AN HARMONIC ANALYSIS FOR OPERATORS I: FORMAL PROPERTIES

BY

KAREL DELEEUW[1]

1. Introduction

Harmonic analysis deals with objects defined on a group or associated with a group and attempts to represent these objects in terms of objects which behave simply with respect to group translation. For example, in the harmonic analysis of functions on compact commutative groups, the functions which behave simplest with respect to translation are the characters, for these, and their scalar multipliers, are precisely the eigenfunctions of the translation operators.

In this paper we initiate a study of a harmonic analysis for operators on homogeneous Banach spaces on the circle group $\mathbf{T}$. In this case the simple operators will be those which commute with translation (we shall call these *invariant operators*) and also those operators which are the composite of an invariant operator and multiplication by a character of $\mathbf{T}$. These simple operators are precisely the operators T which satisfy, for some integer n a functional equation of the form

$$TR_t = e^{int}R_tT, \quad t \in \mathbf{T},$$

where R_t is the translation operator on $\mathbf{T}$ defined by

$$(R_tf)(s) = f(s - t), \quad s \in \mathbf{T}.$$

The operators we call *invariant* are those which are usually called *multipliers* (see [5]) because they are obtained by multiplication on the Fourier transform. In order to avoid confusion, we shall not use this terminology since we shall also be dealing with operations of multiplication by a function on $\mathbf{T}$.

Although we state and prove our results for the circle group $\mathbf{T}$, analogues of all of the results of Sections 2 through 5 are valid for any compact abelian group.

Some of the results we prove here were announced in [2].

2. Homogeneous spaces: invariant, almost invariant, and simple operators

We shall deal with operators acting on a space of functions on the circle group $\mathbf{T}$. The spaces we shall consider will be general enough to include all of the classical Banach function spaces on $\mathbf{T}$.

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We shall assume henceforth that B is a dense linear subspace of $L^1(\mathbf{T})$ which is translation invariant; i.e., if $f \in B$, then its set $\{R_t f: t \in \mathbf{T}\}$ of translates is contained in B. We shall assume that B is a Banach space under a norm $\|\cdot\|_B$ satisfying $\|f\|_{L^1(\mathbf{T})} \leq \|f\|_B$, $f \in B$, that the norm is *translation invariant*, $\|R_t f\|_B = \|f\|_B, f \in B$, $t \in \mathbf{T}$, and that the functions in B translate continuously,

$$\lim_{t \to 0} \|R_t f - f\|_B = 0. \tag{2.1}$$

So far this is precisely the definition of a homogeneous space of functions on $\mathbf{T}$ in the sense of [4]. We assume in addition one more axiom which is valid in all of the classical function spaces on $\mathbf{T}$, namely that B is closed under multiplication by the characters of $\mathbf{T}$, that is, if $f \in B$ and n is an integer, then the function $M_n f$ defined by

$$(M_n f)(t) = e^{int} f(t), \quad t \in \mathbf{T},$$

is also in B. (By the Closed Graph Theorem, the operator M_n is continuous. We do not assume that M_n is an isometry.)

We denote by $\mathscr{L}$ the Banach algebra of bounded linear operators on B, supplied with the norm $\|\cdot\|_{\mathscr{L}}$. The *invariant operators* in $\mathscr{L}$ are those which commute with translation; i.e., T is an invariant operator if and only if $TR_t = R_t T$, $t \in \mathbf{T}$. We denote the set of invariant operators in $\mathscr{L}$ by $\mathscr{L}_0$. It is easy to check that $\mathscr{L}_0$ is a closed subalgebra of $\mathscr{L}$.

Translation by elements of $\mathbf{T}$ gives rise a group of isometries on $\mathscr{L}$ as follows. For $t \in \mathbf{T}$, define ϕ_t by $\phi_t(T) = R_{-t} T R_t$, $t \in \mathbf{T}$. Then $t \rightsquigarrow \phi_t$ is a representation of the group $\mathbf{T}$ on $\mathscr{L}$ and the invariant operators in $\mathscr{L}$ are precisely the fixed points of this representation.

We call an operator T in $\mathscr{L}$ *almost invariant* if

$$\lim_{t \to 0} \|TR_t - R_t T\|_{\mathscr{L}} = 0. \tag{2.2}$$

We denote the set of almost invariant operators in $\mathscr{L}$ by $\mathscr{L}_\#$. It is simple to check that $\mathscr{L}_\#$ is a closed subalgebra of $\mathscr{L}$.

The following example may help to clarify the definition of almost invariance.

