A FORMULA FOR RAMANUJAN'S τ-FUNCTION

BY

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A recent investigation by Morris Newman shows that $\tau(p) \equiv 0 \bmod p$ for $p = 2, 3, 5, 7$, and 2411 [1]. For his calculations Dr. Newman used a formula for $\tau(n)$ which involves $\sigma_3(n)$, the sum of the cubed divisors of n. Since a table of the exact values of $\sigma_3(n)$ was needed, his search for such p was limited to $p \leq 16{,}067$. This formula for $\tau(n)$ appears in [2] and a corrected version is given in [3]. In [2] there also appears a formula expressing $\sigma_3(n)$ in terms of $\sigma(n)$, the sum of the divisors of n. Thus it seemed possible that the ideas in [2] should lead to a formula for $\tau(n)$ in terms of $\sigma(n)$, which could then be used to extend Newman's search. This is indeed true and as is the case with most formulas in [2], once the formula is known, a simple proof of it can then be given. Using the new formula, Mr. K. Ferguson constructed a table of $\tau(p) \bmod p$ for $3 \leq p \leq 65{,}063$ which contained no new solutions of $\tau(p) \equiv 0 \bmod p$.

Since we have not found our formula in the literature and because it may be useful in other investigations, we state and prove it here.

THEOREM. *Let* $e(z) = \exp(2\pi i z)$ *and let* $\tau(n)$ *be defined by the equation* $\sum_{n=1}^{\infty} \tau(n)e(nz) = e(z)\prod_{n=1}^{\infty}(1 - e(nz))^{24}$. *Let* $\sigma(n)$ *be the sum of the positive divisors of* n. *Then for* $n \geq 1$,

$$\tau(n) = n^4\sigma(n) - 24\sum_{k=1}^{n-1}(35k^4 - 52k^3n + 18k^2n^2)\sigma(k)\sigma(n-k).$$

Proof. Let $\Delta(z) = e(z)\prod_{n=1}^{\infty}(1 - e(nz))^{24}$ and put

$$f(z) = -\log \Delta(z) = -2\pi i z + 24\sum_{n=1}^{\infty}\frac{\sigma(n)e(nz)}{n}.$$

As one easily sees, the theorem will be proved if the validity of the following equation is demonstrated:

$$2^9 \cdot 3\pi^6\Delta(z) = 18[f'''(z)]^2 + f'(z)f^{(5)}(z) - 16f''(z)f^{(4)}(z).$$

Call the right-hand side of this equation $F(z)$. Then $F(z) = F(z + 1)$ and the first Fourier coefficient of F is $2^9 \cdot 3\pi^6$. Hence, this equation will be proved if we show that $F(-1/z) = z^{12}F(z)$, since the space of cusp forms of weight 6 for the modular group is one-dimensional.

Received November 15, 1974.

The definition of f implies that $f(-1/z) = -\log z^{12} + f(z)$. Thus,

$$\begin{aligned}
f'(-1/z) &= -12z + z^2 f'(z),\\
f''(-1/z) &= -12z^2 + 2z^3 f'(z) + z^4 f''(z),\\
f'''(-1/z) &= -24z^3 + 6z^4 f'(z) + 6z^5 f''(z) + z^6 f'''(z),\\
f^{(4)}(-1/z) &= -72z^4 + 24z^5 f'(z) + 36z^6 f''(z) + 12z^7 f'''(z) + z^8 f^{(4)}(z),\\
f^{(5)}(-1/z) &= -288z^5 + 120z^6 f'(z) + 240z^7 f''(z) + 120z^8 f'''(z)\\
&\qquad + 20z^9 f^{(4)}(z) + z^{10} f^{(5)}(z).
\end{aligned}$$

Using these equations, we first find that

$$H(z) = f^{(4)}(z) + f'(z)f'''(z) - \tfrac{3}{2}[f''(z)]^2$$

is a cusp form of weight 4 for the modular group. Thus $H(z) = 0$. This is the well-known differential equation for $\Delta(z)$ which is also proved in [2]. It is pointed out in [2] that $H(z) = 0$ is equivalent to the fact that the Schwarzian derivative $[f]_z = (f'''(z)/f'(z)) - \tfrac{3}{2}(f''(z)/f'(z))^2$ equals $f^{(4)}(z)/[f'(z)]^2$.

If we now insert the above expressions for $f^{(j)}(-1/z)$ in the definition of $F(-1/z)$ and collect like powers of z, we obtain

$$F(-1/z) = z^{12}F(z) - 12z^{11}H'(z) - 48z^{10}H(z) = z^{12}F(z).$$

References

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