GENERALIZATION OF A THEOREM OF HAYMAN TO R^n

BY

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1. Introduction

Suppose that $u(z)$ is subharmonic (s.h.) in a disk $|z| \leq R$ in the plane. Let

$$T(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\theta})\, d\theta,\ u_1(re^{i\theta}) = \sup_{0 \leq t \leq r} u^-(te^{i\theta}),$$

where $u^+(z) = \max\{u(z), 0\}$ and $u^-(z) = -\min\{u(z), 0\}$. Hayman [2, Theorem 4, p. 193] proved the following result.

THEOREM A. *If $u(z)$ is s.h. in $|z| \leq R$, then for $0 < r < R$,*

$$\frac{1}{2\pi} \int_0^{2\pi} u_1(re^{i\theta})\, d\theta \leq \left[1 + \psi\left(\frac{r}{R}\right)\right] \{T(R, u) - u(0)\}, \tag{1.1}$$

where

$$\psi(t) = \frac{(1 - t) \log(1 + (2\pi\sqrt{t})/(1 - t))}{\pi\sqrt{t} \log(1/t)}.$$

This powerful result has some interesting applications, and has been used by Hornblower [3], Hornblower and Thomas [4], and Talpur [5], [6] to show the existence of a sectionally polygonal asymptotic path in a disk or the plane along which $u(z) \to M$, where M is $+\infty$ in the latter case. It is natural to investigate the analogues of this and other results in spaces of higher dimensions. In order to show the existence of an asymptotic path Γ such that $u(x) \to M$ as $x \to \infty$ on Γ, where M is the l.u.b. of $u(x)$ in R^3, the author proved a spatial analogue of Theorem A in Talpur [7], but was able to show the existence of an asymptotic path Γ only with finite M. In spite of this the result is interesting as the constant involved is the best possible. Theorem 1 is a generalization of Theorem A to R^n, $n \geq 3$. This theorem has some interesting consequences.

Suppose that $u(x)$ is s.h. in R^n and

$$u(x) = u(x_1, \ldots, x_n) = u(\rho, \theta_1, \ldots, \theta_{n-1})$$

where $0 < \rho < \infty$, $0 < \theta_i < \pi$ $(i = 1, \ldots, n - 2)$, $0 < \theta_{n-1} \leq 2\pi$.

Let ω_n denote the surface area of the n-dimensional unit sphere. Thus $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

Received July 8, 1974.

† Professor M. N. M. Talpur died in June, 1975.

Let $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = -\min\{u(x), 0\}$.

$$T(R) = T(R, u) = \frac{1}{\omega_n R^{n-1}} \int_{|x|=R} u^+(x)\, d\sigma_R(x),$$

$$m(R) = m(R, u) = \frac{1}{\omega_n R^{n-1}} \int_{|x|=R} u^-(x)\, d\sigma_R(x)$$

where the integration is with respect to $d\sigma_R(x)$, the $(n - 1)$-dimensional surface area element on $|x| = R$.

$$u_1(r) = \sup_{0 \le \rho \le r} u^-(\rho, \theta_1, \ldots, \theta_{n-1}) \quad \text{for fixed } \theta_i,\ i = 1, \ldots, n-1.$$

THEOREM 1. *Suppose that $u(x)$ is s.h. in a neighborhood of a closed ball $|x| \le R$; then with the above notation, for $0 < r < R$,*

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} u_1(r)\, d\sigma_r(x) < \left[\tfrac{1}{2} + C_n + \psi_n\left(\frac{r}{R}\right)\right]\{T(R) - u(0)\}, \tag{1.2}$$

where

$$C_n = \frac{\Gamma(n/2)\Gamma(1/2)}{2\Gamma((n-1)/2)} \quad \text{and} \quad \psi_n(t) = \frac{4C_n\sqrt{t}}{\pi(1 - \sqrt{t})}.$$

From this we deduce Theorem 2.

THEOREM 2. *If $u(x)$ is nonpositive and s.h. in R^n, $n \ge 3$, then*

$$\frac{1}{\omega_n} \int_{|x|=1} \inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})\, d\sigma_1(x) \ge (\tfrac{1}{2} + C_n)u(0) \tag{1.3}$$

We then show by an example that the constant $\frac{1}{2} + C_n$ is the best possible. An immediate consequence of the sharp inequality (1.3) is the following.

