let $u \in R\backslash F$. As u acts trivially on $S_0/Z(S_0)$ and S_0 is of Suzuki B-type, $\langle C_{S_0}(u), \sigma_{q+1}\rangle \geq S_0$ and (3) follows.

Thus we have essentially two cases to consider, depending on whether exp (R) equals 4 or 8.

(4) α_1 is not conjugate to any involution of $R \times_* S_0$.

Proof. Let $u \in R \times_* S_0$ be an involution. Suppose that $u \notin F \times_* S_0$. Then

$R \times_* S_0/Z(S_0)$ is elementary abelian, so $O_2(\langle u, \sigma_{q-1}\rangle)$ is of exponent 2 and thus u is not conjugate to α_1.

(5) $\exp(R) = 4$ and $P > V$.

Proof. Suppose not. Define $V_1 = V = N_P(W_0)$, $V_{1,0} = V_0$ which also equals $C_{V_1}(S_0) \times_* S_0$, and in general $V_k = N_P(V_{k-1,0})$ and $V_{k,0} = C_{V_k}(S_0) \times_* S_0$. Finally, let $R_0 = F$.

(a) Assume $\exp(R) > 4$. Then, if $V_k \leq \langle P \cap G', T\rangle$, $V_{k,0}$ contains a subgroup R_k such that

(i) R_k is a complement in $C_{V_k}(S_0)$ to α_1 containing R_{k-1} and normalized by σ_{q-1},

(ii) R_k is homocyclic and inverted by α_1,

(iii) $V_k = (R_k \times_* S_0) . \langle\alpha_1, \eta\rangle$.

To prove this we use induction on k. The case $k = 1$ has partly been considered in (3), where (i) and (iii) were proved, while (ii) follows from Theorem 2.1. Suppose (a) has been established for all $k \leq h$ and assume $P > V_h$. (Note that $R = R_1$.) Let $v \in N_P(V_h)\backslash V_h$ such that $v^2 \in V_h$. Clearly v acts on $R_h \times_* S_0$. As $\exp(R_h) \geq 8$, $S_0^v \cap R_h = Z(S_0)$, and v acts on $F = \mho^a(R_h \times_* S_0)$ for some a. Thus $S_0 \cap S_0^v > Z(S_0)$. Furthermore, if $s \in (S_0 \cap S_0^v)\backslash Z(S_0)$ and $w \in (R \times_* S_0) . \langle\alpha_1, \xi\rangle$ is an involution centralizing s, then $w \in (R \times_* S_0) . \langle\alpha_1\rangle$. Thus v acts on $(R_h \times_* S_0) . \langle\alpha_1\rangle = V_{h,0}$, which is the crucial point in the proof of (5). Also, $\alpha_1^v = \alpha_1 r_1$ for some $r_1 \in R_h$. Now (a) follows easily by using the arguments proving (1) through (3). v acts on

$$\bigcap_{r=1}^{d} C_{V_h}(v^{-2^r}\alpha_1 v^{2^r}) = S_0 \tag{43}$$

where $2^d = \operatorname{ord}(v)$, and on R_k for all $k \leq h$ as well of course. Now, if $W_0 < W$ and $r(\Omega_1(V_{h+1}/V_h)) = 1$, V_{h+1}/V_h is cyclic by Theorem 1.1. Moreover, $\Omega_1(V_{h+1}) = \Omega_1(W) \leq W_0 . \langle\xi\rangle$ since R_h is homocyclic. As $\Omega_1(R_h) = Z(S_0)$ it is easy to verify as in Lemma 4.9 that $Z(S_0)$ is strongly closed in $V_{h+1,0} = V_{h,0}$ and we reach a contradiction as in Lemma 4.10(ii), since we have assumed that $V_{h+1} \leq \langle P \cap G', T\rangle$. Now (i), (ii) and (iii) follows by exactly the same argument as was used to prove (2) and (3), while (ii) follows from Theorem 2.1.

Thus we may assume that $P = V_k$ for some k. Let P_0 denote $V_{k,0}$ and set $Q = R_k$. Then Q is either homocyclic or of class 2 and exponent 4 (and equal to R).

(b) α_1 has a complement in G.

If not, α_1 is conjugate to some involution in $(Q \times_* S_0) . \langle\eta\rangle$ by Lemma 3.4. Hence α_1 is conjugate to some involution of the form ξus, where $u \in Q$ and $s \in S_0$, by the same argument that proves (4). Then ξ inverts u and s, so ξus

is conjugate to ξu and ξ inverts $H = O_2(\langle u, \sigma_{q-1}\rangle)$, which therefore is homocyclic. If $H \simeq E_{2^n}$, $C_P(\xi u) \geq H \times Z(S_0)$, a contradiction unless $u \in Z(S_0)$. If $\exp(H) = 4$, ξu centralizes the diagonal $D \simeq E_{2^n}$ of H and M_0. But then ξu is conjugate to ξd for some $d \in D$ and $C_P(\xi d) \geq D \times Z(S_0)$, again a contradiction. Finally, if $\exp(H) \geq 8$, ξ inverts F. But then $C_P(\xi u) \geq D \times Z(S_0)$, where $D \simeq E_{2^n}$ is the diagonal of F and M_0, a contradiction. Thus $u \in Z(S_0)$ and α_1 is conjugate to ξ. In particular ord $(\eta) = 2$. But then $(Q \times_* S_0) . \langle \alpha_1 \xi \rangle$ is a complement in P to α_1 as well, so by the same argument α_1 is conjugate to $\alpha_1 \xi$. Now, let $f \in F \backslash Z(S_0)$. Then $f^\xi = fi$ for some $i \in Z(S_0)$. Suppose $i \neq 1$ and let $s \in M_0$ such that $s^2 = i$. Then ξ centralizes fs, so ξ centralizes the normal Sylow 2-subgroup H of $\langle fs, \sigma_{q-1}\rangle$, which is homocyclic. In particular, $\exp(H) = 4$, since otherwise ξ centralizes $H \times Z(S_0) \simeq E_{2^{2n}}$, contrary to the fact that ξ is conjugate to α_1. But α_1 acts on H and $C_H(\alpha_1) = Z(S_0)$. Hence a Sylow 2-subgroup of $C_G(\xi)$ contains a homocyclic subgroup of exponent 4 and order 4^n, and an involution, conjugate to α_1 which acts nontrivially on H. Thus a similar situation occurs in $T \in \mathrm{Syl}_2(C)$. By inspecting T we see that the involution in question must be of the form ξs_0 or $\alpha_1 \xi s_0$ for some $s_0 \in S_0$. Conjugating by an element of S_0 we may therefore assume that the involution has the form ξ or $\alpha_1 \xi$. Moreover, the homocyclic subgroup in question has an intersection with S_0 which contains an element s_1 of order 4. Hence s_1 is inverted by that involution. Thus α_1 inverts some element of order 4 in H, a contradiction since α_1 inverts F. This shows that $i = 1$, so ξ centralizes F. But then $\alpha_1 \xi$ inverts F, a contradiction since α_1 is conjugate to $\alpha_1 \xi$. This proves (b).

Hence α_1 has a complement in G with $(Q \times_* S_0) . \langle \eta \rangle$ as Sylow 2-subgroup (at least we may have chosen notation so). It is now clear that we must proceed by reaching a contradiction of the same nature as that in the proof of Lemma 4.10(ii). However, we can no longer expect to prove by a short argument that $Z(S_0)$ is strongly closed in $Q \times_* S_0$ w.r.t. G due to the fact that involutions of $Q \times_* S_0$ may be squares in $(Q \times_* S_0)$, even if they do not belong to $Z(S_0)$. So we must go the opposite way this time so to speak, namely, prove that ξ "transfers out", in which case it will be trivial to verify that $Z(S_0)$ is strongly closed in $Q \times_* S_0$, and (5) will follow.

