IMMERSIONS UP TO COBORDISM

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Given a compact manifold M^m we ask for the least integer k such that M^n immerses into R^{m+k}. A great deal is known for special classes of manifolds (see Gitler [9]). There is a conjecture that (if $m \geq 2$) M^m immerses in $R^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of ones in the dyadic expansion of m. The original question can be weakened to read: given M^m, find the least integer k such that there is a manifold M' cobordant to M and M' immerses in R^{m+k}.

We shall say that M^m immerses into R^{m+k} up to cobordism. Brown [4], [5] has proved that M^m immerses in $R^{2m-\alpha(m)}$ up to cobordism. Of course, we have lost a lot of geometric information by passing to cobordism, since now $k(M)$ is a function only of the cobordism class of M, and if M is a boundary ($M = RP^{2n+1}$, an odd-dimensional real projective space, for example) then $k(M) = 0$. Even if M is not a boundary, a manifold may immerse up to cobordism into a lower dimensional Euclidean space than M itself: for example, RP^{10} immerses up to cobordism into R^{15}, as we shall see later, but RP^{10} itself immerses into R^{16}, and does not immerse into R^{15} (see Gitler [9]).

The fact that we have lost geometric information by passing to cobordism (and reducing the problem to homotopy theory) should not make us sad: the geometric situation was too complicated, so we would not obtain useful qualitative information if we preserved the complexity of the original problem. The purpose of this note is to convince the reader that even after the reduction there is a lot of structure (possibly even too much?) remaining.

The usual approximation theorems of Thom [16] give a reduction of the problem of immersions up to cobordism to a question of homotopy. Let MO be the Thom spectrum [16] for the orthogonal group, then cobordism classes of compact m-dimensional manifolds correspond to elements of

$$\pi_m(MO) = \varinjlim_n \pi_{n+m}(MO(n)).$$

Let $\lambda_k\colon \pi^{st}_{m+k}(MO(k)) \to \pi_m(MO)$ be the map into the direct limit, where the superscript st denotes stable homotopy. If $x \in \pi_m(MO)$ represents the cobordism class of M, then M is cobordant to an M' which immerses in R^{m+k} if and only if x is in the image of λ_k. The essential point here is the use of the theorem of Hirsch [10] which reduces the question of immersion in R^{m+k} to the geometric dimension of the stable normal bundle of the manifold.

Stated in another way: we define an increasing filtration of $\pi_*(MO)$ by setting

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${}^{geo}F_s\pi_*(M)$ = image of λ_s. Let ${}^{geo}E^0\pi_*(M)$ be the associated graded object, then our original question is equivalent to the following: given $x = [M^m] \neq 0$, what is the k so that the class of x is nonzero in $E_k^0\pi_m(MO)$?

The purpose of this note is to study ${}^{geo}E^0\pi_m(MO)$.

We recall that $\pi_*(MO)$ is a polynomial algebra over Z_2 (Thom [16]) and the Hurewicz homomorphism $\pi_*(MO) \to H_*(MO; Z_2)$ is a monomorphism onto the primitives under the coaction of the algebra A_* (dual of the mod 2 Steenrod algebra [14]). This, as we shall see, presents us with an algebraically obvious set of polynomial generators for $\pi_*(MO)$. The homology $H_*(MO; Z_2)$ is filtered by the images of $H_{*+n}(MO(n); Z_2)$ and this filtration induces the algebraic filtration ${}^{alg}F$ of $\pi_*(MO)$. We shall ask ourselves three questions:

Question 1. Do the algebraically obvious polynomial generators have the smallest algebraic filtration?

This question tacitly hopes that ${}^{alg}E_0\pi_*(M)$ has a very simple-minded structure: namely, take polynomial generators of smallest possible filtration, then monomials in these generators should project into a Z_2-basis of ${}^{alg}E_0\pi_*(MO)$, so if it turns out (as it does) that the algebraically obvious generators do not have minimal filtration (we shall see that this first happens in dimension 11), we can still ask:

Question 2. Is ${}^{alg}E^0\pi_*(MO)$ a polynomial algebra?

The answer is *no*, and the first departure from a polynomial algebra occurs in dimension 10.

Since our algebraically obvious generators turn out not to have the minimal algebraic filtration, we can ask:

Question 3. Do the polynomial generators of Boardman [3], Brown [4], Dold [8], and Kozma [11] have minimal algebraic filtration?

The answer is *no* again, unfortunately. Boardman's generators first fail in dimension 11, Brown's in dimension 6, Dold's in dimension 11, Kozma's in dimension 11.

So far we have been talking only about the algebraic filtration of $\pi_*(MO)$. Since the following diagram commutes (horizontal maps are maps into the direct limit, vertical maps are the Hurewicz homomorphisms into homology over Z_2)

$$\begin{array}{ccc} \pi^{st}_{m+n}(MO(n)) & \longrightarrow & \pi_m(MO) \\ \downarrow h & & \downarrow h \\ \tilde{H}_{m+n}(MO(n)) & \longrightarrow & H_m(MO), \end{array}$$

we have ${}^{geo}F_n \subset {}^{alg}F_n$, so we ask:

Question 4. Is the algebraic filtration the same as the geometric filtration?

The answer is *no* again, a counterexample being furnished by $[RP^{14}]$ which is of algebraic filtration 1, but of geometric filtration at least 2.

