FINITE SIMPLE GROUPS OF 2-RANK 3 WITH ALL 2-LOCAL SUBGROUPS 2-CONSTRAINED

BY

MICHAEL E. O'NAN

Introduction

In this paper we obtain the following:

THEOREM. *Let G be a finite simple group of 2-rank 3 in which all 2-local subgroups are 2-constrained. Then G is isomorphic to one of the groups $L_2(8)$, $U_3(8)$, $Sz(8)$, or $G_2(3)$.*

Here to say that G is of 2-rank 3 means that G has an elementary abelian subgroup of order 8 but none of order 16. Alperin, Brauer, and Gorenstein have determined all simple groups of 2-rank 2.

We note also that Stroth has recently obtained this same result using a different method. In addition, Stroth has determined all finite groups of 2-rank 3 in which some 2-local subgroup is not 2-constrained.

The proof of this theorem is possibly more interesting than its statement. One way to prove the theorem is to use a recent theorem of Gorenstein and Lyons [8], to conclude that either G is known, or G possesses a nonsolvable 2-local subgroup H. Set $\bar{H} = H/O(H)$. If 7 divides the order of $\bar{H}$, then a theorem of Alperin yields the structure of $\bar{H}$. Other results then identify G. If 7 does not divide the order of $\bar{H}$, it is possible to show that $\bar{H}/O_2(\bar{H})$ is a subgroup of the automorphism group of A_5 or A_6, and that $O_2(\bar{H})$ is of restricted type. We do not employ this procedure. Rather we prove analogues of Glauberman's ZJ-theorem. Essentially we prove four propositions which guarantee that G has exactly two conjugacy classes of maximal 2-local subgroups. These are:

PROPOSITION 1. *Let H be a 2-constrained group of 2-rank 3 with $O(H) = 1$. Suppose that 7 divides the order of H. Then, either*

(1) *$O_2(H)$ is an abelian group or a Suzuki 2-group and $H/O_2(H)$ is of odd order, or*

(2) *$O_2(H)$ is homocyclic abelian of rank 3, and $H/O_2(H)$ is isomorphic to $L_3(2)$.*

As stated above, known results will identify G if H is a 2-local subgroup of G. When 7 divides the order of no 2-local of G, we wish to obtain a contradiction. The tools of this attempt are the following three propositions.

Received May 20, 1975.

This research was partially supported by a National Science Foundation grant and by the Alfred Sloan Foundation.

PROPOSITION 2. *Let H be a 2-constrained group with $O(H) = 1$ and $m_2(H)$ at most 3. Suppose that 5 divides the order of H. Let S be a Sylow 2-subgroup of H. Then either*

(1) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$, or*

(2) *$O_2(H)$ is a Sylow 2-subgroup of $U_3(4)$, and $H/O_2(H)$ is a split extension of Z_{15} by a cyclic group of order dividing 4.*

PROPOSITION 3. *Let H be a 2-constrained group of 2-rank 3 with $O(H) = 1$. Let S be a Sylow 2-subgroup of H. Suppose that 3^2 divides the order of H. Then, either*

(1) *$O_2(H)$ is a Suzuki 2-group, and $H/O_2(H)$ has odd order, or*

(2) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$.*

PROPOSITION 4. *Let H be a 2-constrained group of 2-rank 3 and suppose that $O(H) = 1$. Suppose that H has order $2^a 3$, and let S be a Sylow 2-subgroup of H. Then either*

(1) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$, or*

(2) *$A_4 \subseteq H \subseteq S_4 \times Z_2$, or*

(3) *S has a unique normal fours subgroup V and, moreover, V is normal in H.*

After this the result follows fairly easily. Interestingly, no fusion analysis is used.

Section 1

In this section we obtain some structural results about 2-constrained groups of 2-rank 3. We begin by limiting the prime factors of their orders. The first several lemmas serve to bound the 2-rank of certain sections of 2-groups of 2-rank 3.

LEMMA 1.1. *Let Q be a 2-group having an elementary abelian subgroup A of order 2^2 such that A is contained in $Z(Q)$ and Q/A is elementary abelian of order 2^{2k+1} with k an integer. Then:*

(1) *Q has an abelian subgroup B of order 2^3 such that B contains A and $C_Q(B)$ is of index at most 2 in Q.*

(2) *Q has an abelian subgroup of order 2^{k+3}.*

(3) *$m(Q)$ is at least $k + 1$.*

Proof. For (1), take j in A, $j \neq 1$. Set $\bar{Q} = Q/\langle j \rangle$ and $\bar{A} = A/\langle j \rangle$. As $\bar{Q}/\bar{A}$ has order 2^{2k+1}, the classification of extra-special groups implies that $Z(\bar{Q}/\bar{A})$ properly contains $\bar{A}$. Thus, there is an element t in $Q - A$ such that if g is in Q, $t^g = t$ or tj. Then, with $B = \langle A, t \rangle$, (1) follows.

For (2), we use induction. The result is clear if $k = 0$. Otherwise, take B as in (1). Then, there is a subgroup Q_1 of Q with B in $Z(Q_1)$ and Q_1/A of order 2^{2k}. In $\bar{Q}_1 = Q_1/A$, take $\bar{T}_1$ a complement to $\bar{B}$ and take T_1 the preimage of $\bar{T}_1$ in Q_1. Then, T_1/A has order 2^{2k-1}. By induction, T_1 has an abelian subgroup S of order 2^{k+2}. Then, SB is abelian of order 2^{k+3}, and (2) follows. Thus, SB/A is elementary abelian of rank $k + 1$, and it follows that $m(SB)$ and $m(Q)$ are at least $k + 1$. Thus, (3) follows.

LEMMA 1.2. *Let Q be a 2-group with $m(Q) \leq 3$. Suppose that there is a subgroup A of $Z(Q)$ with A and Q/A elementary abelian. Then, Q/A has order at most 2^6.*

Proof. If A has order 2^3, this follows by Lemma 2.2 of [13]. If A has order 2^2, the lemma follows from Lemma 1.1. If A is of order 2, it results from the classification of extra-special 2-groups.

If Q is a 2-group having a subgroup A in its center such that A and Q/A are elementary abelian, we define a mapping $q\colon Q/A \to A$ as follows: if x is in Q, $q(xA) = x^2$. Since A is central and elementary abelian, q is well defined. q is called the squaring map. It is well known that q determines the structure of Q. Since $(xy)^2 = x^2y^2[x, y]$, the function $b(x, y)$ from $Q/A \times Q/A$ into A, defined by $b(x, y) = q(x) + q(y) + q(x + y)$, is bilinear.

LEMMA 1.3. *Let Q be a 2-group with A central in Q and A and Q/A elementary abelian. Let q be the squaring map from Q/A into A. Let B be a subgroup of Q/A having order at least 2^3. Then $\sum_{x \in B} q(x) = 0$.*

Proof. If B has order 2^3, this follows immediately from, and indeed is equivalent to, the statement that $b(x, y)$ is bilinear. If B is of larger order, take B_0 of index 4 in B. Let B_1, B_2, and B_3 be the subgroups of index 2 in B which contain B_0. Then by induction,

$$\sum_{x \in B_1} q(x) + \sum_{x \in B_2} q(x) + \sum_{x \in B_3} q(x) = 0.$$

But $\sum_{x \in B_i} q(x) = \sum_{x \in B_0} q(x) + \sum_{x \in B_i - B_0} q(x)$, and the result follows by substituting.