Now, let us consider possible conjugates of α_1 in P. By (b), every conjugate of α_1 in P is of the form $\alpha_1 us$ or $\alpha_1 \xi us$ for some $u \in Q$, $s \in S_0$. Suppose $\alpha_1 us$ is an involution. Then α_1 inverts u and s modulo $Z(S_0)$, since $Q \cap S = Z(S_0)$ and $[Q, S_0] = \langle 1 \rangle$. Hence $u \in F$, i.e. u is inverted by α_1, so $s \in Z(S_0)$. It follows immediately that

$$C_P(\alpha_1 us) \cap (Q \times_* S_0) = S_0. \tag{44}$$

Next, assume that $\alpha_1 \xi us$ is an involution. Then $\alpha_1 \xi us$ is conjugate to $\alpha_1 \xi u$ in P. Furthermore, $\alpha_1 \xi$ inverts the normal Sylow 2-subgroup H of $\langle u, \sigma_{q-1}\rangle$. Suppose $u \notin Z(S_0)$. If ord $(u) = 2$, $Cp(\alpha_1 \xi u) \geq H \times Z(S_0)$, a contradiction. If $u^2 \neq 1$ let $H_0 \leq H$ be the subgroup of order 4^n and exponent 4. Then

$C_P(\alpha_1\xi u) \geq D \times Z(S_0)$, where $D \simeq E_{2^n}$ is the diagonal of H_0 and M_0, again a contradiction. Thus $u \in Z(S_0)$. In this case $\langle\alpha_1\rangle \times Z(S_0) \times \langle\xi\rangle$ is a maximal elementary abelian subgroup of a Sylow 2-subgroup of $C_G(\alpha_1\xi us)$, and

$$\Omega_1(C_P(\alpha_1\xi us)) \cap (Q \times_* S_0) = Z(S_0). \tag{45}$$

(c) ξ is not conjugate to an involution of $Z(S_0)$.

Suppose $\xi^g = i \in Z(P) \leq Z(S)$ for some $g \in G$. Let C_i denote $C_G(i)$. Now, $C_G(\xi)$ contains as a subgroup $\langle\alpha_1, \xi\rangle \times L$, where $L = C_{U_0}(\xi) \simeq PSL(2, q)$. By Sylow's Theorem we may assume that $(\langle\alpha_1\rangle \times Z(S_0))^g \leq P$. Clearly, as $Z(S_0) \in \mathrm{Syl}_2\,(L)$,

$$Z(S_0)^g \cap O_2(C_i) = \langle 1\rangle. \tag{46}$$

On the other hand, by our determination of conjugates of α_1 in P above,

$$(\langle i\rangle \times Z(S_0)^g) \cap (Q \times_* S_0) = Z(S_0). \tag{47}$$

Furthermore, since L has one conjugacy class of involutions, it follows that there exists an $h \in C_i$ such that $\xi^h \in Z(S_0)$. Now we reach a contradiction exactly as in the second part of the proof of Lemma 4.10(ii).

We may now finish the proof of (5) by "extremal" arguments. It follows immediately from (c) and Lemma 3.2 that $\eta^4 = 1$. Furthermore, if ord $(\eta) = 4$ and $\eta \in P \cap G'$, then, by Lemma 3.3, η is conjugate to an element of $(Q \times_* S_0)\,.\,\langle\xi\rangle$ so ξ is conjugate to an element of $Q \times_* S_0$, while if ord $(\eta) = 2$ and $\xi \in P \cap G'$ this follows from Lemma 3.4. Let in any case ξ^g be an extremal conjugate of ξ in P. Then it follows immediately from our determination of the conjugates of α_1 in P that $\xi^g \in Z(S_0)$, a contradiction by (c). Thus $\eta = 1$. As mentioned earlier this implies that $Z(S_0)$ is strongly closed in P w.r.t. G, contrary to our assumption.

(6) $V = (S_0^1 \times_* S_0^2)\,.\,\langle\alpha_1, \eta\rangle$, where $S_0 = S_0^1 \simeq S_0^2, F \leq S_0^2$ and $|P\colon V| = 2$. Furthermore, there exists a $\kappa \in P$ such that $\alpha_1^\kappa = \alpha_1\xi$ and $(M_0)^\kappa = F$, and α_1 has no conjugate in $P\backslash V$.

Proof. By (4), α_1 is not conjugate to any involution of $R \times_* S_0$. By (5) there is a $p \in N_P(V)\backslash V$ such that $p^2 \in V$. As exp $(R) = 4$, $p^{-1}\alpha_1 p$ does not belong to $(R \times_* S_0)\,.\,\langle\alpha_1\rangle$, although clearly $p \in N_P(R \times_* S_0)$. Furthermore, if ord $(\eta) > 2$, $(R \times_* S_0)\,.\,\langle\xi\rangle$ is normalized by p, while if ord $(\eta) = 2$ we may assume this to be the case. Thus $p^{-1}\alpha_1 p = \alpha_1\xi us$ for some $u \in R$, $s \in S_0$. As in the proof of (5), this however forces u to lie in $Z(S_0)$, and $\alpha_1\xi us$ is conjugate to $\alpha_1\xi$ in P. It now follows that V has the claimed structure and that α_1 and $\alpha_1\xi$ are conjugate in $N_P(V)$ by some κ where $\kappa^2 \in V$. Also, $N_P(V) = \langle V, \kappa\rangle$ (note that $V = N_P(V_0)$). Now, if $P > N_P(V)$, there exists an involution $v \in N_P(V)\backslash V$ such that v is conjugate to α_1 and

$$C_{N_P(V)}(v) = \langle v\rangle \times C_V(v) \simeq T = \langle\alpha_1\rangle \times S.$$

This implies that v centralizes $Z(S_0)$. Moreover, as $F^v = M_0$, v centralizes the diagonal of F and M_0, which is isomorphic to E_{2^n}, a contradiction to the assumption that v is conjugate to α_1. Thus we have established (6).

(7) Either $P/(S_0^1 \times_* S_0^2) \cdot \langle \alpha_1 \rangle$ is cyclic or there exists a $\nu \in P \backslash V$ such that $\alpha_1^\nu = \alpha_1 \xi$, $\nu^2 \in \langle \xi \rangle$, $[\eta, \nu] \in \langle \xi \rangle$ and ν centralizes $Z(S_0)$.

Proof. Let $S_{12} = S_0^1 \times_* S_0^2$ and suppose $P/S_{12} \cdot \langle \alpha_1 \rangle$ is not cyclic. Then $\kappa^2 \in S_{12} \cdot \langle \alpha_1, \eta^2 \rangle$. We have furthermore chosen κ such that $\kappa^{-1}\alpha_1\kappa = \alpha_1\xi$. As $\kappa^2 \in V$, $\kappa^{-2}\alpha_1\kappa^2 = \alpha_1 f$ for some $f \in F$. Let $s \in S_0^2$ such that $\alpha_1^s = \alpha_1 f$. Then $(\kappa s)^{-1}\alpha_1(\kappa s) = \alpha_1\xi$ and $(\kappa s)^{-2}\alpha_1(\kappa s)^2 = \alpha_1$. Let κs be denoted μ. Then $\alpha_1^\mu = \alpha_1\xi$ and $\xi^\mu = \xi$. Thus

$$\mu^2 \in N = C_P(\alpha_1) \cap C_P(\xi) = \langle \alpha_1 \rangle \times Z(S_0) \cdot \langle \eta \rangle. \tag{48}$$

Moreover, μ acts on $C_G(\alpha_1) \cap C_G(\xi)$ which contains a normal subgroup $H_1 = \langle \alpha_1 \rangle \times L \cdot \langle \eta \rangle$ of odd index, where $L \simeq PSL(2, q)$. Let $H_2 = H_1 \cdot \langle \mu \rangle$, $H_0 = L \cdot C_{H_1}(L)$. Then $H_2/H_0 \simeq \langle \eta \rangle / \langle \xi \rangle$ as the outer automorphism group of L is cyclic and $\langle \eta \rangle / \langle \xi \rangle$ acts faithfully (as field automorphisms) on L. Hence $\mu\eta^r \in H_0$ for some $r \in \mathbf{N}$. However, as μ acts on N and $\mu^2 \in N$,

$$(\mu\eta^r)^2 \in H_0 \cap N = \langle \alpha_1 \rangle \times Z(S_0) \times \langle \xi \rangle. \tag{49}$$

Now all involutions of $\langle \alpha_1 \rangle \times Z(S_0) \times \langle \xi \rangle \backslash Z(S_0) \times \langle \xi \rangle$ are conjugate to α_1, and therefore

$$(\mu\eta^r)^2 \in Z(S_0) \times \langle \xi \rangle. \tag{50}$$

On the other hand, as $\mu\eta^r \in H_0$ and $(\mu\eta^r)^4 = 1$, $\mu\eta^r$ acts on L as an inner automorphism of order less than or equal to 2. Thus $(\mu\eta^r)^2 \in \langle \xi \rangle$ and (7) follows with $\nu = \mu\eta^r$.