LEMMA 2.1. *Let B be $C(\mathbf{T})$ or one of the $L^p(\mathbf{T})$. Let $\phi \in L^\infty(\mathbf{T})$. Define the operator T on B by $Tf = \phi \cdot f, f \in B$. Then the following are equivalent:*

(a) *T is almost invariant;*

(b) *ϕ is equivalent in $L^\infty(\mathbf{T})$ to a continuous function.*

Proof. For $t \in \mathbf{T}$, let T_t be the operator on B defined by

$$T_t f = (R_t \phi) \cdot f, \quad f \in B.$$

A simple computation shows that $R_t T = T_t R_t$. Thus

$$\|TR_t - R_t T\|_{\mathscr{L}} = \|TR_t - T_t R_t\|_{\mathscr{L}} = \|(T - T_t)R_t\|_{\mathscr{L}} = \|T - T_t\|_{\mathscr{L}}$$

since R_t is an isometry. But $\|T - T_t\|_{\mathscr{L}} = \|\phi - R_t\phi\|_\infty$, where $\|\cdot\|_\infty$ is the L^∞ norm, since for a multiplication operator in $\mathscr{L}$, the operator norm is the same as the $L^\infty(\mathbf{T})$ norm. Thus

$$\lim_{t\to 0} \|TR_t - R_tT\| = 0$$

if and only if

$$\lim_{t\to 0} \|\phi - R_t\phi\|_\infty. \tag{2.3}$$

But (2.3) is true if and only if ϕ is equivalent in $L^\infty(\mathbf{T})$ to a continuous function.

The following characterization of almost invariance will prove to be useful in the future. Because of this characterization we shall say that the almost invariant operators are those which *translate continuously.*

LEMMA 2.2. *Let $T \in \mathscr{L}$. Then the following are equivalent:*

(a) $\lim_{t\to 0} \|TR_t - R_tT\|_{\mathscr{L}} = 0$;

(b) $t \rightsquigarrow R_{-t}TR_t$ *is continuous from* $\mathbf{T}$ *to the norm topology of* $\mathscr{L}$.

Proof. Let $s, t \in \mathbf{T}$. Then

$$R_{-s}TR_s - R_{-t}TR_t = R_{-s}(TR_{s-t} - R_{s-t}T)R_t.$$

Thus, since R_{-s} and R_t are isometries,

$$\|R_{-s}TR_s - R_{-t}TR_t\|_{\mathscr{L}} = \|TR_{s-t} - R_{s-t}T\|_{\mathscr{L}}. \tag{2.4}$$

(2.4) shows that (a) and (b) are equivalent.

Using arguments such as those of Theorem 10.2.1 of [3], it is not hard to see that almost invariance is equivalent to the apparently weaker condition:

$$\{R_{-t}TR_t\colon t \in \mathbf{T}\} \text{ is separable in the norm topology of } \mathscr{L}.$$

Each invariant operator on B is clearly almost invariant. We shall see that the almost invariant operators are those in the closed linear subspace of $\mathscr{L}$ spanned by the operators we introduce next, the *simple operators.*

For n an integer, we denote by $\mathscr{L}_n$ the subset of $\mathscr{L}$ consisting of all operators T satisfying the functional equation

$$TR_t = e^{int}R_tT, \quad t \in \mathbf{T}. \tag{2.5}$$

We call an operator in $\mathscr{L}$ *simple* if it is in one of the $\mathscr{L}_n$. An easy computation shows that the multiplication operator M_n defined by

$$(M_nf)(t) = e^{int}f(t), \quad t \in \mathbf{T}, \tag{2.6}$$

satisfies the functional equation (2.5) and thus is in $\mathscr{L}_n$.

The following summarizes the basic properties of the $\mathscr{L}_n$.

LEMMA 2.3. *Each $\mathscr{L}_n$ is a closed linear subspace of $\mathscr{L}$. Each $\mathscr{L}_n$ is contained in $\mathscr{L}_\#$. If $n \neq m$, then $\mathscr{L}_n \cap \mathscr{L}_m = \{0\}$. If $U \in \mathscr{L}_n$ and $V \in \mathscr{L}_m$, then, $UV \in \mathscr{L}_{n+m}$.*

Proof. These facts all follow immediately from the functional equation (2.5). For example, if $U \in \mathscr{L}_n$ and $V \in \mathscr{L}_m$, then

$$R_{-t}UVR_t = R_{-t}UR_tR_{-t}VR_t = (e^{int}U)\cdot(e^{int}V) = e^{i(n+m)t}UV.$$

Lemma 2.3 yields the following corollary, which characterizes the simple operators in terms of invariant operators and the multiplication operators M_n defined by (2.6).