COROLLARY. *Suppose that $u(x)$ is s.h. in R^n, $n \ge 3$, and bounded above there. Then on almost all lines through a given point, $u(x)$ is bounded below except when u is $-\infty$ at that point.*

2. Preliminary results

Our method of proof is similar to that of Hayman. For our proof we shall need two lemmas. The first lemma is a version of the Riesz decomposition theorem which represents $u(x)$ in $|x| < R$ in terms of the values of $u(x)$ on $|x| = R$, and the Riesz measure μ of $u(x)$ in $|x| < R$. (See for instance Brelot [1, Chapter 4, §3].) The second lemma is on some estimates of kernels. We first introduce some notation.

Let $K(R, x, \xi)$ denote the Poisson kernel for $|x| < R$, and so

$$K(R, x, \xi) = \frac{R^{n-2}(R^2 - |x|^2)}{|\xi - x|^n}.$$

If θ_1 is the angle between x and ξ with $|\xi| = R$, and $|x| = \rho$, then

$$K(R, x, \xi) = K(R, \rho, \theta_1) = \frac{R^{n-2}(R^2 - \rho^2)}{(R^2 + \rho^2 - 2R\rho\cos\theta_1)^{n/2}}. \tag{2.1}$$

Let $k(R, r, \theta_1) = \sup_{0 \le \rho \le r} K(R, \rho, \theta_1)$.

Let $G(R, x, \xi)$ be the Green's function for Laplace's equation in an n-dimensional ($n \geq 3$) sphere of radius R. Then

$$G(R, x, \xi) = \frac{1}{|x - \xi|^{n-2}} - \frac{R^{n-2}}{|\xi|^{n-2}|x - \xi'|^{n-2}},$$

where ξ' is the point inverse to ξ in the R-hypersphere.

If $|x| = \rho$ and $|\xi| = r_\mu$ and θ_1 is the angle between x and ξ, then

$$\begin{aligned} G(R, x, \xi) &= G(R, \rho, r_\mu, \theta_1) \\ &= \frac{1}{(\rho^2 + r_\mu^2 - 2\rho r_\mu \cos\theta_1)^{(n-2)/2}} \\ &\qquad - \frac{R^{n-2}}{(R^4 + \rho^2 r_\mu^2 - 2R^2\rho r_\mu\cos\theta_1)^{(n-2)/2}} \end{aligned} \tag{2.2}$$

Let $g(R, r, r_\mu, \theta_1) = \sup_{0 \le \rho \le r} G(R, \rho, r_\mu, \theta_1)$.

LEMMA 1. *Suppose that $u(x)$ is s.h. in R^n, $n \geq 3$. For every $R > 0$, there exists a unique nonnegative measure $\mu(e)$ defined for all Borel measurable sets e in R^n and finite on compact sets, such that for all x in $|x| < R$, we have,*

$$u(x) = \frac{1}{\omega_n R^{n-1}} \int_{|\xi| = R} u(\xi) K(R, x, \xi)\, d\sigma_R(\xi) - \int_{|\xi| < R} G(R, x, \xi)\, d\mu(\xi). \tag{2.3}$$

LEMMA 2. *With the above notation, we have:*

(i) *In R^3,*

$$\frac{1}{\omega_3 r^2} \int_{|x| = r} k(R, r, \theta_1)\, d\sigma_r(x) < 1 + \tfrac{1}{2}\log\frac{R + r}{R - r} \tag{2.4}$$

In R^n, $n \geq 4$,

$$\frac{1}{\omega_n r^{n-1}} \int_{|x| = r} k(R, r, \theta_1)\, d\sigma_r(x) < \frac{3}{2} + \frac{2C_n}{\pi}\log\frac{R + r}{R - r} \tag{2.5}$$

(ii) *In R^3,*

$$\frac{1}{\omega_3 r^2} \int_{|x| = r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x) < \left(\frac{1}{2} + \frac{\pi}{4}\right)\frac{1}{r_\mu} - \frac{1}{R} + \frac{r^2 r_\mu^2}{8R^5} \tag{2.6}$$

In R^n, $n \geq 4$,

$$\frac{1}{\omega_n r^{n-1}} \int_{|x| = r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x) < \frac{1}{2}\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) + \frac{C_n}{r_\mu^{n-2}} \tag{2.7}$$