We note that $\nu\alpha_1$ is an involution if $\nu^2 = \xi$.

(8) α_1 has a complement G_0 in G.

Proof. Either $P/S_{12} \cdot \langle \alpha_1 \rangle$ is cyclic, in which case $\langle S_{12}, \kappa \rangle$ is a complement in P to α_1, or $S_{12} \cdot \langle \eta, \nu \rangle$ is a complement. In either case α_1 has a complement in P with no conjugate, as we have seen in (6), and (7) follows from Lemma 2.4.

Thus a Sylow 2-subgroup P_0 of G_0 is either $S_{12} \cdot \langle \kappa \rangle$ (in the cyclic case) or $S_{12} \cdot \langle \eta, \nu \rangle$, without loss of generality.

(9) ξ is not conjugate to any involution in S_{12}.

Proof. Suppose $\xi^g \in S_{12}$. As ξ acts on S_0^1 and S_0^2 as a field automorphism, $C_{S_{12}}(\xi) = D \times Z(S_0)$, where $D \simeq E_{2^n}$ is the diagonal of F and M_0. Now, if $\xi^g \in S_{12} \backslash Z(S_0)$, $C_{S_{12}}(\xi^g)$ contains a subgroup $M_{01} \times_* M_{02}$ of index q, where $M_{0i} \leq S_0^i$ is maximal abelian. It follows immediately that ξ is not extremal in P. Let $\xi^h \in P$ be extremal. Then $C_P(\xi^h)$ contains a conjugate of α_1. However, by (6) we have that if $\alpha_1^a \in P$ for some $a \in G$, then $\alpha_1^a = \alpha_1^p$ for some $p \in P$. Thus we may assume that $\alpha_1 \in C_P(\xi^h)$. Hence $\xi^h \in \langle \alpha_1 \rangle \times S$. Now, as ξ is not extremal in P and any involution in $\langle \alpha_1 \rangle \times S$ is conjugate to either α_1, ξ

or an involution in $Z(S_0)$, it follows that $\xi^h \in Z(S_0)$. Let $i \in Z(P)$. As in the proof of 5(c), which only depends on the determination of possible conjugates of α_1 in P, there exists an $h \in C_i = C_G(i)$ such that $\xi^h \in Z(S_0)$. Now, as in the proof of Lemmas 4.3(ii) and 4.10(ii), for every element a and subgroup A of C_i, let $\bar{a}$ and $\bar{A}$ denote the corresponding element and subgroup of $\bar{C}_i = C_i/\langle i \rangle$. Let $s \in M_0$ such that $s^2 = i$. Then by Sylow's Theorem, we may assume that

$$(\langle \bar{\alpha}_1 \rangle \times \langle \bar{s} \rangle \times Z(S_0)^- \times \langle \bar{\xi} \rangle)^h \leq \bar{P}. \tag{51}$$

As $s^h \in C_P(\alpha_1^h)$, $s^h \in S_{12} \cdot \langle \eta \rangle$. However, as $\xi^h \in Z(S_0)$, this implies that $s^h \in S_{12} \cdot \langle \xi \rangle$. Consequently, ξ^h centralizes s^h, a contradiction.

(10) $r(\Omega_1(P_0/S_{12})) > 1$.

Proof. It follows immediately from (9) and Lemma 3.3 that P_0/S_{12} is not cyclic. Assume in the following that $r(\Omega_1(P_0/S_{12})) = 1$. Then $\operatorname{ord}(v) = \operatorname{ord}(\eta) = 4$ since $[v, \eta] \in \langle \xi \rangle$ i.e. $\langle \eta, v \rangle \simeq Q_8$. Furthermore, by Lemma 3.3, η is conjugate to an element ζ of $S_{12} \cdot \langle v \rangle$ and by (9), $S_{12} \cdot \langle v \rangle = S_{12} \cdot \langle \zeta \rangle$. Then $\zeta^2 = \xi f m$ for some $f \in F$, $m \in M_0$, which is conjugate to ξ in S_{12}, so we may as well assume that $\zeta^2 = \xi$. Let $\zeta = v s_1 s_2$, where $s_1 \in S_0$ and $s_2 \in S_0^v$. Then

$$\zeta^2 = (v s_1 s_2)^2 = \xi s_1^v s_2 s_2^v s_1 \mod Z(S_0). \tag{52}$$

Thus $s_1^v = s_2 \mod Z(S_0)$. So $\zeta = v s_1 s_1^v i$ for some $i \in Z(S_0)$. Consequently,

$$\zeta^2 = (v s_1 s_1^v)^2 = \xi s_1^v s_1^\xi s_1 s_1^v.$$

In particular, $s_1^\xi = s_1 \mod Z(S_0)$, i.e. $s_1^\xi = s_1^{-1}$. But then

$$s_1^v(v s_1 s_1^v i) s_1^{-v} = v s_1^\xi s_1 i = vi, \tag{53}$$

i.e. we may assume that $\zeta = vi$. Now, as

$$C_P(\xi) = ((Z(S_0) \times D) \cdot \langle \eta, v \rangle) \cdot \langle \alpha_1 \rangle, \tag{54}$$

we have

$$C_P(vi) = (Z(S_0) \times D_1) \cdot (\langle vi \rangle \times_* C_{\langle \eta \alpha_1 \rangle}(vi)) \tag{55}$$

where $D_1 \leq D$. Also,

$$C_P(\eta) = Z \cdot \langle \alpha_1 \rangle \times \langle \eta \rangle \tag{56}$$

where $Z \leq Z(S_0) \times D$ is of order 2^n. Thus η is not extremal in P. Since η centralizes α_1 and all conjugates of α_1 in P lies in $S_{12} \cdot \langle \eta, \alpha_1 \rangle$ and furthermore the centralizer of any conjugate of α_1 in P is contained in V, it now follows easily that $\eta^g \in S_{12} \cdot \langle \xi \rangle$ if η^g is an extremal conjugate of η in P. By (9), this is a contradiction.

(11) Contradiction.

By (7) and (10), either $v^2 = 1$ or $[\eta, v] = 1$. Moreover, if $v^2 \neq 1$, $\eta^2 \neq 1$ by (10). But in the latter case $\eta_2 v$ is an involution where $\eta_2 \in \langle \eta \rangle$ is of order 4.