LEMMA 1.4. *Let P be a 2-group and f an automorphism of P of odd prime order p. Let T be a minimal f-invariant subgroup of P on which f acts nontrivially.*

(1) *If the smallest nonzero positive integer k such that p divides $2^k - 1$ is odd, then T is elementary abelian of rank k.*

(2) *If $p = 3$, T is elementary abelian of order 2^2 or Q_8.*

(3) *If $p = 5$, T is elementary abelian of order 2^4, $Q_8 * D_8$, or is isomorphic to a Sylow 2-subgroup of $U_3(4)$.*

Proof. First suppose that T is elementary abelian. Let k be the smallest nonzero positive integer such that p divides $2^k - 1$. Then it is well known and, in any case, follows easily from Schur's lemma [6, p. 76] that T has order 2^k.

Next suppose that T is not elementary abelian. Then, T is special by [6, p.

183], and f acts irreducibly of $T/Z(T)$ and centralizes $Z(T)$. By the first paragraph $T/Z(T)$ has order 2^k. Moreover, if B is of index 2 in $Z(T)$, T/B is extraspecial. As $Z(T) \neq 1$, it follows that k is even. Thus, (1) follows.

If $p = 3$, then $\langle f \rangle$ is transitive on $(T/Z(T))^{\#}$ and centralizes $Z(T)$. Thus, all elements of T have the same square j in T. Thus, $T/\langle j \rangle$ is elementary abelian. Since T is special, $Z(T) = \langle j \rangle$.

If $p = 5$, then $\langle f \rangle$ has 3 orbits on $(T/Z(T))^{\#}$. Therefore, at most 3 elements of $Z(T)$ are squares in T. In addition, by Lemma 1.3, if exactly 3 elements of $Z(T)$ are squares in T, the sum of the 3 elements is 0. Thus, there is a subgroup V of $Z(T)$ such that V has order at most 4 and V contains the squares of all elements of T. Thus, T/V is elementary abelian, and so $V = Z(T)$. Then, by [7, Lemma 3.9], (3) follows.

LEMMA 1.5. *Let C be a critical subgroup of a 2-group P. Set $A = [C, C]$ and let C_0 be the preimage of $\Omega_1(C/A)$. Then:*

(1) *A and C_0 are characteristic in P.*

(2) *A and C_0/A are elementary abelian.*

(3) *A is contained in $Z(P)$.*

(4) *If f is an automorphism of P of odd order and f centralizes C_0, $f = 1$.*

(5) *Let f be an automorphism of P of odd prime order p. Let k be the smallest positive integer such that p divides $2^k - 1$. Suppose that k is odd and f centralizes A. Then, C_0 has rank at least $m(A) + k$.*

Proof. (1) and (2) are immediate from the properties of the critical subgroup [6, p. 185]. By that reference, $[P, C] \subseteq Z(C)$. Thus, if $x \in P$, and $y, z \in C$, $y^x = ya$ and $z^x = zb$, with $a, b \in Z(C)$. Therefore, $[y, z]^x = [y^x, z^x] = [ya, zb] = [y, z]$. Thus, (3) follows. (4) follows from the properties of the critical subgroup and [6, p. 178]. To prove (5), let B be a minimal f-invariant subgroup of C_0 on which f acts nontrivially. By Lemma 1.4, B is elementary abelian of rank k. Since f centralizes $A \subseteq Z(P)$, AB is elementary abelian of rank $m(A) + k$.

LEMMA 1.6. *Let P be a 2-group of rank at most 3. Then, the odd part of the order of* Aut (P) *divides* $3^4 \cdot 5 \cdot 7$.

Proof. Take C, C_0, and A as in Lemma 1.5. By Lemma 1.2, C_0/A has order at most 2^6. From the order of $L_6(2)$, the lemma follows providing we eliminate the possibilities that 7^2 or 31 divides the order of Aut (P). In the latter case, however, an element f of order 7 or 31 centralizes A, contrary to Lemma 1.5 (5) and $m(P) \leq 3$.

We shall next obtain some fairly precise structural information for 2-constrained groups of 2-rank 3.

PROPOSITION 1. *Let H be a 2-constrained group of 2-rank 3 with $O(H) = 1$. Suppose that 7 divides the order of H. Then, either*

(1) *$O_2(H)$ is an abelian group of a Suzuki 2-group, and $H/O_2(H)$ is of odd order, or*

(2) *$O_2(H)$ is homocyclic abelian of rank 3, and $H/O_2(H)$ is isomorphic to $L_3(2)$.*

Proof. Set $P = O_2(H)$. Take f in H of order 7. Since H is 2-constrained, f does not centralize P. First we show that f does not centralize $Z(P)$. Indeed, suppose that f centralizes $Z(P)$. Let B be a minimal f-invariant subgroup of P on which f acts nontrivially. By Lemma 1.4, B is elementary abelian of order 8, and by choice $B \cap Z(P) = 1$. Thus, $B \cdot Z(P)$ has rank at least 4, a contradiction.

Set $A = \Omega_1(Z(P))$. By the above $|A| \geq 8$. Since $m(P) = 3$, A contains all involutions of P, and P is a Suzuki 2-group or is abelian. By a result of Higman [11], in the first case P has order 2^6 or 2^9.

Next we show that P is a Sylow 2-subgroup of $C_H(A)$.

If P is a Suzuki 2-group of order 2^6, then the squaring map from P/A into A is one-one. Thus, if $d \in C_H(A)$, d centralizes P/A. By Burnside's theorem, it follows that $C_H(A)$ is a 2-group. Thus, $C_H(A) \subseteq P = O_2(H)$. A similar proof is valid when P is abelian. Lastly, suppose that P is a Suzuki 2-group of order 2^9. By Lemma 1.6, 7 does not divide the order of $C_H(A)$. Thus, by the Schur-Zassenhaus theorem, f normalizes some Sylow 2-subgroup S of $C_H(A)$. Since $m(S) = 3$, A contains all involutions of S. Thus, it follows that S is a Suzuki 2-group. Since S has order at most 2^9, $S = P$.

Now $H/C_H(A)$ is some subgroup of $L_3(2)$ of order divisible by 7. If $H/C_H(A)$ is of even order, it follows that $H/C_H(A)$ is isomorphic to $L_3(2)$. Then, (2) follows by a theorem of Alperin [1]. Suppose then that $H/C_H(A)$ is of odd order. Then, since P is a Sylow 2-subgroup of $C_H(A)$, H/P is of odd order, and (1) follows.

Next we obtain analogous results for 2-constrained groups of order divisible by 3 or 5. In the following if Q is a 2-group, Aut* (Q) will denote the group Aut $(Q)/O_2(\text{Aut}\,(Q))$. Consequently, if H is a 2-constrained group with $O(H) = 1$ and Q is a normal and self-centralizing 2-subgroup of H, then $H/O_2(H)$ is a section of Aut* (Q).