COROLLARY 2.4. *Let $T \in \mathscr{L}$. Then the following are equivalent:*
(a) $T \in \mathscr{L}_n$;
(b) *there is* $U \in \mathscr{L}_0$ *so that* $T = UM_n$;
(c) *there is* $V \in \mathscr{L}_0$ *so that* $T = M_nV$.

Proof. A simple computation shows that (b) and (c) both imply (a). Conversely, suppose that $T \in \mathscr{L}_n$. Then, $M_{-n} \in \mathscr{L}_0$, by Lemma 2.3. Thus, $T = M_n(M_{-n}T)$, which shows that (a) implies (c). A similar proof shows that (a) implies (b).

3. The Fourier transform

In this section we introduce a Fourier series for operators in $\mathscr{L}$, the terms of the series being simple operators. If T is an almost invariant operator, we show that its Fourier series is C-1 summable to T in the operator norm. As a consequence we obtain the fact that the space $\mathscr{L}_\#$ of almost invariant operators is the norm closed subalgebra of $\mathscr{L}$ generated by the invariant operators in $\mathscr{L}_0$ and the multiplication operators M_n defined by (2.6). If T is an arbitrary operator in $\mathscr{L}$ we are able to show that its Fourier series is C-1 summable to T in the strong operator topology of $\mathscr{L}$.

For each operator T in $\mathscr{L}$, and each integer n, we define the operator $\pi_n(T)$ by the vector valued integral

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_t\,dt. \tag{3.1}$$

A word of interpretation of (3.1) is necessary. If T is almost invariant, the integrand of (3.1) is continuous from $\mathbf{T}$ into the norm topology of $\mathscr{L}$ because of Lemma 2.2, so the integral makes sense. (See the appendix of [4]) for the basic properties of vector valued integrals.)

If T is a general operator it is necessary to define $\pi_n(T)$ indirectly. For this we need the following:

LEMMA 3.1. *Let $T \in \mathscr{L}$, $f \in B$. Then the mapping $s \rightsquigarrow R_{-s}TR_sf$ is continuous from $\mathbf{T}$ to the norm topology of B.*

Proof. We shall prove that this map is continuous at a point t of $\mathbf{T}$. Let $s \in \mathbf{T}$. Then

$$\begin{aligned} R_{-s}TR_sf - R_{-t}TR_tf &= [R_{-s}TR_sf - R_{-s}TR_tf] + [R_{-s}TR_tf - R_{-t}TR_tf] \\ &= [R_{-s}TR_s](f - R_{t-s}f) + R_{-s}[(TR_tf) - R_{s-t}(TR_tf)]. \end{aligned}$$

As a consequence,

$$\|R_{-s}TR_s f - R_{-t}TR_t f\|_B \le \|T\|_{\mathscr{L}}\|f - R_{t-s}f\|_B + \|g - R_{s-t}g\|_B, \tag{3.2}$$

where $g = TR_t f$. Since the functions in B translate continuously (as defined by (2.1)), we have

$$\lim_{s\to t} \|f - R_{t-s}f\|_B = 0 \tag{3.3}$$

and

$$\lim_{s\to t} \|g - R_{s-t}g\|_B = 0. \tag{3.4}$$

The lemma now follows from (3.2), (3.3), and (3.4).

We now return to our definition of $\pi_n(T)$. Choose any $T \in \mathscr{L}$ and any $f \in B$. Because of Lemma 3.1, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_t f\,dt \tag{3.5}$$

makes sense. We define $[\pi_n(T)](f)$ to be the value of the integral (3.5). (This is clearly consistent with our definition of $\pi_n(T)$ in the case that T is almost invariant.) It is immediate from elementary properties of the integral that $\pi_n(T)$ is a bounded linear operator in B and that

$$\|\pi_n(T)\|_{\mathscr{L}} \le \|T\|_{\mathscr{L}}. \tag{3.6}$$

The following summarizes the basic properties of the map π_n.

PROPOSITION 3.2. *π_n is a projection of $\mathscr{L}$ onto $\mathscr{L}_n$.*

Proof. It is clear from the linearity properties of the integral that π_n is linear. First we show that π_n takes $\mathscr{L}$ into $\mathscr{L}_n$. Choose any $s \in \mathbf{T}$. Then

$$\begin{aligned} R_{-s}(\pi_n(T))R_s f &= R_{-s}\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_tR_s f\,dt\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-s}R_{-t}TR_tR_s f\,dt \\ &= \frac{1}{2\pi}e^{ins}\int_{-\pi}^{+\pi} e^{-in(s+t)}R_{-(s+t)}TR_{s+t}f\,dt \\ &= e^{ins}\pi_n(T)f, \end{aligned}$$

using the translation invariance of Lebesgue measure. Thus

$$R_{-s}(\pi_n(T))R_s = e^{ins}\pi_n(T),$$

and since s was arbitrary, $\pi_n(T) \in \mathscr{L}_n$.