Thus either $v^2 = 1$ and $[\eta, v] = \xi$, $v^2 = 1$ and $P_0/S_{12} \simeq \langle\xi\rangle \times \langle v\rangle$ or $v^2 \neq 1$ and $P_0/S_{12} \simeq \langle\eta\rangle \times \langle\eta v\rangle$. In the latter case $\eta_2 \notin P \cap G'$ by Lemma 3.3 and (9). In the former case we apply Lemma 3.3. Assume $\eta \in P \cap G'$. Then η^{2^r} is conjugate to an element of $P_0 \backslash S_{12} . \langle\eta^2, v\rangle$, again by (9). But any element of $P_0\backslash S_{12} . \langle\eta^2, v\rangle$ has order larger than or equal to ord (η) unless $\eta^4 = 1$, as $(\eta v)^2 = \eta^2\xi$. Thus we may assume in any case that $\langle\eta, v\rangle$ is isomorphic to either $Z_2 \times Z_2$ or D_8. Consider the case $\langle\eta, v\rangle \simeq D_8$. Then ξ is conjugate to an involution in $P_0\backslash S_{12} . \langle\xi, \eta v\rangle$ unless $\eta \notin P \cap G'$. Assume therefore that ξ is conjugate to vs_1s_2 for some $s_i \in S_0$. Then $s_1^v = s_2 \bmod Z(S_0)$, as v inverts s_1s_2, and s_1s_2 is an involution. Moreover, $|C_{S_{12}}(v)| = 2^{3n}$, and vs_1s_2 is conjugate to vi by s_1, where $i = s_1^vs_2 \in Z(S_0)$. Again, ξ is not extremal, and we easily reach a contradiction. Thus we have reduced to the case $\langle\eta, v\rangle \simeq Z_2 \times Z_2$. But then ξ has an extremal conjugate in $S_{12} . \langle v\rangle$ by Lemma 3.4. However, this forces the extremal conjugate to lie in S_{12}, as no element of $S_{12} . \langle v\rangle\backslash S_{12}$ centralizes any conjugate of α_1 in P. This final contradiction proves Lemma 4.12.

LEMMA 4.13. *Suppose F is elementary abelian and $E^g \cap F \neq \langle 1\rangle$ for some $g \in G$. Then $Z(S_0)$ is strongly closed in P with respect to G.*

Proof. If $E^g \cap F \neq \langle 1\rangle$ for some $g \in G$, it follows that $E \simeq E_{2^n}$ and in fact that $F = E^g \times Z(S_0)$, since $F \simeq E_{2^{2n}}$, from our basic assumptions on E. Furthermore, this implies that ord $(\eta) \leq 2$, as η_2 acts nontrivially on $Z(S_0)$.

First we claim that either $Z(S_0)$ is strongly closed in P w.r.t. G, or the weak closure of E in W w.r.t. $N_G(W)$ is equal to $\langle E, E^g\rangle = F . E$. Suppose $E^a \leq W$ for some $a \in G$. Then $E^a \leq W_0$ since $|E| > 2$. Moreover, either $E^a \leq F . E$ or $E^a \cap (F . E) = \langle 1\rangle$. So assume the latter case occurs. Then, if $\alpha \in E^\#$, $\alpha^a = \beta vs$ for some $\beta \in E^\#$, $v \in E^g$ and $s \in S_0$. Thus $M_s = C_{S_0}(s) \leq C_W(\alpha^a)$, and consequently

$$E^a = E_a = \langle\alpha_1v_1s_1, \ldots, \alpha_nv_ns_n\rangle \tag{57}$$

where $E^g = \langle v_1, \ldots, v_n\rangle$ and $M_s = \langle s_1, \ldots, s_n\rangle$, since E_a is an elementary abelian subgroup of W_0 whose centralizer in W_0 is isomorphic to $E_{2^n} \times S_B(2^n)$. We note that $\langle E, E^g\rangle$ is isomorphic to $PSL(3, 2^n)_2$. Suppose $\eta \neq 1$, i.e. $\eta = \xi$ is of order 2. Now, if furthermore $a \in N_G(W)$, $\xi^a = \xi\gamma v_as_a$ for some $\gamma \in E$, $v_a \in E^g$ and $s_a \in S_0$. However, as ξ inverts γv_as_a, ξ inverts s_a, so we may as well assume that $s_a = 1$. If $v_a \neq 1$, $\xi\gamma$ centralizes F. Then $\alpha_k^{v_a} = \alpha_ks_k^2$ for all k, $1 \leq k \leq n$, since $\xi\gamma v_a$ centralizes E^a. But then v_a belongs to $v_kZ(S_0)$ for all k, a contradiction since $|E| > 2$. So $v_a = 1$, and therefore $v_k^{\xi\gamma} = v_ks_k^2$ for all k, so $v_k^{\alpha_k\xi\gamma} = v_k$ for all k. But then $\alpha_k\xi\gamma$ centralizes F for all k, since $\alpha_k\xi\gamma \in C_G(\sigma_{q-1})$, again a contradiction. Thus $\eta = 1$. But now, by Lemma 4.9, $Z(S_0)$ is strongly closed if $P = W = W_0$ so we may assume that $V = N_P(W_0) > W_0$. Let $v \in V\backslash W_0$ such that $v^2 \in W_0$. Suppose $E^v \cap (F . E) = \langle 1\rangle$. Then, using the above notation

$$E^v = \langle\alpha_1v_1s_1, \ldots, \alpha_nv_ns_n\rangle. \tag{58}$$

Hence $\alpha^v = \alpha v s$ for some $\alpha \in E^\#$, $v \in E^g$ and $s \in M_s$. Now, as $v^2 \in W_0$, $\alpha^v v^v s^v = \alpha v s v^v s^v = \alpha i$ for some $i \in Z(S_0)$. But then $v = v^v \bmod Z(S_0)$, so $v \in N_P(F)$. However, $N_P(F) = W_0$ as $F = E^g \times Z(S_0)$, a contradiction. Thus $E^v \leq \langle E^g, E \rangle$. Consequently, $V = P$ and $|V: W_0| = 2$. Finally, let $a \in N_G(W)$ such that $E^a \cap \langle E, E^v \rangle = \langle 1 \rangle$. Then, from what we have just seen,

$$av^{-1}a^{-1}(E \times Z(S_0))ava^{-1} = E^v \times Z(S_0). \tag{59}$$

Let $\alpha \in E^\#$. Then $a^{-1}\alpha a = \beta v s$ for some $\beta \in E^\#$, $v \in E^{v\#}$ and $s \in S_0 \backslash Z(S_0)$. After possibly replacing a by $\sigma_{q-1}^{-k} a \sigma_{q-1}^{k}$ for some k we may assume that $v^{-1}\beta v v = v\beta$. Thus, if $[\beta, v] = i$,

$$av^{-1}a^{-1}\alpha ava^{-1} = av^{-1}\beta v s v a^{-1} = a\beta v s^v i a^{-1} = \alpha(s^{-1}s^v i)^{a^{-1}} \tag{60}$$

which belongs to $E^v \times Z(S_0)$. Thus ss^v is of order 4, i.e. v acts nontrivially on $S_0/Z(S_0)$. In particular, s does not lie in the maximal abelian subgroup of S_0 normalized by v. Since a was arbitrary, σ_{q+1} does not act transitively on the set of maximal abelian subgroups of S_0. Hence n is odd, and $E \times Z(S_0)$ has exactly $2((q+1)3^{-1}+1)$ conjugate subgroups in W under the action of $N_G(W)$, a contradiction since $(q+1)3^{-1}+1$ is an even number and $|P: W| = 2$.