We refer the reader to Brown [4], [5] for an elegant geometric proof that

$$^{\text{geo}}F_{n-\alpha(n)} = \pi_n(MO)$$

and to Burlet [7], Salmonsen [15] on the homotopy of $MO(n)$ and $MSO(n)$, to Wells [17] on cobordism of immersions. I am indebted to all of the above authors (and to J. F. Adams, T. tom Dieck, R. K. Lashof, and M. Mahowald) for conversations and letters.

The paper is organized as follows: Section 1 describes the algebraically obvious generators for $\pi_*(MO)$, Section 2 describes the structure of $^{\text{alg}}E^0_*\pi_*(M)$ in total degree ≤ 14, Section 3 inspects the generators of Boardman, Brown, Dold, and Kozma, and Section 4 shows that $^{\text{alg}}F \neq {}^{\text{geo}}F$ in an infinite number of dimensions ≥ 14.

1. Algebraically obvious generators for $\pi_*(MO)$

We recall (see [13], for example) that $H_*(MO)$ (Z_2 coefficients here and later) is a polynomial algebra $Z_2[b_1, \ldots, b_n, \ldots]$ on generators b_n coming from $H_{n+1}(MO(1))$ with coaction of A_* (dual of the Steenrod algebra over Z_2, see Milnor [14]) given by $\mu_* b_n = \sum_{s=o}^{n} \gamma_{s+1}^{(n+1)} \otimes b_s$, where $\gamma_s^{(n)} \in A_{*n-s}$ satisfy the relations

$$\gamma_1^{(s+1)} = \begin{cases} \xi_r & \text{if } s = 2^r - 1, \\ 0 & \text{if } s \neq 2^r - 1, \end{cases}$$

where ξ_r are the Milnor [14] generators, and the $\gamma_j^{(i)}$ are determined by $\gamma_n^{(n)} = 1$ and the Cartan relations: for each pair of natural numbers i, j we have

$$\gamma_{i+j}^{(n)} = \sum_{n=s+t} \gamma_i^{(s)} \gamma_j^{(t)},$$

(see [13]).

Let $N_* = Z_2[u_2, u_4, \ldots, u_n, \ldots]$, $n \neq 2^r - 1$ and define an isomorphism of algebras and comodules over A_*

$$f\colon H_*(MO) \to A_* \otimes N_*$$

where the coaction in $A_* \otimes N_*$ is $\phi_* \otimes 1$, $\phi_*\colon A_* \to A_* \otimes A_*$ being the coproduct. The map f is determined by the algebra homomorphism

$$\mathbf{f} = (\eta_* \otimes 1)f\colon H_*(MO) \to N_*$$

where $\eta_*\colon A_* \to Z_2$ is the augmentation and $\mathbf{f}(b_n) = u_n$ if $n \neq 2^r - 1$, $\mathbf{f}(b_n) = 0$ if $n = 2^r - 1$ for some r (see Liulevicius [12], as well as correction in [13]). Of course, the image of $\pi_*(MO)$ is precisely $f^{-1}(1 \otimes N_*)$, and we call the corresponding polynomial generators of $\pi_*(MO)$ *algebraically obvious*. Let us simplify notation by identifying $H_*(MO)$ with $A_* \otimes N_*$ under f, thus identifying $\pi_*(MO)$ with $1 \otimes N_*$. Table 1.1 gives the expression for u_n in terms of the polynomial generators b_i for $n \leq 18$.

TABLE 1.1

Generator	Algebraic degree	Expression
ξ_1	1	b_1
u_2	1	b_2
ξ_2	1	$b_3 +$
	2	$+ b_1 b_2$
u_4	1	b_4
	3	$+ b_1^2 b_2$
u_5	1	b_5
	2	$+ b_1 b_4 + b_2 b_3$
	3	$+ b_1 b_2^2$
u_6	1	b_6
ξ_3	1	b_7
	2	$+ b_3 b_4 + b_1 b_6$
	3	$+ b_1 b_2 b_4 + b_1^2 b_5$
	4	$+ b_1^2 b_2 b_3 + b_1^3 b_4$
	5	$+ b_1^3 b_2^2$
u_8	1	b_8
	3	$+ b_2 b_3^2 + b_1^2 b_6$
	5	$+ b_1^2 b_2^3 + b_1^4 b_4$
	7	$+ b_1^6 b_2$
u_9	1	b_9
	2	$+ b_3 b_6 + b_2 b_7 + b_1 b_8$
	3	$+ b_2 b_3 b_4$
	4	$+ b_1 b_2^2 b_4 + b_1^2 b_2 b_5$
	5	$+ b_1^2 b_2^2 b_3 + b_1^3 b_2 b_4 + b_1^4 b_5$
	6	$+ b_1^4 b_2 b_3 + b_1^3 b_2^3 + b_1^5 b_4$
	7	$+ b_1^5 b_2^2$
u_{10}	1	b_{10}
	5	$+ b_1^4 b_6$
u_{11}	1	b_{11}
	2	$+ b_4 b_7 + b_3 b_8 + b_1 b_{10}$
	3	$+ b_3 b_4^2 + b_3^2 b_5 + b_1 b_4 b_6 + b_1 b_2 b_8 + b_1^2 b_9$
	4	$+ b_2 b_3^3 + b_1 b_3^2 b_4 + b_1 b_2 b_4^2 + b_1^2 b_4 b_5 +$
		$+ b_1^2 b_3 b_6 + b_1^2 b_2 b_7 + b_1^3 b_8$
	5	$+ b_1 b_2^2 b_3^2 + b_1^3 b_4^2 + b_1^2 b_2^2 b_5$
	6	$+ b_1^2 b_2^3 b_3 + b_1^3 b_2^2 b_4 + b_1^4 b_2 b_5$
	7	$+ b_1^3 b_2^4 + b_1^4 b_2^2 b_3 + b_1^5 b_2 b_4$
	8	$+ b_1^5 b_2^3$
u_{12}	1	b_{12}
	3	$+ b_3^2 b_6 + b_1^2 b_{10}$
	5	$+ b_1^2 b_2^2 b_6$
u_{13}	1	b_{13}
	2	$+ b_6 b_7 + b_3 b_{10} + b_1 b_{12}$
	3	$+ b_3 b_4 b_6 + b_1 b_6^2 + b_1 b_2 b_{10}$
	4	$+ b_1 b_2 b_4 b_6 + b_1^2 b_5 b_6$
	5	$+ b_1^2 b_2 b_3 b_6 + b_1^3 b_4 b_6$
	6	$+ b_1^3 b_2^2 b_6$
u_{14}	1	b_{14}