LEMMA 1.7. *Let Q be a 2-group with $m(Q) \leq 3$. Suppose that $Q/Z(Q)$ is elementary abelian. Set $A = \Omega_1(Z(Q))$.*

Suppose that Q admits an automorphism f of order 5 and an automorphism g of odd order which does not centralize A. Take $T = [Q, f]$ and $R = C_Q(f)$. Then:

(1) *$Q = TR$ and $[T, R] = 1$.*

(2) *T is isomorphic to a Sylow 2-subgroup of $U_3(4)$.*

(3) *$T \cap R = Z(T) = [T, T]$.*

(4) Aut* (Q) *has abelian Sylow 2-subgroups.*

(5) *Either*

(a) *$Q = T$ or $T \times Z_2$, or*

(b) *R and AT are characteristic in Q and, moreover, there is an element j in $A - Z(T)$ such that $\langle j \rangle$ is characteristic in Q.*

Proof. Since $Q/Z(Q)$ is elementary abelian, $[Q, Q] \subseteq A$. Take Q_0 to be the preimage in Q of $\Omega_1(Q/A)$. By Lemma 1.2, $m(Q_0/A)$ is at most 6. Therefore, $m(Q/Z(Q))$ is at most 6.

Take T to be a minimal f-invariant subgroup of Q on which f acts non-trivially. Using Lemma 1.4 and $m(Q) \leq 3$, it follows that T is isomorphic to $Q_8 * D_8$ or a Sylow 2-subgroup of $U_3(4)$.

Since $m(Q) \leq 3$, f centralizes $Z(Q)$. Since T is special, $Z(T) = \Phi(T)$. Since $\Phi(Q) \subseteq Z(Q)$, $Z(T) \subseteq Z(Q)$. Since $Z(T)$ is elementary, $Z(T) \subseteq A$. Since f centralizes $Z(Q)$ and $C_T(f) = Z(T)$, $T \cap Z(Q) = Z(T)$.

Since $m(Q/Z(Q)) \leq 6$, T covers $[\bar{Q}, f]$, where $\bar{Q} = Q/Z(Q)$. Thus, $[Q, f] \subseteq T \cdot Z(Q)$. Since f centralizes $Z(Q)$, $[Q, f] = T$. By [6, p. 18], T is normal in Q. Set $R = C_Q(f)$. Then, by [6, p. 180], $Q = TR$.

We claim next that R centralizes T.

Recall that if L is a group, K a normal subgroup of L, and x an element of L, then $|C_{L/K}(\bar{x})| \leq |C_L(x)|$.

Take x in R. Then, $C_T(x)$ is f-invariant. Since f acts irreducibly on $T/Z(T) = \bar{T}$, x centralizes $\bar{T}$. Since $|C_T(x)| \geq |C_{\bar{T}}(x)|$, $C_T(x)$ has order at least 2^4. Since $C_T(x)$ is f-invariant, $C_T(x) = T$, and (1) follows.

Next suppose that T is isomorphic to $Q_8 * D_8$. Then, if A has order 2^3, take t an involution of $T - Z(T)$. Then, $\langle t, A\rangle$ has order 16 and is elementary abelian, in contradiction to $m(Q) \leq 3$. Thus, $|A| \leq 4$. As g does not centralize A, $|A| = 4$ and $C_A(g) = 1$. Since $Z(T) \subseteq A$, $Z(T)^g \neq Z(T)$. But both T and T^g are normal in Q. Therefore, if $T \cap T^g \neq 1$, $Z(T) \subseteq T^g$. But then, as $Z(T) \subseteq Z(Q)$, $Z(T) = Z(T)^g$, a contradiction. Thus, $T \cap T^g = 1$, and TT^g has rank 4, a contradiction. Thus, (2) follows. Since T is special and R centralizes T, (3) is immediate.

Next we shall show that if $Z(T) = A$, then $T = Q$. So we suppose that $Z(T) = A$ and Q properly contains T.

Since $T/Z(T)$ is elementary abelian and $Z(T) \subseteq A$, $T \subseteq Q_0$. Moreover, $\Phi(Q) \subseteq Z(Q)$ and $T \cap Z(Q) = Z(T)$. Thus, in the group Q/A no element of T/A is a square. Thus, T is properly contained in Q_0.

First we show that Q_0 admits no automorphism h of order 3 such that h acts freely on $\bar{Q}_0 = Q_0/A$. Indeed, if so $\bar{Q}_0$ has order 2^6. Now Q_0 has a subgroup Q_1 with $A \subset Q_1$ and Q_1/A of order 2^5. By Lemma 1.1, there is an involution j in $Q_1 - A$. Let $\bar{V}$ be a minimal h-invariant subgroup of $\bar{Q}_0$ which contains $\bar{j}$. Then, the preimage of $\bar{V}$ is elementary abelian of order 16, a contradiction.

Since g does not centralize A and A is of order 4, we may assume without loss that g has order a power of 3. If g has order 9, then both g and g^3 act freely on $\bar{Q}_0$, in contradiction to the last paragraph. Thus, g has order 3 and $C_{\bar{Q}_0}(g) \neq 1$. Since $C_A(g) = 1$, $AC_{Q_0}(g)$ is elementary abelian. Consequently, $C_{Q_0}(g)$ has order 2. It follows that Q/A is abelian of type $(2, 2, 2, 2, 2^k)$.

Consequently, R/A is cyclic of order 2^k, and R is abelian of type $(2, 2, 2^k)$ or $(2, 2^{k+1})$. Now R centralizes T and $Q = RT$. Since R is abelian, $R = Z(Q)$. Thus, if R is of type $(2, 2, 2^k)$, we have a contradiction to the fact that A is of rank 2. If R is of type $(2, 2^{k+1})$ and $k \neq 0$, g does not act freely on A, again a contradiction. Thus, $Q = T$.

In the remainder of the proof, then, we assume that A properly contains $Z(T)$. If $R = A$, the $Q = T \times Z_2$. Thus, we suppose that R properly contains A, and prove (b) of (5).

Set $R_0 = R \cap Q_0$. Then, $R_0 \cap T = Z(T)$, and $\bar{Q}_0$ is a direct sum of $\bar{T}$ and $\bar{R}_0$. Since $A \subseteq Z(Q)$, all involutions of Q lie in A.

Next we show that if $x \in Q_0 - (TA)$, then $x^2 \notin Z(T)$. First, suppose that $x \in R_0 - A$. If $x^2 \in Z(T)$, there is an element y^2 of $T - Z(T)$ with $y^2 = x^2$. Then, $(xy)^2 = x^2y^2 = 1$, and $xy \notin A$, a contradiction. Next suppose that $x \in Q_0 - (TA)$. Then, $x = ab$ with a in T and b in $R_0 - A$. Thus, $x^2 = a^2b^2$. Now $a^2 \in Z(T)$ and $b^2 \notin Z(T)$. So $x^2 \notin Z(T)$.