Assume therefore in the following that the weak closure of E in W w.r.t. $N_G(W)$ is equal to $\langle E, E^g \rangle$. It immediately follows that either $Z(S_0)$ is strongly closed in G, or $|V: W| = 2$, where $V = N_G(W)$. Assume therefore that the latter case occurs. Then actually $P = V$, as clearly $\langle E, E^g \rangle \times_* S_0$ is normal in $N_P(V)$. Let $P = \langle W, v \rangle$, where $v^2 \in W$. We may as well assume without loss of generality that v centralizes ξ, since v centralizes $\xi\gamma \bmod Z(S_0)$ for some $\gamma \in E$: $\xi^v = \xi\gamma v$ for some $\gamma \in E$, $v \in E^v$. But then $\gamma^v = v \bmod Z(S_0)$, so $(\xi\gamma)^v = \xi\gamma \bmod Z(S_0)$. Thus, without loss of generality, $C_V(\xi) = \langle \xi \rangle \times \langle E, v \rangle$. Every involution of $\langle E, E^v \rangle$ is conjugate either to α_1 or to i_1. Moreover, if $\delta \in \langle E, E^v \rangle$ is conjugate to α_1, δ is conjugate to δi for any $i \in Z(S_0)$. In particular α_1 is conjugate to $\alpha_1\xi$ if ξ is conjugate to an involution in $Z(S_0)$. Since v centralizes ξ, it easily follows by Lemma 3.3 that $v^2 \in W_0$. Assume $\xi \in P \cap G'$. By Lemma 3.4, there exists an extremal conjugate $\xi^h \in W_0 \,.\, \langle v \rangle$ of ξ for some $h \in G$. Clearly $\xi^h \notin W_0$. Moreover, $\langle E, E^v \rangle^h$ is normal in $W_0 \,.\, \langle \xi^h \rangle$, in fact $(E \times Z(S_0))^h$ is normal in $W_0 \,.\, \langle v \rangle$ and $C_V(\xi^h)$ contains an element interchanging $(E \times Z(S_0))^h$ and $(E^v \times Z(S_0))^h$. Hence $\langle E, E^v \rangle^h \leq W_0$. As E^h and E^{vh} are of the form considered in (59) this implies that v acts trivially on $S_0/Z(S_0)$. By symmetry, $\xi\xi^h$ acts trivially on $S_0/Z(S_0)$ as well, a contradiction. Thus $\xi \notin P \cap G'$.

Finally suppose $v \in P \cap G'$. First we claim that $Z(S_0)$ is strongly closed in W_0 w.r.t. G. We only have to consider involutions of the form $\tau = \alpha v s$ for some $\alpha \in E^\#$, $v \in E^{v\#}$ and $s \in S_0$. But as mentioned earlier, $C_{W_0}(\tau) = E_1 \times S_1$, where $E_1 \simeq E_{2^n}$ and $S_1 \simeq S_B(q)$ ($\simeq S_0$ as a 2-group). But if $(E_1 \times S_1)^g$ is a subgroup of P for some $g \in G$, it follows immediately by the structure of P that $E_1 \cap Z(S_0) = \langle 1 \rangle$. Hence $\tau^g \notin Z(S_0)$ and it follows that $Z(S_0)$ is strongly closed in W_0 w.r.t. G. So in order to finish the proof we may assume that v is an involution conjugate to those of $Z(S_0)$. Now, $\sigma_{q-1}^{-1}v\sigma_{q-1}v$ acts trivially on $S_0/Z(S_0)$, and $\sigma_{q-1}^{-1}v\sigma_{q-1}v \in N_G(E)$ as v is an involution. Hence

$$\sigma_{q-1}^{-1}v\sigma_{q-1}v = \rho c$$

where $\rho^{q-1} \in C$, $c \in C$ and ρ acts trivially on $Z(S_0)$. On the other hand, as ρ^{q-1} acts trivially or as an inner automorphism on U_0, the structure of Aut (U_0) implies that ρ itself must act as an inner automorphism. Since our only constraint on c is that it must lie in C, we may also assume that $\rho^{q-1} = 1$. But then $\rho \in C_G(U_0)$, as $\rho \in C_G(Z(S_0))$. Furthermore, as $\sigma_{q-1}^{-1}v\sigma_{q-1}v$ acts trivially on $S_0/Z(S_0)$, $\sigma_{q-1}^{-1}v\sigma_{q-1}v = c_0s_0$ for some $c_0 \in C_G(S_0)$ and $s_0 \in S_0$. Moreover, as $\rho c = c_0s_0$, $c_0 = \rho c_1$ for some

$$c_1 \in C \cap C_G(S_0) = E \times Z(S_0).$$

But then, if $c_1 = \alpha i$, where $\alpha \in E$ and $i \in Z(S_0)$, $c_0s_0 = \rho_1 s_1$, where $\rho_1 = \rho\alpha$ and $s_1 = is_0$. Also $\rho_1^{q-1} = 1$ and $\rho_1 \in C_G(U_0)$.

If v acts nontrivially on $S_0/Z(S_0)$, v inverts some maximal abelian subgroup of S_0 by Theorem 1.1. Consider the case when v acts trivially on $S_0/Z(S_0)$. If $s_1 \in Z(S_0)$, v and σ_{q-1} centralize each other mod $C_G(S_0)$. As v acts trivially on $S_0/Z(S_0)$, v centralizes some s in $S_0\backslash Z(S_0)$. But then v centralizes $C_{S_0}(s)$ since σ_{q-1} acts on $C_{S_0}(s)$, and we reach a contradiction exactly as in the proof of Lemma 4.9. Thus s_1 is of order 4. Since v inverts $\rho_1 s_1$, $s_1^v \in S_0$ and $\rho_1 \in C_G(S_0)$, it follows that v inverts both ρ_1 and s_1. In particular, v inverts $C_{S_0}(s_1)$.

Thus v inverts some maximal abelian subgroup M of S_0. Let H be the homocyclic subgroup of $\langle E, E^v\rangle$ of exponent 4 inverted by v. Then v centralizes the diagonal $D \simeq E_{2^n}$ of H and M, and $D \times Z(S_0) \times \langle v\rangle$ is a maximal elementary abelian subgroup of P. Moreover, if v acts nontrivially on $S_0/Z(S_0)$ it follows immediately that v is conjugate to vd for all $d \in D \times Z(S_0)$. However, this is also true if v acts trivially on $S_0/Z(S_0)$. Let $\rho_1 = (\rho_1^k)^2$. Then it follows from the equation $\sigma_{q-1}^{-1}v\sigma_{q-1}v = \rho_1 s_1$ that

$$(\rho_1^{-k})^2\sigma_{q-1}^{-1}v\sigma_{q-1} = \rho_1^{-k}\sigma_{q-1}^{-1}v\sigma_{q-1}\rho_1^k = s_1 v \tag{61}$$

since ρ_1 is inverted by v and $\rho_1 \in C_G(U_0)$. Then v is conjugate to vs for all $s \in M$. Since on the other hand v is conjugate to vh for all $h \in H$ in $\langle E, v\rangle$, the assertion follows. Now let $v^g = i \in Z(S_0)$. By Sylow's Theorem we may assume that $(D \times Z(S_0) \times \langle v\rangle)^g \leq P$. Then

$$(D \times Z(S_0) \times \langle v\rangle)^g \cap W_0 = Z(S_0) \times D_1 \tag{62}$$

where $D_1 \cap \langle E, E^v\rangle = \langle 1\rangle$. Moreover, as $|D| > 2$, $d^g \in W_0$ for some $d \in D^{\#}$. Hence $d^g \in (D_1 \times Z(S_0))\backslash Z(S_0)$, as $Z(S_0)$ is strongly closed in W_0 w.r.t. G, and d^g is conjugate to $d^g i$. But this is a contradiction since $d^g i = (dv)^g$ is conjugate to v.

LEMMA 4.14. *Suppose F is elementary abelian and assume furthermore that $Z(S_0)$ is not strongly closed in P. Then:*

(i) *V contains a normal subgroup $V_0 > W_0$, which is a complement in V to $\langle\eta\rangle$, such that V_0/W_0 is isomorphic to $E_{2^{2n}}$. Moreover, $\sigma_{q^2-1} \in N_G(V_0)$ and σ_{q^2-1} acts irreducibly and faithfully on V_0/W_0.*

(ii) *$C_{V_0}(F) = R$ is a complement in V_0 to E, and R/F is isomorphic to $E_{2^{4n}}$.*

(iii) *The weak closure of E in V_0 is contained in W_0.*

Proof. Let $F = Z_0 \times Z(S_0)$ such that $\sigma_{q-1} \in N_G(Z_0)$.