We first prove (i). We note that for $\pi/2 \leq \theta_1 < \pi$, $K(R, \rho, \theta_1)$ is a decreasing function of ρ. For $0 < \theta_1 < \pi/2$, the function

$$\frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho \cos \theta_1}$$

increases from 1 to cosec θ_1 as ρ increases from 0 to $R \cos \theta_1/(1 + \sin \theta_1)$, and then decreases again. Also

$$\frac{R^{n-2}}{(R^2 + \rho^2 - 2R\rho \cos \theta_1)^{(n-2)/2}}$$

increases from 1 to $\operatorname{cosec}^{n-2} \theta_1$ as ρ increases from 0 to $R \cos \theta_1$ and then decreases again. If θ_0 is the number in the range $0 < \theta_0 < \pi/2$, given by $R \cos \theta_0/(1 + \sin \theta_0) = r$, i.e.

$$\theta_0 = 2 \cot^{-1} \frac{R + r}{R - r},$$

then

$$k(R, r, \theta_1) = \sup_{0 \leq \rho \leq r} K(R, \rho, \theta_1) \leq \begin{cases} K(R, r, \theta_1) & \text{for } 0 < \theta_1 < \theta_0 \\ \operatorname{cosec}^{n-1} \theta & \text{for } \theta_0 \leq \theta_1 < \pi/2. \\ 1 & \text{for } \pi/2 \leq \theta_1 < \pi \end{cases}$$

In polar coordinates let

$$x_i = \rho \sin \theta_1 \cdots \sin \theta_{i-1} \cos \theta_i \qquad (i = 1, \ldots, n - 1)$$

$$x_n = \rho \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1}.$$

Then

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} k(R, r, \theta_1)\, d\sigma_r(x)$$

$$= \frac{1}{\omega_n} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-3} \theta_2 \cdots \sin \theta_{n-2}\, d\theta_2 \cdots d\theta_{n-1} \int_0^{\pi} k(R, r, \theta_1) \sin^{n-2} \theta_1\, d\theta_1$$

$$= \frac{\Gamma(n/2)}{2\pi^{n/2}} \cdot 2\pi \cdot \prod_{i=1}^{n-3} \int_0^{\pi} \sin^i \theta_{n-1-i}\, d\theta_{n-1-i} \int_0^{\pi} k(R, r, \theta_1) \sin^{n-2} \theta_1\, d\theta_1$$

$$\leq \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n - 1)/2)} \left[\int_0^{\theta_0} K(R, r, \theta_1) \sin^{n-2} \theta_1\, d\theta_1 + \int_{\theta_0}^{\pi/2} \frac{d\theta_1}{\sin \theta_1} + \int_{\pi/2}^{\pi} \sin^{n-2} \theta_1\, d\theta_1 \right].$$

In R^3,

$$\frac{1}{\omega_3 r^2} \int_{|x|=r} k(R, r, \theta_1)\, d\sigma_r(x) \leq \frac{1}{2}\left[1 + \frac{R - \sqrt{(R^2 + r^2)}}{r} + \log \frac{R + r}{R - r} + 1\right]$$

$$< 1 + \tfrac{1}{2} \log \frac{R + r}{R - r}.$$

In R^n, $n \geq 4$,

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} k(R, r, \theta_1)\, d\sigma_r(x)$$

$$< \frac{2C_n}{\pi}\left[\int_0^{\pi} K(R, r, \theta_1) \sin^{n-2}\theta_1\, d\theta_1 + \log\frac{R+r}{R-r} + \frac{\Gamma((n-1)/2)\Gamma(1/2)}{2\Gamma(n/2)}\right],$$

$$\leq \frac{3}{2} + \frac{2C_n}{\pi}\log\frac{R+r}{R-r}$$

since $K(R, r, \theta_1) > 0$ in $(0, \pi)$.