The computations are made easier by the following

LEMMA 1.2. *The coefficients* $\gamma_s^{(m)}$ *satisfy the relations*

$$\gamma_{2s}^{(2n)} = (\gamma_s^{(n)})^2, \quad \gamma_{2s+1}^{(2n)} = 0, \quad \gamma_{2s}^{(2n+1)} = \gamma_{2s}^{(2n)}.$$

For example, we have (using $f = 1$ and omitting tensor products):

$$u_{12} = b_{12} + \gamma_{12}^{(1)} + \gamma_{10}^{(3)}u_2 + \gamma_8^{(5)}u_4 + \gamma_6^{(7)}u_6 + \gamma_4^{(9)}u_8 + \gamma_2^{(11)}u_{10},$$

$\gamma_{12}^{(1)} = 0$, $\gamma_{10}^{(3)} = \gamma_{10}^{(2)} = (\gamma_5^{(1)})^2 = 0$, $\gamma_8^{(5)} = \gamma_8^{(4)} = (\gamma_2^{(1)})^4 = 0$, $\gamma_6^{(7)} = \gamma_6^{(6)} = (\gamma_3^{(3)})^2 = \xi_1^6 + \xi_2^2$, $\gamma_4^{(9)} = \gamma_4^{(8)} = (\gamma_1^{(2)})^4 = 0$, $\gamma_2^{(11)} = \gamma_2^{(10)} = (\gamma_1^{(5)})^2 = \xi_1^2$, the last since

$$\gamma_1^{(5)} = \gamma_0^{(4)}\gamma_1^{(1)} + \gamma_1^{(4)}\gamma_0^{(1)} = 1 \cdot \xi_1 + 0 \cdot 1 = \xi_1.$$

Substituting for ξ_1, ξ_2, u_6, u_{10}, we have

$$\begin{aligned} u_{12} &= b_{12} + (b_1^6 + b_3^2 + b_1^2b_2^2)b_6 + b_1^2(b_{10} + b_1^4b_6) \\ &= b_{12} + b_3^2b_6 + b_1^2b_{10} + b_1^2b_2^2b_6, \end{aligned}$$

as given in the table.

It is easy to compute the algebraic filtration of a given element in $H_*(MO; Z_2)$, since the image of $\tilde{H}_*(MO(n); Z_2)$ is precisely the subspace spanned by all monomials b^E of algebraic degree $\leq n$ (see [13] for example).

The lemma also shows that in order to solve for u_{2m} we only have to know the expression for u_{2n}, $n < m$ and ξ_r with $2^r - 1 \leq m$ since $\gamma_{2s}^{(2n+1)} = (\gamma_s^{(n)})^2$, and the exponents of monomials in ξ with nonzero coefficients in $(\gamma_s^{(n)})^2$ are all even. This explains why in the following table we have been able to go up to u_{42}.

TABLE 1.3

Algebraic filtration of polynomial generators

Generators	Algebraic filtration	Top term	Generators	Algebraic filtration	Top term
ξ_1	1	b_1	ξ_4	12	$b_1^9b_2^3$
u_2	1	b_2	u_{16}	15	$b_1^{14}b_2$
			u_{17}	15	$b_1^{13}b_2^2$
ξ_2	2	b_1b_2	u_{18}	13	$b_1^{12}b_6$
u_4	3	$b_1^2b_2$	u_{20}	13	$b_1^{10}b_2^2b_6$
u_5	3	$b_1b_2^2$	u_{22}	9	$b_1^8b_{14}$
u_6	1	b_6	u_{24}	15	$b_1^{10}b_2^4b_6$
			u_{26}	11	$b_1^6b_2^2b_{14}$
ξ_3	5	$b_1^3b_2^2$	u_{28}	11	$b_1^6b_2^4b_{14}$
u_8	7	$b_1^6b_2$	u_{30}	1	b_{30}
u_9	7	$b_1^5b_2^2$	u_{32}	31	$b_1^{30}b_2$
u_{10}	5	$b_1^4b_6$	u_{34}	29	$b_1^{28}b_6$
u_{11}	8	$b_1^5b_2^3$	u_{36}	29	$b_1^{26}b_2^2b_6$
u_{12}	5	$b_1^2b_2^2b_6$	u_{38}	31	$b_1^{28}b_2^2b_6$
u_{13}	6	$b_1^3b_2^2b_6$	u_{40}	33	$b_1^{30}b_2^2b_6$
u_{14}	1	b_{14}	u_{42}	35	$b_1^{32}b_2^2b_6$