(i) By Lemma 4.10(iv), $E \times Z(S_0)$ is not weakly closed in W. Furthermore, $F \cap E^g = \langle 1\rangle$ for all $g \in G$ by the previous lemma. Suppose $W = P$. Then by Lemma 4.10(ii), $|E|$ is larger than 2. Consequently, if $(E \times Z(S_0))^g \leq W$, then actually $(E \times Z(S_0))^g \leq W_0$. So let $(E \times Z(S_0))^g$ be a subgroup of W_0 which is not contained in T_0 for some $g \in G$. If $\alpha \in E^{\#}$, $\alpha^g = \beta zs$ for some $\beta \in E^{\#}$, $z \in Z_0$ and $s \in S_0$, where $s^2 = (\beta z)^2$. Let $s \in M_s = C_{S_0}(s)$. Then M_s is contained in $C_G(\alpha^g)$, so

$$E^g = \langle \alpha_1 z_1 s_1, \ldots, \alpha_m z_m s_m\rangle \tag{63}$$

exactly as in (57), where

$$Z_0 = \langle z_1, \ldots, z_m, \ldots, z_n\rangle \tag{64}$$

and

$$M_s = \langle s_1, \ldots, s_m, \ldots, s_n\rangle \tag{65}$$

Now, by Lemmas 4.9 and 4.10(iii), we may assume that ord (η) is equal to 2. Furthermore, as $E^g \times Z(S_0) \trianglelefteq W_0$, we may assume $W_0^{g^{-1}} \leq W$, in which case it immediately follows that $W_0^{g^{-1}} \leq W_0$. Thus we may assume that $g \in N_G(W_0)$. Now, if $Z(S_0)$ is not strongly closed in P w.r.t. G, by Lemma 3.4, there exists an extremal conjugate ξ^h, $h \in G$, of ξ in W_0. It easily follows that $\xi^h \in Z(S_0)$ as $|E| > 2$. Furthermore we may assume by Sylow's Theorem that $E^h \leq P$. But then

$$(E \times \langle\xi\rangle)^{gh} \leq E \times Z(S_0) \tag{66}$$

and we reach a contradiction by exactly the same argument which proved Lemma 4.2, since $|E| > 2$. Thus $V > W$, as by assumption $Z(S_0)$ is not strongly closed in P.

Let $v \in V\backslash W$ such that $v^2 \in W$. Then

$$E^v = \langle \alpha_1 z_1 s_1, \ldots, \alpha_m z_m s_m\rangle \tag{67}$$

where we have used the notation of (63), (64) and (65). First we note that v acts on $F \times_* S_0$ and trivially on $W_0/F \times_* S_0$. If not, let $\alpha \in E^{\#}$ such that $\alpha^v = \beta zs$ where $\beta \in E\backslash\langle\alpha\rangle$, $z \in Z_0$ and $s \in S_0$. Let $\alpha\beta z_1 s_1 \in E^v$ for suitable $z_1 \in Z_0$ and $s_1 \in S_0$ by (67). As $\alpha^{v^2} = \alpha i$ for some $i \in Z(S_0)$, it follows that $(\alpha\beta z_1 s_1)^v = \beta z_1 s_1 \alpha i = (\alpha\beta z_1 s_1)^{z_0}$ for some $z_0 \in Z_0$, a contradiction as $\alpha\beta z_1 s_1$ is conjugate to an involution of E. Now, by counting conjugates of $E \times Z(S_0)$ in W_0 under the action of $\langle W, \sigma_{q-1}, v\rangle$, we find that

$$|\langle W, \sigma_{q-1}, v\rangle : \langle W, \sigma_{q-1}\rangle| = q.$$

Suppose $\Omega_1(V/W_0) = 1$. As v does not normalize $\langle\sigma_{q-1}\rangle \cdot \langle\eta\rangle$, a Sylow 2-

subgroup $\bar{Q}$ of $\langle W, \sigma_{q-1}, v\rangle/W_0$ is quaternion, and $\langle\bar{\eta}\rangle = W/W_0 \trianglelefteq \bar{Q}$ if we assume $W/W_0 \leq \bar{Q}$. In particular, we may assume that $v^2 = \xi$. On the other hand, as $q \geq 4$, $\bar{\eta}$ is a square in $\bar{Q}$ as well since $\langle\bar{\eta}\rangle \trianglelefteq \bar{Q}$. This is easily seen to be impossible. Thus $r(\Omega_1(V/W_0)) > 1$. Assume therefore that $v^2 \in W_0$. As $\sigma_{q-1} \in N_G(W_0 . \langle\xi\rangle)$, $\sigma_{q-1} \in C_G(\xi)$ and $\sigma_{q+1} \in N_G(W_0)$ while σ_{q+1} is inverted by ξ, we first replace v by $v_0 = vv^{\sigma_{q-1}}$, which is equal to $v^{\sigma_{q-1}{}^k}$ modulo $W_0 . \langle\xi\rangle$ for some $k \in \mathbf{N}$. It is now easy to verify, just as in previous similar cases, that $N = \langle W_0, v_0, \sigma_{q^2-1}\rangle$ is 2-closed since v_0 acts trivially on $W_0/F \times_* S_0$, and we obtain (i) with $V_0 = V \cap N$, since σ_{q^2-1} acts irreducibly on $S_0/Z(S_0)$. Let $\sigma = \sigma_{q+1}^r$. Then $O_2(W_0, v_0^\sigma, \sigma_{q-1})/W_0$ is elementary abelian and isomorphic to $Z(S_0)$ as a σ_{q-1}-module for any r. It also follows that N/W_0 is elementary abelian. As σ_{q^2-1} acts irreducibly on $S_0/Z(S_0)$, even if 3 divides $q + 1$, (i) follows.

(ii) As we have just seen, $V_0/Z_0 \times S_0 \simeq E_{2^{2n}} \times E_{2^m}$. Thus E has a complement R under the action of σ_{q^2-1} containing $Z_0 \times S_0$. Now the action of σ_{q+1} and the fact that σ_{q+1} centralizes F implies that $R \leq C_{V_0}(F)$, and as E acts nontrivially on F, R actually equals $C_{V_0}(F)$. Moreover, R acts on $Z_0 \times S_0/Z_0 \simeq S_0$, so, by Theorem 1.1, R centralizes S_0 mod F. Hence $u^2 \in C_R(S_0) = F$ if $u \in R$. In particular, $R/F \simeq E_{2^{4n}}$.

(iii) Suppose $E^g \leq V_0$ for some $g \in G$. Then $E^g \cap R$ is trivial by (ii). Thus an element of E^g is of the form αu for some $\alpha \in E^\#$, $u \in R$. Suppose $u \notin Z_0 \times S_0$. Then α^u is equal to αzs for some $z \in Z_0^\#$ and $s \in S_0 \backslash Z(S_0)$, so $u^2 = zs$, which contradicts (ii).

COROLLARY. *Assume in addition to the assumptions of Lemma* 4.14 *that* $|E| > 2$. *Then* $V = P$.

Proof. Suppose $|E| > 2$. Then, if $E^g \leq V$, it follows that $E^g \leq V_0$. Thus the corollary follows from (iii).

LEMMA 4.15. *Suppose we are in the situation of Lemma* 4.14.

(i) *Suppose* $\Omega_1(R) = F$. *Then* $R . \langle\alpha_1\rangle \simeq PSU(3, 2^n)_2 \wr Z_2$.

(ii) *Suppose* $\Omega_1(R) > F$. *Then* $R \simeq PSL(3, 2^{2n})_2$.

Proof. Let $\bar{R}_0$ be a complement in $\bar{R}$ of $S_0 \times Z_0/F$ under the action of σ_{q^2-1} and $\bar{R}_0 = R_0/F$ such that $\Omega_1(R_0) > F$ if $\Omega_1(R) > F$. Let $N_0 = N_{R_0}(M_0)$. Then it easily follows from (67) that $|N_0| = 2^{3n}$. Also we may assume without loss of generality that $Z_0 = R_0^2$ if R_0 is not elementary abelian. Suppose $\Omega_1(R_0) > F$. Then $\bar{\bar{R}}_0 = R_0/Z(S_0)$ is either elementary abelian or isomorphic to $PSL(3, 2^n)_2$ by Lemma 2.5. Thus $\bar{\bar{R}}_0$ is either elementary abelian or contains exactly two maximal elementary abelian subgroups. However, as σ_{q^2-1} acts irreducibly on $\bar{\bar{R}}_0/Z_0 = \bar{R}_0$, this is impossible. Thus $\bar{\bar{R}}_0$ is elementary abelian. Since $R_0^2 \leq Z_0$ this implies that R_0 is elementary abelian.