To complete the proof of Proposition 3.2 it remains to show that $\pi_n(T) = T$ if $T \in \mathscr{L}_n$. Choose any $T \in \mathscr{L}_n, f \in B$. Then

$$\begin{aligned}\pi_n(T)f &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_t f\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}(e^{int}T)f\,dt \\ &= Tf.\end{aligned}$$

Thus $\pi_n(T) = T$, which completes the proof of the proposition.

We will associate with each $T \in \mathscr{L}$ the formal series

$$\sum_{-\infty}^{+\infty} \pi_n(T). \tag{3.7}$$

The series (3.7) will be called the *Fourier series* of the operator T.

That our definition yields an extension of the usual notion of Fourier series is seen by the following:

PROPOSITION 3.3. *Suppose that $B = C(\mathbf{T})$ and that ϕ is a function in B. Let M_ϕ be the multiplication operator defined by $M_\phi(f) = \phi \cdot f$, $f \in B$. Then the Fourier series of the operator M_ϕ is $\sum_{-\infty}^{+\infty} \hat{\phi}(n)M_n$, where*

$$\hat{\phi}(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}\phi(t)\,dt.$$

Proof. Let $s \in \mathbf{T}$. Then

$$\begin{aligned}[[\pi_n(M_\phi)]f](s) &= \left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}M_\phi R_t f\,dt\right](s) \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}[R_{-t}M_\phi R_t f](s)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}\phi(s + t)f(s)\,dt \\ &= f(s)\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{ins}e^{-in(s+t)}\phi(s + t)\,dt \\ &= f(s)e^{ins}\hat{\phi}(n) = \hat{\phi}(n)(M_n f)(s).\end{aligned}$$

We show next that the Fourier series of an operator T is C-1 summable to T in the operator norm if T is almost invariant and is C-1 summable to T in the strong operator topology for general T.

PROPOSITION 3.4. *Let T be an almost invariant operator. Then its Fourier series $\sum_{-\infty}^{+\infty} \pi_n(T)$ is C-1 summable to T in the operator norm.*

Proof. If $\Phi_t: \mathscr{L}_\# \to \mathscr{L}_\#$ is defined by $\Phi_t(T) = R_{-t}TR_t$, then $t \rightsquigarrow \Phi_t$ is a representation of the group $\mathbf{T}$ on the Banach space $\mathscr{L}_\#$. Every element of $\mathscr{L}_\#$ *translates continuously*; i.e.,

$$\lim_{t\to 0} \|\Phi_t(T) - T\|_{\mathscr{L}} = 0$$

Such a situation is covered by the discussion of [1]. Theorem 1.1 of that paper yields Proposition 3.4 as a special case.

For completeness we include another proof of Proposition 3.4. The Nth C-1 sum of the Fourier series of T is

$$\begin{aligned} \sum_{-N}^{+N}\left(1 - \frac{|n|}{N+1}\right)\pi_n(T) &= \sum_{-N}^{+N}\left(1 - \frac{|n|}{N+1}\right)\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_t\,dt\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[\sum_{-N}^{+N}\left(1 - \frac{|n|}{N+1}\right)e^{-int}\right]R_{-t}TR_t\,dt \qquad (3.8) \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} K_N(t)R_{-t}TR_t\,dt, \end{aligned}$$

where K_N is the Nth Fejer kernel. That the last term in (3.5) converges in the norm $\|\cdot\|_{\mathscr{L}}$ to T now follows by standard arguments (see for example p. 10 of [4]) since $\{K_N\}$ is an approximate identity and

$$\lim_{t\to 0} \|R_{-t}TR_t - T\|_{\mathscr{L}} = 0.$$

COROLLARY 3.5. *$\mathscr{L}_\#$, the space of almost invariant operators in $\mathscr{L}$, is the norm closed subalgebra of $\mathscr{L}$ generated by the invariant operators and the multiplication operators M_n.*

Proof. This is immediate from Proposition 3.2, Proposition 3.4, and Corollary 2.4.

We also have an analogue of the Riemann-Lebesgue lemma.

COROLLARY 3.6. *Let T be an almost invariant operator. Then*

$$\lim_{|n|\to\infty} \|\pi_n(T)\|_{\mathscr{L}} = 0. \qquad (3.9)$$

Proof. (3.9) is clearly true for T a finite sum of simple operators. By Proposition 3.4, such operators are norm dense in $\mathscr{L}_\#$. (3.6) then shows that (3.9) holds for every T in $\mathscr{L}_\#$.