Now we can show that AT is characteristic in Q.

Let h be an automorphism of Q. If $Z(T)^h = Z(T)$, then all elements of T^h have squares lying in $Z(T)$. Thus, by the last paragraph, $T^h \subseteq AT$. Since $A^h = A$, $(AT)^h = AT$. Thus, we may suppose that $Z(T)^h \neq Z(T)$. Since A has order 2^3, $C = Z(T) \cap Z(T)^h$ is of order 2. As $\bar{Q}_0$ has order at most 2^6, $\bar{T} \cap \bar{T}^h$ has order at least 2^2. Therefore, there is a subgroup $\bar{V}$ of $\bar{T}$ such that $\bar{V}$ has order 4 and all squares of elements in $\bar{V}$ lie in C. It follows that T has a subgroup D which is quaternion of order 8. But this implies that T/C has an elementary abelian subgroup of order 8, namely $\langle Z(\bar{T}), \bar{D}\rangle$. This contradicts the fact that T/C is isomorphic to $Q_8 * D_8$.

Since $R = C_Q(AT)$, it follows that R also is characteristic in Q. Thus, $R_0 = R \cap Q_0$ is characteristic in Q. If R_0/A has order 2, then R_0 is abelian of type (4, 2, 2) and clearly a j as in (5) exists. If R_0/A has order 2^2, one of the following holds:

(a) Three distinct elements of $A - Z(T)$ are squares in R_0 and a unique element j of $A - Z(T)$ is not a square in R_0.

(b) One element of $A - Z(T)$ is a square in two cosets of R_0/A and a unique element j of $A - Z(T)$ is a square in one coset of R_0/A.

(c) A single element j of $A - Z(T)$ is a square in R_0.

Thus, in all cases (5) follows.

To prove (4), observe that Aut* (AT) is a split extension of Z_{15} by Z_4, Aut* (R) is some subgroup of S_3, and Aut* (Q) is a subgroup of the direct product of Aut* (AT) and Aut* (R).

We now have:

PROPOSITION 2. *Let H be a 2-constrained group with $O(H) = 1$ and $m_2(H)$ at most 3. Suppose that 5 divides the order of H. Let S be a Sylow 2-subgroup of H. Then either*

(1) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$, or*

(2) $O_2(H) = T$ *is a Sylow 2-subgroup of* $U_3(4)$, *and* $H/O_2(H)$ *is a split extension of* Z_{15} *by a cyclic group of order at most* 4.

Proof. Set $P = O_2(H)$ and take C and C_0 as in Lemma 1.5. Set $A = \Omega_1(Z(C_0))$. Since H is 2-constrained, $B = \Omega_1(Z(S))$ is contained in P. Since C is critical, $B \subseteq C_0$ and $B \subseteq A$. Take f in H of order 5. Now if all elements of odd order act trivially on A, then B is contained in the center of H. Since B is characteristic in S, the lemma follows with $D = B$. Thus, we may suppose that some element g of odd order in H acts nontrivially on A. Thus, the group C satisfies the hypotheses of the previous lemma.

First suppose that C does not satisfy (5) (a) of the previous lemma. Take R and T as in that lemma.

By the lemma, $A = \langle Z(T), j\rangle$. By Lemma 1.5(3), $[C, C] \subseteq Z(P)$. Since $Z(T) = [T, T] \subseteq [C, C]$, $Z(T) \subseteq Z(P)$. Also, as $\langle j\rangle$ char C, $j \in Z(P)$. Thus, all involutions of P lie in A. We claim that A is characteristic in S. By (4) of the last lemma, H/P has abelian Sylow 2-subgroups. Thus, S/P is abelian, and P contains the commutator subgroup S' of S. Let h be an automorphism of S. Since $Z(T) \subseteq S'$, $Z(T)^h \subseteq S' \subseteq P$. Thus, $Z(T)^h \subseteq A$. Since $\langle j\rangle$ char C and $C \lhd S$, $j \in B$. Since B char S, j^h lies in B. As $B \subseteq A$, $j^h \in A$. Since A is generated by $Z(T)$ and j, $A^h = A$. Thus, A is characteristic in S. Since $\langle j\rangle$ char C, j lies in $Z(H)$. Thus, the proposition follows with $D = A$.

Thus, in the remainder of this proof we may suppose that $C = T$ or $T \times Z_2$, where T is a Sylow 2-subgroup of $U_3(4)$. Since Aut* (C) is an extension of Z_{15} by Z_4, we may choose an element g of order 3 in H so that g commutes with f.

First we shall show that $C_P(f) = A$. Set $E = C_P(f)$. Since $T = [C, f]$, T is E-invariant. Then, an argument of the last lemma shows that E centralizes T. Thus, E/A acts on A and centralizes $Z(T)$. On the other hand, $C = TA$ and C is self-centralizing, as C is critical in P. Consequently, E/A acts faithfully on A while centralizing $Z(T)$. Thus, E/A has order 2^2 at most. Since g centralizes f, E is g-invariant. Since g does not centralize $Z(T)$, g acts faithfully on E/A. Thus, E/A has order exactly 2^2. Set $\bar{E} = E/Z(T)$. Then, $\bar{E}$ is isomorphic to Q_8 or is elementary abelian of order 8.

In the first case, $\bar{E} = \langle \bar{x}, \bar{y}\rangle$, with $\bar{x}$ and $\bar{y}$ of order 4. Then, the preimages of $\bar{x}$ and $\bar{y}$ are each contained in abelian groups of order 16 which contain A. So A is contained in $Z(E)$, a contradiction if $E \supset A$. In the second case, it follows easily that $[\bar{E}, g]$ has as preimage in E an abelian group V of type (4, 4). But T also possesses an abelian group U of type (4, 4). Moreover, V and U commute and intersect in $Z(T)$. Therefore, VU has rank 4, a contradiction. It follows that $C_P(f) = A$.

Next we show that $A \subseteq Z(P)$. Since A is abelian of order at most 8, $C_P(A)$ has index at most 8 in P. Since f is of order 5, f centralizes $P/C_P(A)$. Thus, $P = C_P(A)C_P(f)$, and so $P = C_P(A)$.

Now $C = TA$. Suppose first that $A = Z(T)$. Then $C = T$, and we shall show that $P = T$. From this, conclusion (2) of the proposition follows.

Now $C_{Z(T)}(g) = 1$ and $Z(T)$ contains all involutions of T. Thus, $C_T(g) = 1$. Let $F = C_P(g)$. Then F is f-invariant, since f centralizes g. Since $F \cap A = 1$, $C_F(f) = 1$. Thus, if $F \neq 1$, $m(F)$ is at least 4, a contradiction. So $F = 1$.

By [4, p. 90], P has class at most 2. Since $Z(P)$ is f-invariant of rank at most 3, $Z(P)$ is centralized by f. Consequently, $Z(P) = Z(T) = A$. Thus, $\bar{P} = P/Z(T)$ is abelian of rank at most 6. Since $C_{\bar{P}}(f) = 1$, $\bar{P}$ has rank 4. Since $T = [P, f]$, $\bar{P}$ is elementary abelian and $P = T$.