We now prove (ii). We note that for θ_1 in $(\pi/2, \pi)$, $G(R, \rho, r_\mu, \theta_1)$ decreases with increasing ρ so that $G(R, \rho, r_\mu, \theta_1)$ attains its maximum value at $\rho = 0$. For $0 < \theta_1 < \pi/2$; we consider the two terms of $G(R, \rho, r_\mu, \theta_1)$ separately. For $0 < \theta_1 < \pi/2$,

$$\sup_{0 \leq \rho \leq r} \frac{1}{(\rho^2 + r_\mu^2 - 2\rho r_\mu \cos\theta)^{(n-2)/2}} = \frac{\operatorname{cosec}^{n-2}\theta_1}{r_\mu^{n-2}}.$$

We note further that for $0 < \theta_1 < \pi/2$,

$$\frac{R^{n-2}}{(R^4 + \rho^2 r_\mu^2 - 2R^2\rho r_\mu \cos\theta_1)^{(n-2)/2}}$$

increases as ρ increases from 0 to $(R^2\cos\theta_1)/r_\mu$ and then decreases. Thus the minimum value in the interval is attained at $\rho = 0$ or $\rho = r$ and is

$$\frac{1}{R^{n-2}} \quad \text{or} \quad \frac{R^{n-2}}{(R^4 + r^2 r_\mu^2 - 2R^2 r r_\mu \cos\theta_1)^{(n-2)/2}}$$

respectively, the latter value being the minimum if $r > (2R^2\cos\theta_1)/r_\mu$.

Therefore using polar coordinates we have as in (i),

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x)$$

$$= \frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \int_0^{\pi} g(R, r, r_\mu, \theta_1) \sin^{n-2}\theta_1\, d\theta_1$$

$$= \frac{2C_n}{\pi}\left[\int_0^{\pi/2} \frac{1}{r_\mu^{n-2}}\, d\theta_1 + \int_{\pi/2}^{\pi} \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) \sin^{n-2}\theta_1\, d\theta_1 - I_1\right]$$

where

$$I_1 = \int_0^{\cos^{-1} r r_\mu/2R^2} \frac{\sin^{n-2}\theta_1}{R^{n-2}}\, d\theta_1$$

$$+ \int_{\cos^{-1} r r_\mu/2R^2}^{\pi/2} \frac{R^{n-2}\sin^{n-2}\theta_1\, d\theta_1}{(R^4 + r^2 r_\mu^2 - 2R^2 r r_\mu \cos\theta)^{(n-2)/2}}.$$

Since I_1 is rather tedious we evaluate it for $n = 3$. For $n > 3$, we take I_1 to be zero. For $n = 3$,

$$I_1 = \frac{1}{R} - \frac{rr_\mu}{2R^3} + \frac{1}{rr_\mu R}\left[(R^4 + r^2r_\mu^2)^{1/2} - R^2\right] > \frac{1}{R} - \frac{r^2r_\mu^2}{8R^5}.$$

In R^n, $n \geq 4$, since $I_1 > 0$, we have

$$\frac{1}{\omega_n r^{n-1}}\int_{|x|=r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x) < \frac{C_n}{r_\mu^{n-2}} + \frac{1}{2}\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right).$$

This completes the proof of Lemma 2.

3. Proof of Theorem 1

We saw in Lemma 2 (ii) that in R^3,

$$I = \frac{1}{\omega_3 r^2}\int_{|x|=r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x) < \frac{(1/2 + \pi/4)}{r_\mu} - \frac{1}{R} + \frac{r^2r_\mu^2}{8R^5}$$

and in R^n, $n \geq 4$,

$$I = \frac{1}{\omega_n r^{n-1}}\int_{|x|=r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(x) < \frac{1}{2}\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) + \frac{C_n}{r_\mu^{n-2}}.$$

This gives

$$I < \left(\frac{1}{r_\mu} - \frac{1}{R}\right)\left[\frac{1}{2} + \frac{\pi}{4} + \frac{Rr_\mu}{R - r_\mu}\left\{\frac{\pi/4 - 1/2}{R} + \frac{r^2r_\mu^2}{8R^5}\right\}\right]$$

in R^3 and

$$I < \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right)\left[\frac{1}{2} + C_n + \frac{C_n r_\mu^{n-2}}{R^{n-2} - r_\mu^{n-2}}\right]$$

in R^n for $n \geq 4$.