Remark. By looking at Table 1.3 one is tempted to conjecture that algebraic filtration of an element in $\pi_*(MO)$ is detected by

$$Z_2[b_1, b_2, b_6, b_{14}, b_{30}, \ldots].$$

This certainly seems to be the case for ξ_r and u_n, but unfortunately breaks down in $\pi_{10}(MO)$.

2. Structure of ${}^{\mathrm{alg}}E^0\pi(MO)$

The computations in Section 1 are already enough to show that the answer to Question 1 is "no", since $[RP^{2n}]$ are known to be indecomposable.

PROPOSITION 2.1. *Given n, let r be the smallest integer such that* $2n + 1 < 2^r$; *then the algebraic filtration of* $[RP^{2n}]$ *is* $2^r - 2n - 1$.

Proof. By choice of r, if $x \in H^1(RP^{2n})$ is the generator, $(1 + x)^{2^r} = 1$, so $(1 + x)^{-2n-1} = (1 + x)^{2^r-2n-1}$ and the coefficient of x^{2^r-2n-1} is 1, and this power of x is nonzero.

Remark. According to the Proposition 2.1, alg. filt. $[RP^{12}] = 16 - 13 = 3$, but alg. filt. $u_{12} = 5$. The discrepancy is even more dramatic later—for example: alg. filt. $[RP^{28}] = 32 - 29 = 3$, but alg. filt. $u_{28} = 11$. Notice that through dimension 10, however, the filtrations of u_{2n} are the same as those of $[RP^{2n}]$. Indeed we have Table 2.2.

TABLE 2.2

$[RP^{2n}]$ in terms of u_n's

$$\begin{aligned}
[RP^2] &= u_2,\\
[RP^4] &= u_4 + u_2^2,\\
[RP^6] &= u_6,\\
[RP^8] &= u_8 + u_4^2 + u_2^2u_4 + u_2^4,\\
[RP^{10}] &= u_{10} + u_2^5,\\
[RP^{12}] &= u_{12} + u_6^2 + u_4^3 + u_2^2u_8 + u_2u_5^2.
\end{aligned}$$

Conjecture. The algebraic filtration of $[RP^{2n}]$ is the best possible for a polynomial generator of $\pi_*(MO)$ in dimension $2n$.

The following table exhibits a Z_2-basis for ${}^{\mathrm{alg}}E^0\pi(MO)$ and shows that the answer to Question 2 is "no," the first counterexample occuring in dimension 10.

TABLE 2.3

A basis for ${}^{\mathrm{alg}}E^0\pi_k(MO)$ for $k \leq 14$

Dimension	Basis: class of	Filtration
2	u_2	1
4	u_4	3
	u_2^2	2
5	u_5	3
6	u_2u_4	4
	u_2^3	3
	u_6	1
7	u_2u_5	4
8	u_8	7
	u_4^2	6
	$u_2^2u_4$	5
	u_2^4	4
	u_2u_6	2
9	u_9	7
	u_4u_5	6
	$u_2^2u_5$	5
10	u_2u_8	8
	$u_2u_4^2$	7
	$u_2^3u_4$	6
	u_2^5	5
	u_{10}	5
	u_4u_6	4
	$\alpha_{10} = u_2^3u_4 + u_5^2$	4 (drop of 2)
	$u_2^2u_6$	3
11	u_2u_9	8
	$u_2u_4u_5$	7
	$u_2^3u_5$	6
	$v_{11} = u_2u_9 + u_{11}$	5 (drop of 3)
	u_5u_6	4
12	u_4u_8	10
	$u_2^2u_8$	9
	$u_2^2u_4^2$	8
	$u_2^4u_4$	7
	u_2^6	6
	u_2u_{10}	6
	$u_2\alpha_{10}$	5
	$u_2u_4u_6$	5
	$u_2^3u_6$	4
	$\alpha_{12} = u_2^2u_8 + u_4^3 + u_2u_5^2 + u_2u_4u_6$	3 (drop of 6)
	$v_{12} = u_{12} + u_2u_4u_6$	3 (drop of 2)
	u_6^2	2
13	u_5u_8	10
	$u_4^2u_5$	9
	$u_2^2u_4u_5$	8
	$u_2^4u_5$	7
	u_2v_{11}	6

TABLE 2.3 (*continued*)