Next suppose that A has order 2^3. Let j be an involution of S. Now H/P is a subgroup of the semidirect product of Z_{15} by Z_4. Thus, any involution of H/P centralizes $\bar{g}$, the homomorphic image of g in H/P. Thus, $\bar{j} = jP$ centralizes $\bar{g}$. From the action of g on A, it follows that j centralizes A. Consequently, $j \in A$. Thus, A contains all involutions of S. Clearly, $A \cap Z(H) \neq 1$, and the proposition follows with $D = A$.

LEMMA 1.8. (1) *Let P be a 2-group of rank at most* 3. *Suppose that* Aut (P) *has an abelian subgroup B of type* (3, 3). *Let b_1, b_2, b_3 represent* 3 *distinct cyclic subgroups of B of order* 3, *and suppose that for $i = 1, 2, 3$, $C_P(b_i)$ is of rank* 1. *Then, $\Omega_1(C_P(b_i)) \subseteq Z(P)$, for $i = 1, 2, 3$.*

(2) *Let P be a 2-group of rank at most* 3. *Suppose that P admits a group B of automorphisms of order* 9 *which does not centralize $\Omega_1(Z(P))$. Then, $\Omega_1(Z(P))$ has order 2^3.*

Proof. For (1), we proceed by induction on the order of P. By Burnside's theorem, B acts faithfully on $\bar{P}$, the Frattini factor group of P. Then there are B-submodules V_1 and V_2 of $\bar{P}$, with V_1 and V_2 of order 4, and $\bar{P} = V_1 \oplus V_2 \oplus U$, where U is B-invariant. Let R_1 be the preimage of $V_1 + U$ and R_2 be that of $V_2 + U$. Since $C_P(b_i)$ has rank 1, $\Omega_1(C_P(b_i))$ lies in R_1 and R_2. Now if B acts faithfully on R_1 and R_2,

$$\Omega_1(C_P(b_i)) \subseteq Z(R_1) \cap Z(R_2) \subseteq Z(P).$$

Therefore, it suffices to treat the case in which some element of B, say b, centralizes R_1, where $b \neq 1$. Let Q_0 be a minimal b-invariant subgroup of P on which b acts nontrivially. Then Q_0 is isomorphic to an elementary abelian group of order 4 or Q_8. Then Q_0 covers V_2, and so $P = R_1 Q_0$. Since b centralizes R_1, $Q_0 = [P, b]$. It follows that Q_0 is normal in P. Now if Q_0 is quaternion of order 8, the unique involution j of Q_0 is central in P. Since $Q_0 = [P, b]$, B normalizes Q_0 and centralizes j. Thus, j lies in $C_P(b_i)$ for all i, and (1) follows. Thus, we may suppose that Q_0 is elementary abelian of order 4. Then, as both R_1 and Q_0 are normal subgroups of P, it follows that $P = R_1 \times Q_0$. Since P has 2-rank 3 or less, R_1 has rank 1. Since P admits a group of automorphisms of type (3, 3), R_1 is isomorphic to Q_8. Thus, $\Omega_1(R_1)$ is contained in $Z(P)$ and centralizes b_1, b_2, b_3.

Next we prove (2). Let $A = \Omega_1(Z(P))$. If (2) fails, A has order 2^2. If B is of type (3, 3), some b in $B^{\#}$ centralizes A, but B itself does not centralize A. Then, if b_1, b_2, b_3 represent the remaining cyclic subgroups of B, $C_A(b_i) = 1$. Since

$m(P)$ is at most 3, $C_P(b_i)$ must be of rank 1. Now the first part gives a contradiction.

Thus, we may suppose that B is cyclic of order 9. Take C and C_0 as in Lemma 1.3. Then, $\bar{C}_0 = C_0/C_0'$ has order at most 2^6. Since $\bar{C}_0$ admits the action of B, $\bar{C}_0$ has rank exactly 6 and B acts irreducibly on $\bar{C}_0$. Now A is contained in C and C_0, as C is critical. Since B is irreducible on $\bar{C}_0$, A is contained in C_0'. Thus, C_0' has order at least 2^2. If C_0' has order exactly 2^2, then there is some involution j in $C_0 - C_0'$. Let g generate the subgroup of B or order 3. Then, the preimage of a minimal g-invariant subgroup of $\bar{C}_0$ which contains $\bar{j}$ is elementary of order 16, a contradiction. Consequently, C_0' has order 8. Lemma 1.5 now implies that C_0' is contained in $Z(P)$, and the result follows.

LEMMA 1.9. *Let Q be a 2-group with $m(Q) = 3$ and suppose that $Z(Q)$ contains an elementary abelian group A of order 2^3 such that Q/A is elementary abelian. Let L be the subgroup of* Aut (Q) *which centralizes A. Then:*

(1) *If L has an abelian subgroup of type* (3, 3, 3), *then Q is a direct product of 3 copies of Q_8.*

(2) *L contains no extra-special group of order* 3^3 *and exponent* 3.

Proof. First suppose that B is a subgroup of L of type (3, 3, 3). By Lemma 1.2 and the action of B, $\bar{Q} = Q/A$ has order 2^6. Also, $\bar{Q} = V_1 \oplus V_2 \oplus V_3$, where V_1, V_2, and V_3 are B-invariant of order 2^2 and there are elements b_1, b_2, b_3 of B such that $[\bar{Q}, b_i] = V_i$. Now if Q_i is a minimal b_i-invariant subgroup of Q on which b_i acts nontrivially, Q_i is either elementary abelian of order 2^2 or a quaternion group of order 8.

In the first case, since b_i centralizes A, Q_iA has rank 5, a contradiction. Therefore, Q_i is quaternion of order 8 and Q_i covers V_i. Consequently, $Q_i = [Q, b_i]$ and Q_i is a normal subgroup of Q. Moreover, Q_i is b_j-invariant. Now if $i \neq j$, $Q_i \cap Q_j$ is contained in A. Thus, b_i centralizes Q_j, if $i \neq j$. Then, as $Q_j = [Q, b_j]$, and b_j centralizes Q_i, Q_j centralizes Q_i. Thus, Q_iQ_j is isomorphic to $Q_8 * Q_8$ or $Q_8 \times Q_8$. In the first case, some involution of Q does not lie in A, a contradiction. Thus, $Q_1Q_2 = Q_8 \times Q_8$, and Q_3 centralizes Q_1Q_2. If $(Q_1Q_2) \cap Q_3 \neq 1$, there is an element x in Q_3 and an element y in Q_1Q_2 with $x^2 = y^2 \neq 1$. Then $(xy)^2 = 1$, and xy is not in A, a contradiction. Thus, (1) follows.

Next take B in L to be extra-special of order 3^3 and exponent 3. Again by the action of B, $\bar{Q}$ has order 2^6. Let q be a quadratic form on $\bar{Q}$ preserved by B. We shall show that q is uniquely determined.