We next set $R_1 = (rR)^{1/2}$ and suppose first that $0 < r_\mu \leq R_1$. Then in R^3,

$$I < \left(\frac{1}{r_\mu} - \frac{1}{R}\right)\left[\frac{1}{2} + \frac{\pi}{4} + \frac{R\sqrt{rR}}{R - \sqrt{rR}}\left\{\frac{\pi/4 - 1/2}{R} + \frac{r^3}{8R^4}\right\}\right]$$

$$= \left(\frac{1}{r_\mu} - \frac{1}{R}\right)\left[\frac{1}{2} + \frac{\pi}{4} + f\left(\frac{r}{R}\right)\right],$$

where

$$f(t) = \frac{\sqrt{t}}{1 - \sqrt{t}}\left\{\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{t^3}{8}\right\}.$$

Since

$$f(t) < \frac{\sqrt{t}}{1 - \sqrt{t}}\left\{\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{1}{8}\right\} < \frac{\pi}{4}\cdot\frac{\sqrt{t}}{1 - \sqrt{t}} < \frac{4}{\pi}\cdot C_3\frac{\sqrt{t}}{1 - \sqrt{t}} = \psi_3(t),$$

we have in R^3,

$$I < \left(\frac{1}{r_\mu} - \frac{1}{R}\right)\left\{\frac{1}{2} + \frac{\pi}{4} + \psi_3\left(\frac{r}{R}\right)\right\} \tag{3.1}$$

Also in R^n for $n \geq 4$, we have

$$I < \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right)\left[\tfrac{1}{2} + C_n + \phi_n\left(\frac{r}{R}\right)\right],$$

where

$$\phi_n(t) = C_n \frac{t^{(n-2)/2}}{1 - t^{(n-2)/2}} < \frac{4C_n\sqrt{t}}{\pi(1 - \sqrt{t})} = \psi_n(t),$$

since $0 < t < 1$. And so

$$I < \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right)\left[\tfrac{1}{2} + C_n + \psi_n\left(\frac{r}{R}\right)\right]. \tag{3.2}$$

Next if $R_1 < r_\mu < R$, we note that $G(R, \rho, r_\mu, \theta_1)$ is a positive harmonic function of $(\rho, \theta_1, \ldots, \theta_{n-1})$ for $0 \leq \rho \leq R_1$ and the Riesz measure μ vanishes inside the R_1-hypersphere. Therefore from Lemma 1, we have

$$G(R, \rho, r_\mu, \theta_1) = \frac{1}{\omega_n R_1^{n-1}} \int_{|\zeta| = R_1} G(R, R_1, r_\mu, \theta_1) K(R_1, \rho, \theta_1)\, d\sigma_{R_1}(\zeta)$$

and

$$g(R, r, r_\mu, \theta_1) \leq \frac{1}{\omega_n R_1^{n-1}} \int_{|\zeta| = R_1} k(R_1, r, \theta_1) G(R, R_1, r_\mu, \theta_1)\, d\sigma_{R_1}(\zeta).$$

Therefore

$$I \leq \frac{1}{\omega_n r^{n-1}} \int_{|\xi| = r} \left\{\frac{1}{\omega_n R_1^{n-1}} \int_{|\zeta| = R_1} k(R_1, r, \theta_1) G(R, R_1, r_\mu, \theta_1)\, d\sigma_{R_1}(\zeta)\right\} d\sigma_r(\xi).$$

Inverting the order of integration, which is justified since the integrands are positive, we have

$$I \leq \frac{1}{\omega_n r^{n-1}} \int_{|\xi| = r} k(R_1, r, \theta_1)\, d\sigma_r(\xi) \left\{\frac{1}{\omega_n R_1^{n-1}} \int_{|\zeta| = R_1} G(R, R_1, r_\mu, \theta_1)\, d\sigma_{R_1}(\zeta)\right\}.$$

From the harmonicity of $G(R, \rho, r_\mu, \theta_1)$ it follows that the average on the surface of the R_1-hypersphere equals

$$G(R, 0, \xi) = \frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}.$$

Hence from (2.4) and (2.5), we have in R^3,

$$I < \left(1 + \tfrac{1}{2} \log \frac{R_1 + r}{R_1 + r}\right)\left(\frac{1}{r_\mu} - \frac{1}{R}\right) \tag{3.3}$$

and in R^n, $n \geq 4$,

$$I < \left(\frac{3}{2} + \frac{2C_n}{\pi} \log \frac{R_1 + r}{R_1 - r}\right)\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right). \tag{3.4}$$

We now show that (3.1) and (3.2) always hold. Since $\log x < \frac{1}{2}(x - 1/x)$, where $x > 1$, we have

$$\log \frac{1 + \sqrt{t}}{1 - \sqrt{t}} < \frac{1}{2}\left(\frac{1 + \sqrt{t}}{1 - \sqrt{t}} - \frac{1 - \sqrt{t}}{1 + \sqrt{t}}\right) = \frac{2\sqrt{t}}{1 - t}.$$