As a final consequence of Proposition 3.4 we obtain the fact that an almost invariant operator is compact if and only if each term of its Fourier series is compact. This will be applied later in Section 6 to yield a generalization of the F. and M. Riesz Theorem.

COROLLARY 3.7. *Let T be an almost invariant operator. Then the following are equivalent:*

(a) *T is a compact operator;*

(b) *$\pi_n(T)$ is a compact operator for each n.*

Proof. (a) implies (b). If T is compact, each $R_{-t}TR_t$ is compact. $\pi_n(T)$, since it is defined by (3.1) will be in the norm closed linear subspace of $\mathscr{L}$ spanned by the $R_{-t}TR_t$ and consequently will itself be compact.

(b) implies (a). This is immediate from Proposition 3.4, since the limit in the norm topology of compact operators must be compact.

Finally we point out that the Fourier series of a general T in $\mathscr{L}$ is C-1 summable to T in the strong operator topology of $\mathscr{L}$. Equivalently,

PROPOSITION 3.8. *Let $T \in \mathscr{L}$, $f \in B$. Then the series $\sum_{-\infty}^{+\infty} \pi_n(T)f$ is C-1 summable to Tf in the norm of B.*

Proof. Similar to that of Proposition 3.4.

COROLLARY 3.9. *Let $T \in \mathscr{L}, f \in B$. Then* $\lim_{|n| \to \infty} \|[\pi_n(T)]f\|_B = 0$.

Proof. This follows from Proposition 3.8 in the same way that Corollary 3.6 followed from Proposition 3.4.

4. The Fourier transform takes multiplication into convolution

In the next two sections we establish some of the formal properties of the Fourier series defined in the preceding section.

First we introduce some notation. For $T \in \mathscr{L}$, n an integer, we define $\hat{T}(n)$ to be the operator $\pi_n(T)$. We call $\hat{T}$ the *Fourier transform* of the operator T. Note that $\hat{T}$ is an $\mathscr{L}$-valued function defined on the integers and that $\hat{T}(n) \in \mathscr{L}_n$ for each n. $\hat{T}$ is a bounded function because of (3.6),

$$\|\hat{T}(n)\|_{\mathscr{L}} \leq \|T\|_{\mathscr{L}}, \quad \text{all } n.$$

Finally, if T is an almost invariant operator, $\lim_{|n| \to \infty} \|\hat{T}(n)\|_{\mathscr{L}} = 0$, because of Corollary 3.6.

We show in the next proposition that the Fourier transform takes operator multiplication into convolution. That this result includes the fact that the Fourier transform of the product of two functions is the convolution of their transforms follows from Proposition 3.3.

PROPOSITION 4.1. *Let S and T be almost invariant operators. Then the series $\sum_{m=-\infty}^{m=+\infty} \hat{S}(n-m)\hat{T}(m)$ is C-1 summable to the operator $\widehat{ST}(n)$ in the operator norm.*

Proof. We have

$$(4.1)\qquad \begin{aligned}(ST)^\wedge(n) = \pi_n(ST) &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}STR_t\,dt \\ &= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}S\left[\sum_{-N}^{+N}\left(1-\frac{|m|}{N+1}\right)\pi_m(T)\right]R_t\,dt\end{aligned}$$

because of Proposition 3.2, the limit referring to the operator norm. (4.1) is equal to

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}\left[\sum_{-N}^{+N}\left(1-\frac{|m|}{N+1}\right)R_{-t}SR_tR_{-t}\pi_m(T)R_t\right]dt$$

which is the same as

$$(4.2)\qquad \lim_{N\to\infty}\sum_{-N}^{+N}\left(1-\frac{|m|}{N+1}\right)\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}SR_te^{imt}\pi_n(T)\,dt$$

because each $\pi_m(T)$ is in $\mathscr{L}_m$ and thus satisfies $R_{-t}\pi_m(T)R_t = e^{imt}\pi_m(T)$. Finally, (4.2) is equal to

$$\lim_{N\to\infty}\sum_{-N}^{+N}\left(1-\frac{|m|}{N+1}\right)\pi_{n-m}(S)\pi_m(T),$$

because of the definition of $\pi_{n-m}(S)$, which completes the proof.

For general operators, we have the analogue of Proposition 4.1 in the strong operators topology.

PROPOSITION 4.2. *Let S and T be in $\mathscr{L}$, $f \in B$. Then the series*

$$\sum_{m=-\infty}^{m=+\infty} \hat{S}(n-m)\hat{T}(m)f$$

is C-1 summable to $[(ST)^\wedge(n)]f$ in the norm topology of B.