If $\Omega_1(R_0) > F$ it therefore follows, by Lemma 2.5, that N_0 is elementary abelian and hence that $\langle M_0, N_0\rangle \simeq E_{2^n} \times PSL(3, 2^n)_2$.

If $\Omega_1(R) = F$, let $t \in N_0$ be an element of order 4. We may assume without loss of generality that t centralizes s where $\alpha^t = \alpha z s$, $z \in Z_0$, $s \in S_0$; if $s^t \neq s$ we replace R_0 by a complement containing ts', where $s' \in S_0$ such that $s^{ts'} = s$. So in this case $\langle M_0, N_0 \rangle$ is homocyclic of order 2^{4n} and rank $2n$, using the argument in the first part of the proof of Lemma 2.5. Now, as σ_{q^2-1} acts irreducibly on $\bar{R}_0$, we use the idea in the proof of Lemma 2.5, namely, we show that all commutators are uniquely determined. If $\Omega_1(R) > F$, R_0 is elementary abelian. If $\Omega_1(R) = F$, R_0 is equal to $R_1 \times Z(S_0)$, where $\Omega_1(R_1) = Z_0$, and, by Lemma 2.5, R_1 is isomorphic to $PSU(3, 2^n)_2$. Finally, if $u \in R_0$ and $\alpha \in E^{\#}$, let $\alpha^u = \alpha z s$ where $z \in Z_0$, $s \in S_0$. Then

$$(\alpha^{u^\rho})^u = \alpha^u z^\rho u^{-1} s^\rho u \quad \text{where} \quad \rho = \sigma_{q^2-1}^k \text{ for any } k,$$

an equation which determines $u^{-1}s^\rho u$ uniquely, and the lemma follows.

COROLLARY. *Suppose $Z(S_0)$ is not strongly closed. Then $|E| = 2$.*

Proof. Obvious.

LEMMA 4.16. *Suppose $\Omega_1(R) = F$. Then F is strongly closed.*

Proof. First we consider the case when $P = V$. It follows immediately that F is strongly closed in P w.r.t. G if $V = V_0$. So we may assume that $\eta \neq 1$ and that $P \cap G'$ is not contained in V_0. Let $R = R_1 \times R_2$, where $R_i \simeq S_0$ and $R_1^{\alpha_1} = R_2$. We may assume without loss of generality that η normalizes $\bar{R}_i = R_iF/F$. In particular, ξ centralizes F. Furthermore, ξ is conjugate to an involution of F unless $P \cap G'$ is contained in $V_0 . \langle \xi \rangle$ by Lemma 3.3, while if $P \cap G' = V_0 . \langle \xi \rangle$ this immediately follows from Lemma 3.4, since in that case V_0 is a complement to ξ in P and ξ centralizes F. It follows immediately that the extremal conjugate of ξ in P lies in $Z(S_0)$. As we know the structure of R completely, we easily apply the argument of the proof of Lemma 4.10(ii) and reach a contradiction.

Assume therefore in the following that $V < P$. In particular, $\eta \neq 1$. Le) $V_1 = N_P(V)$, $V_{r+1} = N_P(V_r)$. From the structure of R it follows that $N_P(R$t contains a subgroup P_0 of index 2 such that P_0 normalizes $Z(R_i)$ and

$$\bar{\bar{R}}_i = R_i \times Z(R_{3-i})/Z(R_{3-i}), \quad i = 1, 2$$

and that $N_P(R) = P_0 . \langle \alpha_1 \rangle$. Also, as η acts as a field automorphism on S_0, we may, after possibly change of notation, assume that $\eta \in P_0$ and that η acts as a field automorphism on $\bar{\bar{R}}_i$, $i = 1, 2$. It easily follows by induction that $|V_{r-1} : V_r| \leq 2$. Let $\gamma_{r+1} \in V_{r+1} \backslash V_r$.

(1) We may choose γ_1 in P_0 such that one of the following cases occurs:

(a) $V_1 = (R . (\langle \eta \rangle \times \langle \gamma_1 \rangle)) . \langle \alpha_1 \rangle$, $\gamma_1^2 = 1$, $\gamma_1 \in C_G(L)$ and $[\gamma_1, \alpha_1] = \xi$.

(b) $\text{Ord}(\eta) = 2$, $V_1 = (R . \langle \gamma_1 \rangle) . \langle \alpha_1 \rangle$, $\gamma_1^2 = \xi$ and $[\gamma_1, \alpha_1] = \xi$.

To see this we first observe that $R \trianglelefteq V_1$ and $R . \langle \eta \rangle = P_0 \cap V_1 \trianglelefteq V_1$. Hence $C_{V_1}(\xi)$ is not contained in V, so it easily follows that we may choose γ_1

in $C_P(\xi)$ such that $\gamma_1^{-1}\alpha_1\gamma_1 = \alpha_1\xi$, i.e. γ_1 acts on $C_G(\alpha_1) \cap C_G(\xi)$. This allows us to assume exactly as in the proof of (7) in Lemma 4.12 that either $\gamma_1^2 = \eta$ or $\gamma_1^2 \in \langle\xi\rangle$ and $[\gamma_1, \eta] \in \langle\xi\rangle$. Furthermore, after possibly replacing γ_1 by $\gamma_1\alpha_1$ we may assume that $\gamma_1 \in P_0$. Hence $[\eta, \gamma_1] = 1$ by Theorem 1.1. Finally, if $\gamma_1^2 = \eta$ and ord $(\eta) > 2$, clearly $P = V_1$. But then, by Lemma 3.4, $\alpha_1 \notin P \cap G'$ and consequently, by Lemma 3.2, $\gamma_1\alpha_1, \gamma_1 \notin P \cap G'$ so we are back to $P = V$, which has been considered above.

(2) If $V_{r-1} < P$, $r \geq 2$, then γ_r may be chosen in P_0 such that one of the following cases occurs:

(a) $V_r = (R \,.\, (\langle\eta\rangle \times \langle\gamma_r\rangle)) \,.\, \langle\alpha_1\rangle$, $\gamma_r^2 = \gamma_{r-1}$, $\gamma_r^{-1}\alpha_1\gamma_r = \alpha_1\eta_r\gamma_{r-1}$.

(b) $V_r = (R \,.\, \langle\gamma_r\rangle) \,.\, \langle\alpha_1\rangle$, $\gamma_r^2 = \gamma_{r-1}$ and $\gamma_r^{-1}\alpha_1\gamma_r = \alpha_1\gamma_{r-1}$.

The proof of course goes by induction. First we consider case (a). Assume (2a) has been established for all $r \leq h$ and that $P > V_h$. In particular, ord $(\eta) = 2^k >$ ord $(\gamma_h) = 2^h$. Again we may assume without loss of generality that $\gamma_{h+1} \in P_0$, $\gamma_{h+1}^{-1}\alpha_1\gamma_{h+1} = \alpha_1\eta_{h+1}\gamma_h$ and that $\gamma_{h+1} \in C_G(\eta_{h+1}\gamma_h)$. Thus it easily follows that γ_{h+1} acts on $C_G(\eta_{h+1}) \cap C_G(\alpha_1)$. This allows us to assume, as above, either that $\gamma_{h+1}^2 = \eta$ or that $\gamma_{h+1}^2 \in \langle\eta_{h+1}\rangle \times \langle\gamma_h\rangle$. If however $\gamma_{h+1}^2 = \eta$, γ_{h+1}^2 centralizes α_1, so $\eta_{h+1}\gamma_h$ is inverted by γ_{h+1}. This is only possible if $h = 0$, i.e. $\eta = \xi$ and we are in case (b). Thus $\gamma_{h+1}^2 \in \langle\eta_{h+1}\rangle \times \langle\gamma_h\rangle$ and it follows without loss of generality, using Theorem 1.1, that we may assume that $\gamma_{h+1}^2 = \gamma_h$ and $[\eta, \gamma_{h+1}] = 1$, proving (a). Case (b) is even easier, and we leave the proof to the reader.

Before we continue, we note that any involution of $P_0 \backslash R$ is conjugate either to ξ or to γ_1. We therefore obtain

(3) F is strongly closed in P_0 w.r.t. G, and $\alpha_1 \notin G'$.

The proof is obtained in the same fashion as many times earlier. If γ_1 is an involution, $\gamma_1 \in C_G(L)$ by (1), so both ξ and γ_1 has L in their centralizer and our method applies to both involutions. That $\alpha_1 \notin G'$ follows immediately from the fact that $\Omega_1(P_0) \leq C_P(F)$.

(4) F is strongly closed in P w.r.t. G.

This is clear by (2) and (3) if P_0/R is cyclic. Assume therefore that we are in case (2a). By (3), it suffices to prove that

$$\Omega_1(G^{(\infty)} \cap P) \leq R \,.\, (\langle\xi\rangle \times \langle\gamma_1\rangle). \tag{68}$$

Only the following four cases may occur:

I. $P \cap G^{(\infty)} = R \,.\, (\langle\eta_{k_1}\rangle \times \langle\gamma_{r_1}\rangle)$,

II. $P \cap G^{(\infty)} = R \,.\, (\langle\eta_{k_1}\rangle \times \langle\gamma_{r_1}\alpha_1\rangle)$,

III. $P \cap G^{(\infty)} = R \,.\, (\langle\eta_{k_1}\alpha_1, \gamma_{r_1}\rangle)$,

IV. $P \cap G^{(\infty)} = R \,.\, (\langle\eta_{k_1}\alpha_1, \gamma_{r_1}\alpha_1\rangle)$, for suitable $k_1, r_1 \in \mathbf{N}$, $k_1 > r_1$. Note that $\eta_{k_2}\gamma_{r_1}\alpha_1$ is an involution if and only if $k_2 = r_1 + 1$.

Case II. Suppose $k_1 > r_1$. As $\alpha_1\gamma_{r_1}\alpha_1 = \eta_r^{-1}\gamma_r^{-1}$, let

$$P_1 = \langle R, \eta_{k_1-1}, \gamma_{r_1-1}, \eta_{r_1+1}\alpha_1\gamma_{r_1}\rangle. \tag{69}$$

Then P_1 is a maximal subgroup of $P \cap G^{(\infty)}$ not containing η_{k_1} and $\exp(P_1/R) = 2^{k_1-1}$, while every element in $P \backslash P_1$ has order 2^{k_1} mod R. Hence η_{k_1} transfers out by Lemma 3.3 and (4), a contradiction. Thus $k_1 = r_1$. Now let

$$P_1 = R\,.\,(\langle \eta_{r_1}\rangle \times \langle \gamma_{r_1-1}\rangle) \tag{70}$$

where this time $\alpha_1\gamma_{r_1} \notin P_1$. As every element of $P\backslash P_1$ has order 2^{r_1+1} mod R in this case, we have reached a contradiction again.

Case III. It easily follows in the same way here that $k_1 \leq r_1 + 1$. If $k_1 = r_1 + 1$, let

$$P_1 = \langle R, \eta_{r_1}, \gamma_{r_1}\eta_{r_1+1}\alpha_1, \gamma_{r_1-1}\rangle. \tag{71}$$

Then every element of $P\backslash P_1$ has order 2^{r_1+1} $(\eta_{r_1+1}\alpha_1)$ or 2^r $(\gamma_{r_1} = \gamma_{r_1}\eta_{r_1+1}\alpha_1\,.\,(\eta_{r_1+1}\alpha_1)^{-1})$ mod R. It now easily follows that ξ is conjugate to γ_1 in $G^{(\infty)}$, say $\xi^g = \gamma_1$, by our remark on conjugacy classes of involutions in P_0. Let $\alpha_1^g = \alpha_1 x$, $x \in G^{(\infty)}$. Then

$$\gamma_1\alpha_1 x\gamma_1 = \alpha_1\xi\gamma_1 x\gamma_1, \tag{72}$$

i.e. $\gamma_1 x\gamma_1 = \xi x$. Hence $\alpha_1\xi$ centralizes x, so α_1 centralizes $\gamma_1 x\gamma_1 = \xi x$. But then α_1 centralizes x, so ξ centralizes x and consequently x is an involution in $L\,.\,\langle\xi\rangle$, a contradiction since $\gamma_1 \in C_G(L\,.\,\langle\xi\rangle)$ while $\gamma_1 x\gamma_1 = \xi x$. Thus $k_1 = r_1$, in which case (68) holds.

Case IV. This case is easily taken care of by referring to Lemma 3.3 unless $k_1 \leq r_1 + 1$ in which case it immediately follows that (68) holds.

Remark. Lemma 4.16 deals with cases as $G = U_0^1 \wr Z_2$, where $U_0^1 \simeq U_0$, $G = U^1 \quad Z_2$, where $U^1 \simeq U$, and the "twisted wreath product" $G = (U_0^1 \quad Z_2)\,.\,Z_{2n}$, and variations thereof.

LEMMA 4.17. *Suppose $\Omega_1(R) > F$. Then G contains a normal subgroup H with R as Sylow 2-subgroup.*

Proof. By Lemma 4.14(iii), R does not contain any involution conjugate α_1, so the lemma follows immediately if $\eta = 1$. Assume therefore that $\eta \neq 1$. First we claim that $P = V$. Let $F_0 \times F$ be a maximal elementary abelian subgroup of R, which by Theorem 2.1 is isomorphic to $PSL(3, 2^{2^n})_2$. Now, if ord $(\eta) > 2$, $(F_0 \times F)^\xi = F_0 \times F$, while if ord $(\eta) = 2$ we may as well assume this to be the case. Let $u \in F_0$ such that $u\alpha_1 u = \alpha_1 zs$, where $z \in Z_0$ and $s \in M \neq M_0$. Then ss^ξ is an element of order 4 in M_0. Moreover, u acts trivially on $M^\xi F_0 F/F$. Thus

$$u^\xi u\alpha_1 u u^\xi = \alpha_1 ss^\xi \mod F. \tag{73}$$

But ξ centralizes $u^{\xi}u$. It therefore follows that if $u \in F$ such that $u\alpha_1 u = \alpha_1 zs$ where $s \in M_0$, then $z^{\xi} = zs^2 = z^{\alpha_1}$. Thus $\alpha_1\xi$ centralizes F. In particular, $\alpha_1\xi$ is not conjugate to α_1. Neither is ξ, as $|C_{F_0 \times F}(\xi)| \geq 2^{2n}$. Since all involutions of $P\backslash R$ are conjugate to α_1, $\alpha_1\xi$ or ξ, it follows that $P = V$ and that α_1 has a complement in G with $R . \langle\eta\rangle$ as Sylow 2-subgroup without loss of generality. Finally, if $P \cap G'$ is not contained in R, it easily follows that ξ is conjugate to an involution of $Z(S_0)$ by Lemma 3.3.

Again we are in a situation, where the argument of Lemma 4.10(ii) may be applied to reach a contradiction.

Thus we have shown that if G is a finite group with an involution whose centralizer in G satisfies (*), then either G contains an elementary abelian 2-group which is strongly closed in G, or G contains a normal subgroup H whose Sylow 2-subgroup is isomorphic to that of $PSL(3, 2^{2n})$. This completes the proof of Theorems 1 and 2 as mentioned in the introduction.

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