Note that B has one orbit on $(\bar{Q})^\#$ of length 27 and all remaining orbits of length 9. Moreover, B is irreducible. It follows that q is nondegenerate (if $q \neq 0$), and the orthogonal group determined by q is $O_6^-(2)$, whose commutator subgroup is isomorphic to $PSp(4, 3)$ and has Sylow 3-subgroup Z_3 wr Z_3. It follows quickly that the Sylow 3-subgroup of $O_6^-(2)$ has one conjugacy class of subgroups isomorphic to B, and moreover such a B is transitive on the 27 isotropic vectors with respect to q. Thus, the zeros of q are precisely the points of $\bar{Q}$ in the orbit of B of length 27. Consequently, q is uniquely determined.

Now if q is the squaring map from Q/A into A, and e_1, e_2, e_3 is a basis for A, $q(x) = q_1(x)e_1 + q_2(x)e_2 + q_3(x)e_3$, where q_1, q_2, q_3 are quadratic forms on Q/A. By the above, $q_1 = q_2 = q_3$. Thus, there is some involution in $Q - A$, contrary to $m(Q) = 3$.

The following involves a calculation in a known group and the proof will be omitted.

LEMMA 1.10. *Let L be a subgroup of $L_6(2)$ of order 2^a3^b with $O_2(L) = 1$. Then, L is a subgroup of S_3* wr *S_3 or $GU(3, 2)$, the latter being a split extension of an extra-special group of order 3^3 and exponent 3 by $GL(2, 3)$.*

LEMMA 1.11. *Let H be a 2-constrained group of 2-rank 3 in which $O(H) = 1$. Suppose that $A = \Omega_1(Z(O_2(H)))$ is of order 2^3 and $H/C_H(A)$ is isomorphic to Z_3 or S_3. Let S be a Sylow 2-subgroup of H. Then, either*

$$A_4 \subseteq H \subseteq Z_2 \times S_4,$$

or there is a characteristic subgroup D of S with D normal in H and

$$D \cap Z(H) \neq 1.$$

Proof. Set $P = O_2(H)$ and let R be a Sylow 2-subgroup of $C_H(A)$ which is contained in S. Then, by hypothesis, R has index at most 2 in S. Moreover, A contains all involutions in R.

First suppose that A lies in the Frattini subgroup of S and let h be an automorphism of S. Then, also A^h is contained in the Frattini subgroup of S. Since R is of index 2, $A^h \subseteq R$. Thus, $A^h = A$, and the lemma follows with $D = A$.

Thus, we may suppose that $A \nsubseteq \Phi(S)$. Then, $A \nsubseteq \Phi(P)$. First, suppose that $A \cap \Phi(P) = V$ has order 4. Then, V is characteristic in P and normal in H. By the action of $H/C_H(A)$ on A, $A = \langle j \rangle \oplus V$, where j is central in H.

Take h an automorphism of S. Then, as $V \subseteq \Phi(S)$, $V^h \subseteq \Phi(S) \subseteq R$. Since A contains all involution of R, $V^h \subseteq A$. Also, as j lies in $Z(S)$, j^h lies in $Z(S)$. Since $Z(S) \subseteq P$, as H is 2-constrained, it follows that $j^h \in P$. As $A = \langle V, j \rangle$, $A^h = A$. The lemma follows with $D = A$.

Thus, we may suppose that $A \cap \Phi(P)$ has order 2 at most. Then $P = V \times L$, where L is of rank 1. First, suppose that L has order at least 8 and set $\langle j \rangle = D = \Omega_1(L)$. Since j is the only square in P, D char P. Thus j lies in $Z(H)$. Moreover, if h is an automorphism of S, $L^h \cap P$ has order at least 4 and j is a square in $L^h \cap P$. Thus, $D^h = D$, and the lemma follows. When $|L| \leq 4$, the treatment is similar and not difficult.

PROPOSITION 3. *Let H be a 2-constrained group of 2-rank 3 with $O(H) = 1$. Let S be a Sylow 2-subgroup of H. Suppose that 3^2 divides the order of H. Then either*

(1) *$O_2(H)$ is a Suzuki 2-group and $H/O_2(H)$ is of odd order, or*

(2) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$.*

Proof. Set $P = O_2(H)$ and $A = \Omega_1(Z(P))$. If 5 or 7 divides the order of H, (1) or (2) follows. So we may suppose that H has order 2^a3^b. Let C and C_0 be as in Lemma 1.5. Then C_0/A and $C/Z(C)$ are elementary abelian of order at most 2^6. By [6, p. 185], $[P, C] \subseteq Z(C)$. It follows that $O_2(H/P) = 1$. Then, using Lemma 1.10, it follows that H/P is a subgroup of S_3 wr S_3 or $GU(3, 2)$.

Set $E = \Omega_1(Z(S))$. Since H is 2-constrained, $E \subseteq A$. If all elements of H of odd order act trivially on A, (2) follows with $D = E$. So some 3-element f of H does not centralize A. By Lemma 1.8, A is of order 2^3. Thus, $H/C_H(A)$ is some subgroup of S_4 of order divisible by 3.

First suppose that $H/C_H(A)$ contains A_4. Then by the structure of H/P there is a subgroup B of order 3^3 in H, where B centralizes A and is of exponent 3. By Lemma 1.9, $C_0 = Q_8 \times Q_8 \times Q_8$. Let j_1, j_2, j_3 be the three involutions lying in the direct factors of C_0. The Krull-Schmidt theorem implies that Aut (C_0) permutes j_1, j_2, j_3. Thus, $j_1 j_2 j_3$ is fixed by Aut (C_0), contrary to $A_4 \subseteq H/C_H(A)$.

Now the proposition follows from Lemma 1.11.

PROPOSITION 4. *Let H be a 2-constrained group of 2-rank 3 and suppose that $O(H) = 1$. Suppose that H has order 2^a3, and let S be a Sylow 2-subgroup of H. Then either*

(1) *there is a characteristic subgroup D of S with D normal in H and $D \cap Z(H) \neq 1$, or*

(2) *$A_4 \subseteq H \subseteq S_4 \times Z_2$, or*

(3) *S has a unique normal fours subgroup V with V normal in H.*

Proof. Let $P = O_2(H)$. Then H/P is isomorphic to S_3 or Z_3. Set $A = \Omega_1(Z(P))$, $B = \Omega_1(Z(S))$. Then $B \subseteq A$. If $|A| = 2$, (1) holds with $D = B$. If $|A| = 2^3$, (1) or (2) follows by Lemma 1.11.

Thus, we may assume that $|A| = 2^2$. Set $V = A$. Take f in H for order 3. If f centralizes V, (1) holds with $D = B$. So we suppose that f does not centralize V.

If $V \nsubseteq \Phi(P)$, then the action of f implies that $V \cap \Phi(P) = 1$. Thus, $P = V \times L$, with $m(L) \leq 1$. Since $m(Z(P)) = 2$, $L = 1$, and (2) holds.

Thus, in the remainder we assume that $V \subseteq \Phi(P)$. We suppose that S has a normal fours group U with $U \neq V$ and derive a contradiction from this.