Proof. Similar to the proof of Proposition 4.1.

5. The Fourier transform takes convolution into multiplication

In this section we define the *convolution* $\mu * T$ of a finite Borel measure μ on **T** and an operator T in $\mathscr{L}$ and show that $(\mu * T)^\wedge = \hat{\mu}\cdot\hat{T}$. In the case that T is the operation of multiplication by a function ϕ, $\mu * T$ will be the operation of multiplication by the function $\mu * \phi$, and thus our result includes the fact that the Fourier transform of the convolution of a measure and a function is the product of their transforms.

Let T be an operator in $\mathscr{L}$, μ a finite Borel measure on **T**. We define the operator $\mu * T$ as follows. For each $f \in B$, $(\mu * T)(f)$ is defined by the vector valued integral

$$\int_{-\pi}^{+\pi} R_t T R_{-t} f \, d\mu(t). \tag{5.1}$$

By Lemma 3.1, (5.1) is a well defined element of B. Using linearity properties of the integral it is easy to see that $\mu * T$ is an operator in $\mathscr{L}$. If T is almost invariant, $\mu * T$ can be defined directly by the integral

$$\int_{-\pi}^{+\pi} R_t T R_{-t} \, d\mu(t).$$

$\mu * T$ will then also be almost invariant since each $R_t T R_{-t}$ is in $\mathscr{L}_\#$ and $\mathscr{L}_\#$ is a norm closed linear subspace of $\mathscr{L}$.

The following lemma gives the justification of our definition of convolution.

LEMMA 5.1. *Let $B = C(\mathbf{T})$, $\phi \in B$, T the operator of multiplication by ϕ. Then $\mu * T$ is the operator of multiplication by $\mu * \phi$, where $\mu * \phi$ is the function defined by*

$$(\mu * \phi)(s) = \int_{-\pi}^{+\pi} \phi(s - t) \, d\mu(t).$$

Proof. Let $f \in B$, $s \in \mathbf{T}$. Then

$$\begin{aligned} [(\mu * T)f](s) &= \left[\int_{-\mu}^{+\mu} R_t T R_{-t} f \, d\mu(t)\right](s) \\ &= \int_{-\pi}^{+\pi} [R_t T R_{-t} f](s) \, d\mu(t) \\ &= \int_{-\pi}^{+\pi} (T R_{-t} f)(s - t) \, d\mu(t) \\ &= \int_{-\pi}^{+\pi} \phi(s - t)(R_{-t} f)(s - t) \, d\mu(t) \\ &= \int_{-\pi}^{+\pi} \phi(s - t) f(s) \, d\mu(t) \\ &= f(s) \int_{-\pi}^{+\pi} \phi(s - t) \, d\mu(t) \\ &= (\mu * \phi)(s) \cdot f(s). \end{aligned}$$

Thus $(\mu * T)(f) = (\mu * \phi) \cdot f$.

Our next result shows that the Fourier transform we have defined takes convolution into multiplication.

PROPOSITION 5.2. *Let $T \in \mathscr{L}$, μ a finite Borel measure on* $\mathbf{T}$. *Then*

$$(\mu * T)^\wedge(n) = \hat{\mu}(n)\hat{T}(n), \quad \text{all } n,$$

where $\hat{\mu}$ is the function defined by $\hat{\mu}(n) = \int_{-\pi}^{+\pi} e^{-int}\, d\mu(t)$.

Proof. Let $f \in B$. Then

$$\begin{aligned}
[(\mu * T)^\wedge(n)]f &= [\pi_n(\mu * T)]f \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}(\mu * T)R_t f\, dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}\left[\int_{-\pi}^{+\pi} R_s T R_{-s} R_t f\, d\mu(s)\right] dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}\left[\int_{-\pi}^{+\pi} R_{s-t} T R_{-(s-t)} f\, d\mu(s)\right] dt \\
&= \int_{-\pi}^{+\pi} e^{-ins}\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-in(-s+t)} R_{s-t} T R_{-(s-t)} f\, dt\right] d\mu(s) \\
&= \int_{-\pi}^{+\pi} e^{-ins}[\pi_n(T)f]\, d\mu(s) = \hat{\mu}(n)\hat{T}(n)f.
\end{aligned}$$

This proves that $(\mu * T)^\wedge = \hat{\mu} \cdot \hat{T}$.

The foregoing shows that Fourier Stieltjes transforms are universal "scalar multipliers" for Fourier transforms of operators on B. To be precise, if $T \in \mathscr{L}$ and μ is a finite Borel measure, then there is an operator S in $\mathscr{L}$ so that $\hat{S}(n) = \hat{\mu}(n)\hat{T}(n)$, all n. (And in fact $S = \mu * T$.)