Since $0 < t < 1$, we have $2\sqrt{t}/(1 - t) < 2\sqrt{t}/(1 - \sqrt{t})$. Therefore

$$\log \frac{R_1 + r}{R_1 - r} = \log \frac{\sqrt{(rR)} + r}{\sqrt{(rR)} - r} = \log \frac{1 + \sqrt{t}}{1 - \sqrt{t}} < \frac{2\sqrt{t}}{1 - \sqrt{t}} < \frac{\pi}{2C_n} \psi_n\left(\frac{r}{R}\right).$$

Therefore in R^3,

$$I < \left(1 + \psi_3\left(\frac{r}{R}\right)\right)\left(\frac{1}{r_\mu} - \frac{1}{R}\right) < \left(\frac{1}{2} + \frac{\pi}{4} + \psi_3\left(\frac{r}{R}\right)\right)\left(\frac{1}{r_\mu} - \frac{1}{R}\right)$$

which is (3.1), and in R^n for $n \geq 4$,

$$\begin{aligned} I &< \left(\frac{3}{2} + \psi_n\left(\frac{r}{R}\right)\right)\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) \\ &\leq \left(\tfrac{1}{2} + C_n + \psi_n\left(\frac{r}{R}\right)\right)\left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right), \end{aligned}$$

as $C_n \geq 1$ for $n \geq 4$. We now complete the proof of Theorem 1.

From (2.3), if x is the origin, we have

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{|\xi| = R} u(\xi)\, d\sigma_R(\xi) - \int_{|\xi| < R} \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) d\mu(\xi).$$

With the notation of Section 1, this gives us

$$m(R) + \int_{|\xi| < R} \left(\frac{1}{r_\mu^{n-2}} - \frac{1}{R^{n-2}}\right) d\mu(\xi) = T(R) - u(0). \tag{3.5}$$

Also with that notation (2.3) can be written as

$$\begin{aligned} u^-(x) = {} & \frac{1}{\omega_n R^{n-1}} \int_{|\xi| = R} K(R, x, \xi) u^-(\xi)\, d\sigma_R(\xi) \\ & + \int_{|\xi| < R} G(R, x, \xi)\, d\mu(\xi) \\ & - \left\{\frac{1}{\omega_n R^{n-1}} \int_{|\xi| = R} K(R, x, \xi) u^+(\xi)\, d\sigma_R(\xi) - u^+(x)\right\}. \end{aligned} \tag{3.6}$$

Since $u^+(x)$ is s.h., the last term on the right hand side of (3.6) is positive, and

$$u^-(x) \le \frac{1}{\omega_n R^{n-1}} \int_{|\xi|=R} K(R, x, \xi) u^-(\xi)\, d\sigma_R(\xi) + \int_{|\xi|<R} G(R, x, \xi)\, d\mu(\xi).$$

Since $u_1(r) = \sup_{0 \le \rho \le r} u^-(\rho, \theta_1, \dots, \theta_{n-1})$ for fixed θ_i, $i = 1$ to $n - 1$, we have

$$\begin{aligned} u_1(r) \le \frac{1}{\omega_n R^{n-1}} \int_{|\xi|=R} & k(R, x, \xi) u^-(\xi)\, d\sigma_R(\xi) \\ & + \int_{|\xi|<R} g(R, x, \xi)\, d\mu(\xi) \end{aligned} \tag{3.7}$$

We now operate on both sides of (3.7) by

$$\frac{1}{\omega_n r^{n-1}} \int_{|\xi|=r} d\sigma_r(\xi),$$

and invert the order of integration which is justified since the integrands are positive.

$$\begin{aligned} \frac{1}{\omega_n r^{n-1}} \int_{|\xi|=r} & u_1(r)\, d\sigma_r(\xi) \\ & \le \frac{1}{\omega_n R^{n-1}} \int_{|\xi|=R} \left\{ \frac{1}{\omega_n r^{n-1}} \int_{|\xi|=r} k(R, x, \xi)\, d\sigma_r(\xi) \right\} u^-(\xi)\, d\sigma_R(\xi) \\ & \quad + \int_{|\xi|<R} \left\{ \frac{1}{\omega_n r^{n-1}} \int_{|\xi|<r} g(R, r, r_\mu, \theta_1)\, d\sigma_r(\xi) \right\} d\mu(\xi) \end{aligned} \tag{3.8}$$