First suppose that U does not centralize V. Then, $U \nsubseteq P$. Since $B \subseteq V$, but $V \nsubseteq Z(S)$, it follows that $|B| = 2$. Since U and V are normal in S, $B = U \cap V$. It follows that $[S, U] \subseteq B$, $[S, V] \subseteq B$. Thus, $[S, \langle U, V\rangle] \subseteq B$, and $\langle U, V\rangle$ is dihedral. Set $L = C_S(\langle U, V\rangle)$. Then it follows by [6, p. 195] that $S = \langle U, V\rangle \cdot L$. Clearly, L is normal in S and $L \cap \langle U, V\rangle = B$. Thus, S/L is elementary abelian of order 2^2. It follows that V is not contained in the Frattini subgroup of P, a contradiction. Thus, we may suppose that $[U, V] = 1$, and $P \supseteq U$.

Let $E = \langle U, V\rangle$. Then, E is elementary abelian of order 8. First we show that f normalizes E. Let $F = \langle E, E^f, E^{f^2}\rangle$, and suppose that $E \neq F$. Since E/V lies in $Z(P/V)$, F/V has order 2^3 at most.

If F/V has order 2^2, then f acts freely on F. Since there is an involution in $F - V$, F is elementary abelian of order 16, a contradiction. Thus, F/V has order 2^3. Let $L = [F, f]$. It follows as before that L is abelian of type (4, 4), as f acts freely on L. Also, $C_F(f)$ has order exactly 2. Thus, some involution j of $F - L$ centralizes f. Now the involution j does not centralize L. For if so, it follows that F is abelian and $E = \Omega_1(F)$ is normalized by f. Since $C_L(j)$ is f-invariant, $|C_L(j)| = 4$. Thus, j and all involutions of $F - L$ have four conjugates in F. Thus, we have a contradiction to the fact that F contains the normal fours subgroups U and V, not both of which lie in L. Therefore, f normalizes E.

Thus, E is normal in H. Since V lies in $Z(P)$ and $C_P(U)$ has index at most 2 in P, $C_P(E)$ has index at most 2 in P. If $C_P(E)$ has index exactly 2 in P, then in the group $H/C_P(E)$, $\bar{f}$ normalizes and so centralizes an element of order 2. This contradicts the structure of the group $L_3(2)$. Thus, $C_P(E) = P$, contrary to $m(Z(P)) = 2$. Thus, it follows that V is the unique normal fours subgroup of S.

Section 2

Now let G be a finite simple group all of whose 2-local subgroups are 2-constrained, and suppose that the 2-rank of G is exactly 3. Then a theorem of Gorenstein and Walter [9], together with a theorem of Aschbacher [2], imply that if H is any 2-local subgroup of G, then $O(H) = 1$.

Now let H be some 2-local subgroup of G. If the order of H is divisible by 7, Proposition 1 yields the structure of H. First suppose that H satisfies the first conclusion of that proposition. Then $H = N_G(O_2(H))$ and $H/O_2(H)$ is of odd order. Then H contains a Sylow 2-subgroup of G. From the structure of $O_2(H)$ it follows that $O_2(H)$ contains an elementary abelian 2-subgroup which is strongly closed in $O_2(H)$ with respect to G. By a theorem of Goldschmidt [5], G is isomorphic to one of the groups $L_2(8)$, $U_3(8)$, or $Sz(8)$. Next if H satisfies the second conclusion of Proposition 1, then $O_2(H)$ is a homocyclic abelian group and $H/O_2(H)$ is isomorphic to $L_3(2)$. When $O_2(H)$ is of exponent 4 or more, then the result of [12] shows that G is known. However, not all 2-local subgroups of G are 2-constrained. If, on the other hand, $O_2(H)$ is elementary abelian, a theorem of Harada [10] and a theorem of Gorenstein-Harada [7] imply that G is isomorphic to the group $G_2(3)$. Thus, for the remainder of this section we assume that 7 divides the order of no 2-local subgroup of G, and from this we derive a contradiction.

LEMMA 2.1. *Let L be a maximal 2-local subgroup of G having Sylow 2-subgroup S. Then S is a Sylow 2-subgroup of G and either*

(1) *$L = C_G(j)$, for some involution j of S, or*

(2) *S has a unique normal fours subgroup V, $L = N_G(V)$, and $|L| = 2^k 3$.*

Proof. Set $P = O_2(L)$. By Lemma 1.6, the odd part of the order of L divides $3^4 5$.

First suppose that 5 divides the order of L. Proposition 2 is then applicable to L. In the second conclusion to that proposition, P is isomorphic to a Sylow 2-subgroup of $U_3(4)$ and $Z(P)$ is normal in L. Thus, $L = N_G(Z(P))$. Now S/P is abelian and so it follows that $Z(P)$ contains all involutions of the commutator subgroup S' of S. Thus, $Z(P)$ is characteristic in S, and so S is a Sylow 2-subgroup of G. Clearly, S has sectional 2-rank 4, and a contradiction results applying a theorem of Gorenstein-Harada [7].

Thus, the first conclusion of Proposition 2 is valid in L. Therefore, there is a characteristic subgroup D of S with D normal in L and $D \cap Z(L) \neq 1$. By its maximality, $L = N_G(D)$. Thus, as D is characteristic in S, S is a Sylow 2-subgroup of G, and (1) follows.

If 3^2 divides the order of L, (1) follows as above, using Proposition 3. Then, if exactly 3 divides the order of L, Proposition 4 guarantees that (1) or (2) holds, unless L is some subgroup of $Z_2 \times S_4$. But then G has a self-centralizing subgroup of order 8, and a theorem of Harada [10] yields a contradiction as before.

If $L = S$, then clearly S is a Sylow 2-subgroup of G. What we have proved above shows that S is not a maximum 2-local, unless all 2-local subgroups of G are 2-groups. But in the last case, Frobenius' theorem shows that G is nonsimple.

If all 2-local subgroups of G are solvable, G is known by a theorem of Gorenstein and Lyons [8], and a contradiction results. We take H to be a nonsolvable 2-local subgroup of G, and S a Sylow 2-subgroup of H. Without loss we may suppose that H is a maximal 2-local subgroup of G. Then the last lemma implies that S is a Sylow 2-subgroup of G.

LEMMA 2.2. *If t is an involution of $Z(S)$, $H = C_G(t)$.*

Proof. Since H is nonsolvable, 5 divides the order of H. Let $B = \Omega_1(Z(S))$, $P = O_2(H)$, and take C a critical subgroup of P. If all elements of H of odd order centralize B, the lemma follows. Otherwise, by Lemma 1.7, the structure of C is known. It follows that Aut (C) is solvable, a contradiction.

Now if all maximal 2-local subgroups of G are conjugate to H, G has a strongly embedded subgroup, and Bender's theorem [3] gives a contradiction. Thus, we may suppose that there is a maximal 2-local subgroup M with M not conjugate to H. By Lemma 2.1, and conjugating if necessary, we may suppose that S is contained in M. By Lemma 2.1 and 2.2, S has a unique normal fours subgroup V, where V is normal in M, and M has order $2^k 3$. Let $P = O_2(H)$ and $R = O_2(M)$. Let C be the critical subgroup of P, and take C_0 as in Lemma 1.5.