6. A generalization of the F. and M. Riesz Theorem

In this section we use the harmonic analysis introduced earlier to obtain a generalization of the F. and M. Riesz Theorem. The crucial ingredients are Corollary 3.7, which shows that an almost invariant operator is compact whenever all of the terms of its Fourier series are compact, the fact that the compact invariant operators on $C(\mathbf{T})$ are given by convolution by $L^1(\mathbf{T})$ functions and the classical F. and M. Riesz Theorem.

We shall restrict our attention to the space $C(\mathbf{T})$ of continuous complex valued functions on $\mathbf{T}$. We define $C(\mathbf{T})_+$ and $C(\mathbf{T})_-$ by

$$C(\mathbf{T})_+ = \{f : f \in C(\mathbf{T}), \hat{f}(n) = 0 \text{ if } n < 0\};$$

$$C(\mathbf{T})_- = \{f : f \in C(\mathbf{T}), \hat{f}(n) = 0 \text{ if } n > 0\}.$$

Here is our generalization of the F. and M. Riesz Theorem.

THEOREM 6.1. *Let T be an almost invariant operator on $C(\mathbf{T})$. Assume that*

$$T(C(\mathbf{T})_+) \subseteq C(\mathbf{T})_-. \tag{6.1}$$

Then T must be a compact operator.

We will present the proof of Theorem 6.1 shortly.

First, observe that an arbitrary operator T on $C(\mathbf{T})$ satisfying (6.1) need not be compact. For example, take the operator T defined by

$$(Tf)(t) = f(-t), \quad t \in \mathbf{T}.$$

We next show how Theorem 6.1 contains the classical F. and M. Riesz Theorem. For a finite Borel measure μ on π, define the convolution operator C_μ on $C(\mathbf{T})$ by

$$(C_\mu f)(t) = (\mu * f)(t) = \int_{-\pi}^{+\pi} f(t - s)\, d\mu(s).$$

It is easy to check that $C_\mu(C(\mathbf{T})_+) \subseteq C(\mathbf{T})_-$ if and only if $\hat{\mu}(n) = 0$, $n = 1, 2, 3, 4, \ldots$ and thus the classical F. and M. Riesz Theorem is a special case of Theorem 6.1 because of the following.

LEMMA 6.2. *Let μ be a finite Borel measure on $\mathbf{T}$. Then the following are equivalent:*

(a) *the convolution operator C_μ is compact;*

(b) *μ is absolutely continuous with respect to Lebesgue measure.*

Proof. Denote by $M(\mathbf{T})$ the Banach space of finite Borel measures on $\mathbf{T}$ supplied with the total variation norm $\|\cdot\|_{M(\mathbf{T})}$. For any $\mu \in M(\mathbf{T})$, $s \in \mathbf{T}$, the translated measure μ_s is defined by

$$\int_{-\pi}^{+\pi} g(x)\, d\mu_s(x) = \int_{-\pi}^{+\pi} g(x + s)\, d\mu(x), \quad g \in C(\mathbf{T}).$$

It is well known that μ is absolutely continuous with respect to Lebesgue measure if and only if

$$\lim_{t \to s} \|\mu_t - \mu_s\|_{M(\mathbf{T})} = 0, \quad \text{all } s \in \mathbf{T}. \tag{6.2}$$

For each function f in $C(\mathbf{T})$, we define the function $\tilde{f}$ by $\tilde{f}(t) = f(-t)$, $t \in \mathbf{T}$. We prove first that (b) implies (a). Let μ be absolutely continuous with respect to Lebesgue measure. We shall show that $C\mu$ takes the unit ball of $C(\mathbf{T})$ into a bounded, equicontinuous subset of $C(\mathbf{T})$ and thus (a) follows from Ascoli's Theorem. Boundedness is a consequence of

$$\|C_\mu f\|_{C(\mathbf{T})} = \|\mu * f\|_{C(\mathbf{T})} \leq \|\mu\|_{M(\mathbf{T})} \|f\|_{C(\mathbf{T})}.$$

For the equicontinuity, take any f in the unit ball of $C(\mathbf{T})$, s and t in $\mathbf{T}$. Then

$$\begin{aligned}|(C_\mu f)(s) - (C_\mu f)(t)| &= |(\mu * f)(s) - (\mu * f)(t)| \\ &= |(\mu_s * f)(0) - (\mu_t * f)(0)| \\ &\leq \|\mu_s - \mu_t\|_{M(\mathbf{T})}\|f\|_{C(\mathbf{T})},\end{aligned}$$

so the equicontinuity follows from (6.2).