Dimension	Basis: class of	Filtration
	$\alpha_{13} = u_5u_8 + u_4u_9 + u_2u_{11}$	6 (drop of 4)
	$u_2u_5u_6$	5
	$\beta_{13} = u_4^2u_5 + u_2^2u_9 + u_2^2u_4u_5$	4 (drop of 5)
	$v_{13} = u_5u_8 + u_4u_9 + u_2u_{11} + u_{13}$	3 (drop of 7)
14	$u_2u_4u_8$	11
	$u_2^3u_8$	10
	$u_2^3u_4^2$	9
	$u_2^5u_4$	8
	u_6u_8	8
	$\alpha_{14} = u_2^3u_8 + u_5u_9 + u_2^3u_4^2$	8
	u_2^u	7
	$u_4^2u_6$	7
	$u_4\alpha_{10}$	7
	$u_2^2u_4u_6$	6
	$u_2^2\alpha_{10}$	6
	$u_2^4u_6$	5
	u_2v_{12}	4
	$\beta_{14} = u_2u_4^3 + u_2^3u_8 + u_2^2u_5^2 + u_2u_{12}$	4
	$\gamma_{14} = u_6u_8 + u_4u_{10} + u_2^2u_4u_6$	4
	$u_2u_6^2$	3
	$\delta_{14} = u_4^2u_6 + u_2^7u_{10}$	3
	u_{14}	1

3. Generators for $\pi_*(MO)$

In this section we recall the construction of the polynomial generators for $\pi_*(MO)$ by Boardman [3], Brown [4], [5], Dold [8], Kozma [11] and examine their algebraic filtration.

Boardman generators for $\pi_*(MO)$ are defined in terms of the standard generator $\omega \in MO^1(RP^\infty)$ which under the Boardman homomorphism B (see Adams [2], for example) have image

$$B(\omega) = x + b_1x^2 + \cdots + b_nx^{n+1} + \cdots,$$

where $b_n \in H_n(MO)$ are the generators we have used in Section 1 and $x \in H^1(RP^\infty)$ is the nonzero element in cohomology. Let

$$m: RP^\infty \times RP^\infty \to RP^\infty$$

be the standard multiplication with $m^*x = x \otimes 1 + 1 \otimes x$. Then ω is not primitive under the diagonal in $MO^*(RP^\infty)$ induced by $MO^*(m)$, but there is a unique primitive $\pi \in MO^1(RP^\infty)$ satisfying the condition

$$B(\pi) = x + \alpha_1x^2 + \alpha_2x^4 + \cdots + \alpha_nx^{2^n} + \cdots$$

(Boardman [3], see also Bröcker and tom Dieck [6]); then

$$\pi = \omega + \beta_2\omega^2 + \beta_4\omega^5 + \cdots + \beta_n\omega^{n+1} + \cdots$$

where $n \neq 2^r - 1$ and $\beta_n \in \pi_n(MO)$ are the Boardman polynomial generators. Table 3.1 lists the Hurewicz images of the Boardman generators.

TABLE 3.1

Generator	Block	Hurewicz image	Algebraic filtration
β_2	1	b_2	1
β_4	1	b_4	3
	2	$+ b_2^2$	
	3	$+ b_1^2 b_2$	
β_5	1	b_5	3
	2	$+ b_2 b_3 + b_1 b_4$	
	3	$+ b_1 b_2^2$	
β_6	1	b_6	1
β_8	1	b_8	7
	2	$+ b_4^2$	
	3	$+ b_2 b_3^2 + b_2^2 b_4 + b_1^2 b_6$	
	4	$+ b_2^4$	
	5	$+ b_1^4 b_4$	
	6	$+ b_1^4 b_2^2$	
	7	$+ b_1^6 b_2$	
β_9	1	b_9	7
	2	$+ b_4 b_5 + b_3 b_6 + b_2 b_7 + b_1 b_8$	
	3	$+ b_1 b_4^2$	
	4	0	
	5	$+ b_1^4 b_5$	
	6	$+ b_1^4 b_2 b_3 + b_1^5 b_4$	
	7	$+ b_1^5 b_2^2$	
β_{10}	1	b_{10}	5
	5	$+ b_1^4 b_6 + b_2^5$	
β_{11}	1	b_{11}	6
	2	$+ b_5 b_6 + b_4 b_7 + b_3 b_8 + b_2 b_9 + b_1 b_{10}$	
	3	$+ b_3 b_4^2 + b_3^2 b_5 + b_2^2 b_7 + b_1^2 b_9$	
	4	$+ b_2 b_3^3 + b_2^2 b_3 b_4 + b_1 b_3^2 b_4 + b_1 b_2 b_4^2 + b_2^3 b_5$	
		$+ b_1^2 b_4 b_5 + b_1 b_2^2 b_6 + b_1^2 b_3 b_6$	
	5	$+ b_2^4 b_3 + b_1 b_2^2 b_3^2 + b_1^3 b_4^2$	
	6	$+ b_1 b_2^5$	
β_{12}	1	b_{12}	3
	3	$+ b_4^3 + b_2 b_5^2 + b_3^2 b_6 + b_2^2 b_8 + b_1^2 b_{10}$	
β_{13}	1	b_{13}	3
	2	$+ b_6 b_7 + b_5 b_8 + b_4 b_9 + b_3 b_{10} + b_2 b_{11} + b_1 b_{12}$	
	3	$+ b_1 b_6^2$	

We express in Table 3.2 the β_i in terms of our generators u_i, v_i from Sections 1 and 2.