In view of (2.4) and (3.1) we have in R^3,

$$\begin{aligned} \frac{1}{\omega_3 r^2} \int_{|\xi|=r} & u_1(r)\, d\sigma_r(\xi) \\ & \le \left(1 + \tfrac{1}{2} \log \frac{R+r}{R-r}\right) m(R) \\ & \quad + \left[\frac{1}{2} + \frac{\pi}{4} + \psi_3\left(\frac{r}{R}\right)\right] [T(R) - u(0) - m(R)] \\ & = \left\{\frac{1}{2} + \frac{\pi}{4} + \psi_3\left(\frac{r}{R}\right)\right\} \{T(R) - u(0)\} - \left(\frac{\pi}{4} - \frac{1}{2}\right) m(R) \\ & \quad - \left\{\psi_3\left(\frac{r}{R}\right) - \tfrac{1}{2} \log \frac{R+r}{R-r}\right\} m(R). \end{aligned} \tag{3.9}$$

As above we note that

$$\tfrac{1}{2} \log \frac{R+r}{R-r} = \tfrac{1}{2} \log \frac{1+t}{1-t} < \frac{t}{1-t^2} < \frac{\sqrt{t}}{1-\sqrt{t}} = \psi_3\left(\frac{r}{R}\right).$$

Therefore the second and the third term on the right hand side of (3.9) are positive and

$$\frac{1}{\omega_3 r^2}\int_{|\xi|=r} u_1(r)\,d\sigma_r(\xi) < \left\{\frac{1}{2}+\frac{\pi}{4}+\psi_3\left(\frac{r}{R}\right)\right\}\{T(R)-u(0)\}, \tag{3.10}$$

and in R^n for $n \geq 4$, we have, in view of (2.5), (3.2), and (3.5),

$$\begin{aligned}
\frac{1}{\omega_n r^{n-1}}&\int_{|\xi|=r} u_1(r)\,d\sigma_r(\xi)\\
&< \left(\frac{3}{2}+\frac{2C_n}{\pi}\log\frac{R+r}{R-r}\right)m(R)\\
&\quad + \left[\tfrac{1}{2}+C_n+\psi_n\left(\frac{r}{R}\right)\right][T(R)-u(0)-m(R)] \\
&= \left\{\tfrac{1}{2}+C_n+\psi_n\left(\frac{r}{R}\right)\right\}\{T(R)-u(0)\}-(C_n-1)m(R)\\
&\quad - \left\{\psi_n\left(\frac{r}{R}\right)-\frac{2C_n}{\pi}\log\frac{R+r}{R-r}\right\}m(R).
\end{aligned} \tag{3.11}$$

Again, since $C_n \geq 1$ for $n \geq 4$, and $\psi_n(r/R) = (4C_n/\pi)\psi_3(r/R)$, we see as before that the second and third terms of the right hand side of (3.11) are positive, and

$$\frac{1}{\omega_n r^{n-1}}\int_{|\xi|=r} u_1(r)\,d\sigma_r(\xi) < \left\{\tfrac{1}{2}+C_n+\psi_n\left(\frac{r}{R}\right)\right\}\{T(R)-u(0)\}. \tag{3.12}$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2

We note that if $u(x)$ is nonpositive in $|x| \leq R$, then $u^+(x) = 0$, and $-u_1(r) = \inf_{0\leq\rho\leq r} u(\rho, \theta_1, \ldots, \theta_{n-1})$ for fixed θ_i, $i = 1, \ldots, n-1$, and it follows from Theorem 1, that

$$\begin{aligned}
\frac{1}{\omega_n r^{n-1}}\int_{|x|=r} &\inf_{0\leq\rho\leq r} u(\rho,\theta_1,\ldots,\theta_{n-1})\,d\sigma_r(x)\\
&> \left\{\tfrac{1}{2}+C_n+\psi_n\left(\frac{r}{R}\right)\right\}u(0)
\end{aligned} \tag{4.1}$$

As $u(x)$ is nonpositive in R^n, we can let $R \to +\infty$ in (4.1) and note that $\psi_n(r/R) \to 0$ as $R \to +\infty$. Thus we have from (4.1), that

$$\frac{1}{\omega_n r^{n-1}}\int_{|x|=r} \inf_{0\leq\rho\leq r} u(\rho,\theta_1,\ldots,\theta_{n-1})\,d\sigma_r(x) \geq (\tfrac{1}{2}+C_n)u(0) \tag{4.2}$$

Now (4.2) holds for all r. If we have a sequence of r_n tending to infinity, we have $\{\inf_{0\leq\rho\leq r_n} u(\rho, \theta_1, \ldots, \theta_{n-1})\}$ as a decreasing sequence of negative measurable functions. Hence

$$\frac{1}{\omega_n r^{n-1}} \int_{|x|=r} \inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})\, d\sigma_r(x) \geq [\tfrac{1}{2} + C_n]u(0). \tag{4.3}$$

Since (4.3) holds for all r, we take $r = 1$. Therefore

$$\frac{1}{\omega_n} \int_{|x|=1} \inf_{0\leq\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})\, d\sigma_1(x) \geq [\tfrac{1}{2} + C_n]u(0).$$

This completes the proof of Theorem 2. We show by a simple example that the constant $1/2 + C_n$ is the best possible.

Example. $u(x_1, \ldots, x_n) = -[(x_1 - 1)^2 + x_2^2 + \cdots + x_n^2]^{-(n-2)/2}$ for $(x_1, \ldots, x_n) \neq (1, 0, \ldots, 0)$, $u(1, 0, \ldots, 0) = -\infty$. Thus $u(x_1, \ldots, x_n) < 0$ in R^n.

In polar coordinates

$$u(\rho, \theta_1, \ldots, \theta_{n-1}) = \frac{-1}{(\rho^2 - 2\rho\cos\theta_1 + 1)^{(n-2)/2}}.$$

If $\pi/2 \leq \theta_1 < \pi$, evidently $\inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1}) = -1$ for fixed θ_i, $i = 1, \ldots, n-1$. If $0 < \theta_1 < \pi/2$,

$$\inf_{\cos\theta_1<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1}) = -\operatorname{cosec}^{n-2}\theta_1,$$

and $u(\rho, \theta_1, \ldots, \theta_{n-1})$ is a decreasing function of ρ for $0 \leq \rho \leq \cos\theta_1$ and attains its minimum, $-\operatorname{cosec}^{n-2}\theta_1$, when $\rho = \cos\theta_1$.

Since we are concerned with large values of ρ, we consider $\rho \geq \cos\theta_1$. Then in polar coordinates (as in Lemma 2 (i))

$$\begin{aligned}
&\frac{1}{\omega_n} \int_{|x|=1} \inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})\, d\sigma_1(x) \\
&\quad = \frac{2C_n}{\pi} \int_0^\pi \sin^{n-2}\theta_1 \inf u\, d\theta_1 \\
&\quad = \frac{2C_n}{\pi}\left[\int_0^{\pi/2} - \sin^{n-2}\theta_1 \operatorname{cosec}^{n-2}\theta_1\, d\theta_1 - \int_{\pi/2}^{\pi} \sin^{n-2}\theta_1\, d\theta_1\right] \\
&\quad = \frac{2C_n}{\pi}\left[-\frac{\pi}{2} - \frac{\Gamma((n-1)/2)\Gamma(1/2)}{2\Gamma(n/2)}\right] = -[C_n + \tfrac{1}{2}].
\end{aligned}$$

Since $u(0, \ldots, 0) = -1$, we have

$$\frac{1}{\omega_n} \int_{|x|=1} \inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})\, d\sigma_1(x) = [\tfrac{1}{2} + C_n]u(0, \ldots, 0).$$

This shows that the inequality (1.3) is sharp.

As a corollary, we see that if u is bounded above in R^n by M, then $u - M$ is nonpositive in R^n. By (1.3),

$$\inf_{0<\rho<\infty} u(\rho, \theta_1, \ldots, \theta_{n-1})$$

must be finite on almost all straight lines. Hence $u(\rho, \theta_1, \ldots, \theta_n)$ is bounded below on almost all straight lines.

The results of this paper for R^3 were proved in the author's thesis, completed under the supervision of Professor W. K. Hayman. The author would like to thank Professor Hayman for his help and encouragement. The other part of this work was done when author was a visiting scholar at the University of Illinois. The author would like to thank the Council for International Exchange of Scholars for a grant and the University of Illinois for the scholarly privileges.

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