We next prove the converse, that (a) implies (b). Suppose that μ is a measure in $M(\mathbf{T})$ such that C_μ is a compact operator on $C(\mathbf{T})$. Let S be the unit ball of $C(\mathbf{T})$. Choose any $\varepsilon > 0$. Then, by Ascoli's Theorem, there is a δ so that for all $f \in S$,

$$|(C_\mu \tilde{f})(s) - (C_\mu \tilde{f})(t)| < \varepsilon \quad \text{if } |s - t| < \delta.$$

Choose s and t in $\mathbf{T}$ so that $|s - t| < \delta$. Then

$$\begin{aligned}|(C_\mu \tilde{f})(s) - (C_\mu \tilde{f})(t)| &= |(\mu_s * \tilde{f})(0) - (\mu_t * \tilde{f})(0)| \\ &= \left|\int_{-\pi}^{+\pi} f(x)\, d\mu_s(x) - \int_{-\pi}^{+\pi} f(x)\, d\mu_t(x)\right| \qquad (6.3) \\ &< \varepsilon.\end{aligned}$$

Since

$$\|\mu_s - \mu_t\|_{M(\mathbf{T})} = \sup\left\{\left|\int_{-\pi}^{+\pi} f(x)\, d\mu_s(x) - \int_{-\pi}^{+\pi} f(x)\, d\mu_t(x)\right| : f \in S\right\},$$

$\|\mu_s - \mu_t\|_{M(\mathbf{T})} \leq \varepsilon$ is a consequence of (6.3), which by (6.2) shows that μ must be absolutely continuous with respect to Lebesgue measure. This completes the proof.

Two more lemmas are needed before we are able to complete the proof of Theorem 6.1. In the first, $C(\mathbf{T})$ could be replaced by any homogeneous Banach space of functions in the sense of Section 2.

LEMMA 6.3. *Let E and F be closed translation invariant linear subspaces of $C(\mathbf{T})$, T a bounded linear operator on $C(\mathbf{T})$. If $T(E) \subseteq F$, then $\pi_n(T)(E) \subseteq F$, all n.*

Proof. Let n be a positive integer, $g \in E$. Then

$$\pi_n(T)g = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-int}R_{-t}TR_t g\, dt. \qquad (6.4)$$

Since $g \in E$, for each $t \in \mathbf{T}$, $R_t g \in E$. Thus, by the hypotheses of the lemma, for each $t \in \mathbf{T}$, $R_{-t}TR_t g \in F$. This shows that for each $t \in \mathbf{T}$, the integrand of the integral (6.4) lies in F, so the integral lies in F since F is a closed linear subspace.

LEMMA 6.4. *Let T be a bounded linear operator on $C(\mathbf{T})$ satisfying*

$$T(C(\mathbf{T})_+) \subseteq C(\mathbf{T})_-.$$

Then, for each integer n, $\pi_n(T)$ is an operator of the form $M_n C_\mu$, where M_n is multiplication by the function $e^{in\,\cdot}$ and C_μ is convolution by a measure μ which absolutely continuous with respect to Lebesgue measure.

Proof. It is known that the invariant operators on $C(\mathbf{T})$ are precisely the convolution operators C_μ, for μ a finite Borel measure on $\mathbf{T}$. (See [5].) Let n be an integer. Thus, by Corollary 2.4, $\pi_n(T) = M_n C_\mu$ for some measure μ. By Lemma 6.3, applied to $E = C(\mathbf{T})_+$ and $F = C(\mathbf{T})_-$,

$$(M_n C_\mu)(C(\mathbf{T})_+) \subseteq C(\mathbf{T})_-. \tag{6.5}$$

Since, for each integer n, $C_\mu(e^{im\,\cdot}) = \mu * e^{im\,\cdot} = \hat{\mu}(m)e^{im\,\cdot}$,

$$\hat{\mu}(m) = 0, \quad m = -n+1, -n+2, -n+3, \ldots \tag{6.6}$$

is a consequence of (6.5). By the classical F. and M. Riesz Theorem, (6.6) implies that μ is absolutely continuous with respect to Lebesgue measure. This completes the proof of the lemma.

We are now able to complete the proof of Theorem 6.1. Because of assumption (6.1), Lemma 6.4 and Lemma 6.2 show that each $\pi_n(T)$ is a compact operator. That T is itself compact now follows from Proposition 3.4.

Finally let us observe that the conclusions of this section remain valid if $C(\mathbf{T})$ is replaced by $L^1(\mathbf{T})$.

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