TABLE 3.2

B-generator	Expression	Algebraic filtration	Comment on filtration
β_2	u_2	1	best
β_4	$u_4 + u_2^2$	3	best
β_5	u_5	3	best
β_6	u_6	1	best
β_8	$u_8 + u_4^2 + u_2^2u_4 + u_2^4$	7	best
β_9	$u_9 + u_4u_5$	7	best
β_{10}	$u_{10} + u_2^5$	5	best
β_{11}	$u_{11} + u_2u_9 + u_5u_6 + u_2^2u_5$	6	off by 1
	$= v_{11} + u_5u_6 + u_2^3u_5$		
β_{12}	$v_{12} + \alpha_{12} + u_6^2$	3	best
β_{13}	$u_{13} + u_2u_{11} + u_4u_9 + u_5u_8$	3	best
	$= v_{13}$		

It turns out that $\beta_{2n} = [RP^{2n}]$, so we conjecture that at least the even-dimensional Boardman generators have the minimal algebraic filtration. We now introduce a new set of polynomial generators for $H_*(MO)$ (in analogy with the generators m_k for $H_*(MU; Z)$, see Adams [2]). Let us identify $MO^*(RP^\infty)$ with its image in $H_*(MO) \hat{\otimes} H^*(RP^\infty)$ under the Boardman homomorphism B, so

$$\omega = x + b_1x^2 + \cdots + b_nx^{n+1} + \cdots.$$

Define the elements $m_k \in H_*(MO)$ by setting

$$x = \omega + m_1\omega^2 + \cdots + m_k\omega^{k+1} + \cdots;$$

then $m_k = b_k +$ decomposables, so they form a new set of polynomial generators of $H_*(MO)$. Notice that $m_{2k} = B_{2k}^{-2k-1}$ by the Burmann-Lagrange inversion formula (since $2k + 1 = 1$ in Z_2), so $m_{2k} = h[RP^{2k}]$ (compare Adams [2] for $[CP^n]$). Table 3.3 gives the m_i in terms of b_i (notice that the table also gives b_i in terms of m_i, as well).

It is particularly easy to express the Boardman generators in terms of the polynomial generators m_k. By definition

$$\begin{aligned} x &= \omega + m_1\omega^2 + \cdots + m_k\omega^{k+1} + \cdots, \\ \pi &= \omega + \beta_2\omega^3 + \cdots + \beta_n\omega^{n+1} + \cdots, \quad n+1 \neq 2^r, \\ \pi &= x + \alpha_1x^2 + \alpha_2x^4 + \alpha_3x^8 + \cdots + \alpha_rx^{2^r} + \cdots, \end{aligned}$$

so since $x^{2^r} = \omega^{2^r} + m_1^{2^r}\omega^{2^{r+1}} + \cdots + m_k^{2^r}\omega^{2^r(k+1)} + \cdots$ we get Table 3.4 by equating coefficients.

Remark. Of course, the recursion relations give us immediately that $\beta_{2n} = m_{2n}$.

TABLE 3.3

m_i	Block	Expression in b's	Algebraic filtration
m_1	1	b_1	1
m_2	1	b_2	1
m_3	1	b_3	3
	2	$+ b_1b_2$	
	3	$+ b_1^3$	
m_4	1	b_4	3
	2	$+ b_2^2$	
	3	$+ b_1^2b_2$	
m_5	1	b_5	2
	2	$+ b_1b_4 + b_2b_3$	
m_6	1	b_6	1
m_7	1	b_7	7
	2	$+ b_3b_4 + b_2b_5 + b_1b_6$	
	3	$+ b_2^2b_3 + b_1b_3^2 + b_1^2b_5$	
	4	$+ b_1b_2^3 + b_1^2b_2b_3 + b_1^3b_4$	
	5	$+ b_1^4b_3$	
	6	$+ b_1^5b_2$	
	7	$+ b_1^7$	
m_8	1	b_8	7
	2	$+ b_4^2$	
	3	$+ b_2b_3^2 + b_2^2b_4 + b_1^2b_6$	
	4	$+ b_2^4$	
	5	$+ b_1^4b_4$	
	6	$+ b_1^4b_2^2$	
	7	$+ b_1^6b_2$	
m_9	1	b_9	6
	2	$+ b_4b_5 + b_3b_6 + b_2b_7 + b_1b_8$	
	3	0	
	4	0	
	5	$+ b_1b_2^4 + b_1^4b_5$	
	6	$+ b_1^4b_2b_3 + b_1^5b_4$	
m_{10}	1	b_{10}	
	5	$+ b_2^5 + b_1^4b_6$	
m_{11}	1	b_{11}	
	2	$+ b_5b_6 + b_4b_7 + b_3b_8 + b_2b_9 + b_1b_{10}$	
	3	$+ b_3b_4^2 + b_1b_5^2 + b_3^2b_5 + b_2^2b_7 + b_1^2b_9$	
	4	$+ b_2b_3^3 + b_2^2b_3b_4 + b_1b_3^2b_4 + b_1b_2b_4^2$ $+ b_2^3b_5 + b_1^2b_4b_5 + b_1b_2^2b_6 + b_1^2b_3b_6$ $+ b_1^2b_2b_7 + b_1^3b_8$	

We wish to show that the Boardman generators are algebraically obvious in exactly the same sense as our generators u_n in Section 1. Let H be the Eilenberg-Mac Lane spectrum representing cohomology with Z_2 coefficients, t: $MO \to H$ the Thom class $1 \in H^0(MO)$. We have $t_*b_n = 0$ if $n \neq 2^r - 1$ and $t_*b_{2^r-1} =$

TABLE 3.4

α, β	Expression in m_k
α_1	$= m_1$
β_2	$= m_2$
α_2	$= m_3 + m_1^3$
β_4	$= m_4$
β_5	$= m_5 + m_1 m_2^2$
β_6	$= m_6$
α_3	$= m_7 + m_1 m_3^2 + m_1^4 m_3 + m_1^7$
β_8	$= m_8$
β_9	$= m_9 + m_1 m_4^2$
β_{10}	$= m_{10}$
β_{11}	$= m_{11} + m_1 m_5^2 + m_2^4 m_3 + m_1^3 m_2^4$
β_{12}	$= m_{12}$
β_{13}	$= m_{13} + m_1 m_6^2$
β_{14}	$= m_{14}$
α_4	$= m_{15} + m_1 m_7^2 + m_1^8 m_7 + m_1^{12} m_3$ $+ m_1^9 m_3^2 + m_1^3 m_3^4 + m_3^5 + m_1^{15}$
β_{16}	$= m_{16}$
β_{17}	$= m_{17} + m_1 m_8^2$
β_{18}	$= m_{18}$

ξ_r, by definition of the Milnor generators $\xi_r \in H_*(H) = A_*$, the dual of the Steenrod algebra over Z_2.

LEMMA 3.5. *The map* $t_*\colon H_*(MO) \to A_*$ *is given by* $t_* m_n = 0$ *if* $n \neq 2^r - 1$, $t_* m_{2^r - 1} = c_* \xi_r$, *where* $c_*\colon A_* \to A_*$ *is the conjugation homomorphism.*

Proof. The following diagram is commutative, where B is the B Boardman map and λ is the Milnor homomorphism [14]:

$$\begin{array}{ccc} MO^*(RP^\infty) & \xrightarrow{B} & H_*(MO) \hat{\otimes} H^*(RP^\infty) \\ \downarrow t_* & & \downarrow t_* \otimes 1 \\ H^*(RP^\infty) & \xrightarrow{\lambda} & A_* \hat{\otimes} H^*(RP^\infty) \end{array}$$

that is,

$$\bar{\omega} = \lambda(t_* \omega) = x + \xi_1 x^2 + \cdots + \xi_r x^{2^r} + \cdots$$

(where we omit tensor signs) and

$$x = \bar{\omega} + (t_* m_1)\bar{\omega}^2 + \cdots + (t_* m_k)\bar{\omega}^{k+1} + \cdots,$$

but we have $x = \bar{\omega} + \gamma_1 \bar{\omega}^2 + \cdots + \gamma_s \bar{\omega}^{2s} + \cdots$, where γ_i satisfy the recursion relation

$$\xi_r + \xi_{r-1}^2 \gamma_1 + \xi_{r-2}^4 \gamma_2 + \cdots + \gamma_r = 0,$$

that is, $\gamma_r = c_* \xi_r$, and the lemma follows.

We can now determine the coaction of the m_k:

COROLLARY 3.6. *Let $\mu_*: H_*(MO) \to A_* \otimes H_*(MO)$ be the coaction; then*

$$\mu_* m_n = \sum_i c_*\zeta_i \otimes (M^{2^i})_{n-i},$$

where $M = 1 + m_1 t + m_2 t^2 + \cdots$.

Proof. Proposition 9.4 of Adams [2] with MU replaced by MO and Lemma 3.5.

We now define (as in Section 1) a map $g: H_*(MO) \to A_* \otimes N_*$ of comodule algebras over A_* by letting $\mathbf{g}: H_*(MO) \to N_*$ be the algebra homomorphism defined by setting

$$g(m_n) = \begin{cases} u_n & \text{if } n \neq 2^r - 1 \\ 0 & \text{if } n = 2^r - 1. \end{cases}$$

We then have (and we invite the reader to check this) $g^{-1}(u_n) = \beta_n$, the Boardman generator, and $g^{-1}(c_*\xi_r) = \alpha_r$, where

$$B(\pi) = x + \alpha_1 x^2 + \cdots + \alpha_r x^{2^r} + \cdots.$$

We now describe the generators of $\pi_*(MO)$ constructed by Ilan Kozma [11]. These are elements $T_i \in \pi_i(MO)$ satisfying the recursion relation

$$m_{s-1} = \sum_{id=s} m_{i-1} T^i_{d-1}.$$

TABLE 3.5

Kozma's generators

Generator	Expression in m_k	Algebraic filtration	Comment
T_2	m_2	1	best
T_4	m_4	3	best
T_5	$m_5 + m_1 m_2^2$	3	best
T_6	m_6	1	best
T_8	$m_8 + m_2^4$	7	best
T_9	$m_9 + m_1 m_4^2$	7	best
T_{10}	m_{10}	5	best
T_{11}	$m_{11} + m_2^4 m_3 + m_1^3 m_2^4 + m_1 m_5^2$	6	off by 1
T_{12}	m_{12}	3	best
T_{13}	$m_{13} + m_1 m_6^2$	3	best

Remark. Notice that $\beta_{2n+1} = T_{2n+1}$ for $n \leq 7$, but $T_{17} \neq \beta_{17}$.

The Dold generators $\pi_*(MO)$ are described as follows (Satz 3, p. 32 of Dold [8]—the notation is ours). Let $d_{2n} = [RP^{2n}]$, and if n is odd and $n + 1$ is not a power of 2, let $n = 2^r(2s + 1) - 1$ and set $d_{2n+1} = [P(2^r - 1, s2^r)]$, where

$P(a, b)$ for natural numbers a, b is the smooth manifold of dimension $a + 2b$ obtained as a quotient of $S^a \times CP^b$ under the action of Z_2 by $T(x, y) = (-x, \bar{y})$, where $x \in S^a$ and y represents the homogeneous coordinates of a point in CP^b and bar denotes conjugation.

TABLE 3.6

Dold's generators

Generator	Expression in u_k	Algebraic filtration	Comment
d_2	$m_2 = u_2$	1	best
d_4	$m_4 = u_4 + u_2^2$	3	best
d_5	u_5	3	best
d_6	$m_6 = u_6$	1	best
d_8	$m_8 = u_8 + u_4^2 + u_2^2 u_4 + u_2^4$	7	best
d_9	$u_9 + u_4 u_5$	7	best
d_{10}	$m_{10} = u_{10} + u_2^5$	5	best
d_{11}		7	off by 2
d_{12}	m_{12}	3	best
d_{13}		3	best

Brown's generators are described [4], [5] as follows: if n is even and $n = r_1 + \cdots + r_k$ is the binary expansion of n as a sum of distinct powers of 2, then let $V^n = RP^n$ if $k = 1$ and for $k > 1$ let V^n be a submanifold of

$$K^{n+1} = RP^{r_1} \times \cdots \times RP^{r_{k-1}} \times RP^{r_k+1}$$

dual to $x_1 + \cdots + x_k \in H^1(K^{n+1}; Z_2)$ where x_i is the fundamental class of the ith factor. The cobordism class of V^n gives the even-dimensional Brown generators λ_n for $\pi_*(MO)$. The odd-dimensional generators are obtained by a modification of Dold's construction $P(a, b)$: if N^n is an n-manifold, let $P(m, N)$ be the $(m + 2n)$-manifold obtained from $S^m \times N \times N$ by identifying

TABLE 3.7

Brown's generators

Generator	Expression in u_n	Algebraic filtration	Comment
λ_2	u_2	1	best
λ_4	$u_4 + u_2^2 = m_4$	3	best
$\lambda_5 = d_5$	u_5	3	best
λ_6	$u_6 + u_2 u_4$	4	off by 3
λ_8	m_8	7	best
$\lambda_9 = d_9$	$u_9 + u_4 u_5$	7	best
λ_{10}		8	off by 3

(x, y_1, y_2) with $(-x, y_2, y_1)$. The Brown polynomial generators λ_n are defined in odd dimensions n by letting $n = 2^r(2s + 1) - 1$, $s > 0$ and letting $V^n = P(2^r - 1, V^{2^r s})$, where V^{even} has been defined before, and $\lambda_n = [V^n]$ is the Brown polynomial generator for dimension n. Notice that $P(m, RP^n) = P(m, n)$ of Dold.

4. The algebraic and geometric filtrations do not coincide

We shall show that the algebraic and geometric filtrations of $\pi_*(MO)$ are different by proving that the geometric filtration of $b_{14} = m_{14} = u_{14} = [RP^{14}]$ is greater than one. We show that this is a consequence of the work of J. F. Adams on the nonexistence of elements of Hopf invariant one [1]. The argument was pointed out to me by M. E. Mahowald. Let Y be the fiber of the map $S^0 \to HZ_2$, then the mod -2 homology of Y is just the augmentation ideal with a shift down by one in grading: $H_k(Y) = (\bar{A}_*)_{k-1}$. Let P be the suspension spectrum of RP^∞ and let $J\colon P \to S^0$ be the stable J-map. Then since $J_* = 0$ in homology, it factors through $j\colon P \to Y$.

Lemma 4.1. *Let x_n be the class in $H_n(P)$ dual to $x^n \in H^n(P)$, where x is the non-zero element of $H^1(P)$. Then $j_*(x_n) \neq 0$.*

Corollary 4.2. *If $n = 2^r - 1$, then $j_*(x_n) = [\xi_1^{2^r}]$.*

Proof. The element x_n is primitive for $n + 1 = 2^r$ under the coaction of A_*, but the only primitives of A_* are elements $\xi_1^{2^s}$ (see [1]).

We now notice that $MO(1)$ is homotopy equivalent to RP^∞ and under this homotopy equivalence the element b_n corresponds to x_{n+1}. We shall show that b_{14} is not in the image of the stable Hurewicz homomorphism, but for this it is sufficient to show that $j_* b_{14} = [\xi_1^{16}]$ is not spherical, but this was shown by Adams [1]. Indeed, if we let h_s = class of $[\xi_1^{2^s}] \in \mathrm{Ext}_A^1(Z_2, Z_2)$, then $d_2 h_s = h_0 h_{s-1}^2$ which is nonzero if $s \geq 4$. We have shown:

Corollary 4.3. *If $n = 2^r - 2$, $r \geq 4$, then the geometric filtration of b_n is at least 2.*

Of course, we can ask: what is the geometric dimension of $b_{2^r-2} = [RP^{2^r-2}]$ for $r \geq 4$? We shall return to this question in a later note.

Preliminary examination indicates that ${}^{\text{alg}}F_n = {}^{\text{geo}}F_n$ for $n \leq 10$ and there is reason to suspect that b_{14} is the lowest-dimensional example for which the two filtrations are distinct.

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