LEMMA 2.3. (1) *$Z(P)$ is cyclic.*

(2) *C_0 is isomorphic to $Q_8 * D_8$, $Q_8 * D_8 * Z_4$, or $Q_8 * D_8 * D_8$.*

(3) *V is contained in C_0.*

(4) *$H/O_2(H)$ is isomorphic to S_5.*

(5) *V is not contained in the Frattini subgroup of P.*

Proof. First we claim that H has no elementary abelian normal 2-subgroup of order 4 or greater. Indeed, let E be such an elementary abelian normal subgroup of H. Since E is normal in S, E contains a normal fours subgroup of S. Since S has a unique normal fours group V, it follows that $V \subseteq E$. Since E has order 8 at most, E is centralized by an element of H of order 5. Thus, V is centralized by some element of order 5. This contradicts the fact that $M = N_G(V)$ has order $2^k 3$, and the claim follows.

Now it follows that $Z(P)$, $Z(C_0)$, and C_0' are all cyclic. Also C_0/C_0' is elementary abelian of order at most 2^6. Then the classification of extra-special groups and the fact that Aut (C_0) is of order divisible by 5 gives the above structure for C_0.

Let j be the unique involution in $Z(C_0)$. Now the number of fours subgroups of C_0 which contain j is 5, 15, or 27. according to the structure of C_0. In particular, this number is odd. Thus, one fours group of the above is normalized by S. It follows then that V lies in C_0, yielding (3).

Now Aut* (C_0) is a subgroup of $O_6^-(2) = \text{Aut}\,(PSp(4, 3))$. Thus, $L = H/O_2(H)$ is a subgroup of $O_6^-(2)$ and is nonsolvable. Moreover, $O_2(L) = 1$. Consequently, L is A_5, S_5, or contains some subgroup isomorphic to A_6.

On the other hand, observe that since $M = N_G(V)$ is not contained in H, $N_H(V) = S$.

Suppose first that $K = A_6$ is contained in L. Let X be the orbit of L on the 27 or fewer fours subgroups of C_0 which are conjugate to V. Since S fixes V, $|X|$ is odd. Since $|X| \leq 27$, every element of X is fixed by some element of K of odd order $\neq 1$, contrary to $N_H(V) = S$. Thus, L is A_5 or S_5. If L is A_5, then S is normalized by an element of H of order 3. Since V is characteristic in S, V is normalized by an element of H or order 3, a contradiction. So (4) follows.

Suppose $V \subseteq \Phi(P)$. Then if g is any element of H, $V^g \subseteq \Phi(P)$. As V is normal in P, so is V^g. Therefore, V and V^g commute. Thus, the normal closure of V in H is abelian, contrary to the above.

LEMMA 2.4. $|S| \leq 2^8$.

Proof. Recall $P = O_2(H)$ and $R = O_2(M)$. Now $|S: P| = 2^3$, and $|S: R| \leq 2$. By the last lemma, $V \subseteq P$, but $V \nsubseteq Z(P)$. Moreover, $R = C_S(V)$. Thus, V is not central in S, and so $|S: R| = 2$. Thus, $|R: P \cap R| = 2^2$. Let f be an element of M of order 3.

Now $V \nsubseteq \Phi(P)$ implies that $V \nsubseteq \Phi(R \cap P)$. So there exist $j_1, j_2 \in V^\#$, $j_1 \neq j_2$ such that $j_1, j_2 \notin \Phi(R \cap P)$. Then, $j_1^f, j_2^f \notin \Phi(R \cap P^f)$. Therefore,

$$j_1, j_2, j_1^f, j_2^f \notin \Phi(R \cap P \cap P^f).$$

Since f acts freely on V, it follows that $V \cap \Phi(R \cap P \cap P^f) = 1$. But $R \cap P$ is normal in R as P is normal in S. Therefore, $R \cap P^f$ is normal in R. Thus, $R \cap P \cap P^f$ is normal in R. Consequently, $\Phi(R \cap P \cap P^f)$ is normal in R. Thus, if $\Phi(R \cap P \cap P^f) \neq 1$, there is an involution j in $Z(R) - V$. Thus, there is an abelian subgroup A of order 2^3 in $Z(R)$. Since $|S: R| = 2$, it follows that $Z(S)$ is not cyclic. But $Z(S) \subseteq Z(P)$, and so $Z(P)$ is not cyclic, a contradiction. Thus, it follows that $R \cap P \cap P^f$ is elementary abelian. Consequently, $|R \cap P \cap P^f| \leq 2^3$.

On the other hand, $|R: P \cap R| = 2^2$ implies that $|R: R \cap P^f| = 2^2$. Thus,

$$|R: R \cap P \cap P^f| \leq 2^4.$$

Thus, $|R| \leq 2^7$, and the lemma follows.

We now obtain a final contradiction. Since $H/O_2(H) = S_5$, $O_2(H)$ has order at most 2^5. By Lemma 2.3, $O_2(H) = C_0 = Q_8 * D_8$. Moreover, $O_2(H)$ has exactly 5 subgroups of type (2, 2). Consequently, every subgroup of $O_2(H)$ of type (2, 2) is normalized by some nonidentity element of H of odd order. This contradicts $N_H(V) = S$.

1. J. L. ALPERIN, *Sylow 2-subgroups of 2-rank three*, Finite Groups '72, Proceedings of the Gainesville Conference, North Holland, New York, 1973.
2. M. ASCHBACHER, *Finite groups with a proper 2-generated core*, Trans. Amer. Math. Soc., vol. 197 (1974), pp. 87–112.
3. H. BENDER, *Transitive Gruppen gerader Ordnung, in dene jede Involution genau einen Punkt festlasst*, J. Algebra, vol. 17 (1971), pp. 527–554.
4. W. BURNSIDE, *Theory of groups of finite order*, Dover, New York, 1955.
5. D. GOLDSCHMIDT, *2-fusion in finite groups*, Ann. Math., vol. 99 (1974), pp. 70–117.
6. D. GORENSTEIN, *Finite groups*, Harper and Row, New York, 1968.
7. D. GORENSTEIN AND K. HARADA, *Finite groups whose 2-subgroups are generated by at most 4 elements*, Mem. Amer. Math. Soc., no. 147, Providence, 1974.
8. D. GORENSTEIN AND R. LYONS, *Nonsolvable finite groups with solvable 2-local subgroups*, J. Algebra, to appear.
9. D. GORENSTEIN AND J. WALTER, *Centralizers of involutions in balanced groups*, J. Algebra, vol. 20 (1972), pp. 284–319.
10. K. HARADA, *On finite groups having self-centralizing 2-subgroups of small order*, J. Algebra, vol. 33 (1975), pp. 144–160.
11. G. HIGMAN, *Suzuki 2-groups*, Illinois J. Math., vol. 7 (1963), pp. 79–96.
12. M. O'NAN, *Some evidence for the existence of a new simple group*, Proc. Lon. Math. Soc., to appear.
13. M. O'NAN AND R. SOLOMON, *Simple groups transitive on internal flags*, J. Algebra, to appear.

RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY