BOUNDARY-PRESERVING MAPPINGS OF 3-MANIFOLDS ONTO CUBES-WITH-HANDLES

BY

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1. Introduction

Let M^3 and N^3 be 3-manifolds with boundary. A continuous mapping $f\colon M^3 \to N^3$ is said to be *boundary-preserving* if $f^{-1}(\partial N^3) = \partial M^3$ and $f \mid \partial M^3$ is a homeomorphism, where ∂M^3 and ∂N^3 denote the boundaries of M^3 and N^3 respectively. All manifolds and mappings in this paper will be assumed to be piecewise linear. A *cube-with-handles* is a 3-manifold homeomorphic to a regular neighborhood of a connected finite graph in S^3. A *cube-with-holes* is a 3-manifold homeomorphic to the closure of the complement of a cube-with-handles in S^3. Fox [1] has shown that any compact 3-manifold with connected boundary in S^3 is a cube-with-holes. Lambert [7], and Jaco and McMillan [5] have given examples of cubes-with-holes for which there exist no boundary-preserving mappings onto cubes-with-handles. Jaco and McMillan also give a necessary and sufficient condition on a cube-with-holes for the existence of a boundary-preserving mapping of it onto a cube with-handles. In Theorem 3.1 we generalize this result to compact orientable 3-manifolds with connected boundary. Theorems 3.2 and 3.3 are also concerned with the existence of boundary-preserving mappings onto cubes-with-handles.

Let M^3 and N^3 be orientable 3-manifolds. Let K^3 be a compact submanifold of M^3 which has connected boundary, and let H^3 be a cube-with-handles which is a submanifold of N^3. Let $f\colon M^3 \to N^3$ be a mapping so that $f \mid K^3$ is a boundary-preserving mapping of K^3 onto H^3, and so that $f \mid \mathrm{cl}\,(M^3 - K^3)$ is a homeomorphism. In Theorems 2.2 and 2.3 we show that any degree one mapping between closed 3-manifolds, and any boundary-preserving mapping between compact 3-manifolds with boundary, is homotopic to a mapping satisfying the conditions given for f above. In the closed manifold case, the genus of ∂K^3 is determined by the Heegaard genus of N^3. In Theorem 4.2 we show that the homeomorphism type of K^3, and its embedding in M^3, determine the 3-manifold N^3.

In Section 5, we describe how any genus n cube-with-handles U in S^3, where $\mathrm{cl}\,(S^3 - U) = K^3$ is a boundary-retractable cube-with-holes, gives rise to a homotopy 3-sphere M^3 of Heegaard genus n, and a mapping $f\colon S^3 \to M^3$ so that $f \mid U$ is a homeomorphism. Then we give conditions on U and K^3 which

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imply that M^3 is homeomorphic to S^3. For instance, if K^3 has genus 2 and contains a nontrivial spanning disk, M^3 is homeomorphic to S^3. And if U has genus 2 and contains a nontrivial unknotted simple closed curve, then M^3 is homeomorphic to S^3.

A disk D in a 3-manifold with boundary K^3 is called a *spanning disk* of K^3 if $D \cap \partial M^3 = \partial D$. A *spanning surface* is defined similarly. We will define the *genus* of an orientable 3-manifold with connected boundary to be the genus of the boundary. A *Heegaard splitting* of a closed 3-manifold M^3 is a pair (U, V) where U and V are cubes-with-handles in M^3 such that $M^3 = U \cup V$ and $U \cap V = \partial U = \partial V$. The *Heegaard genus* of M^3 is the genus of U and of V.

Let M^2 be a 2-manifold. We can *attach* a (3-dimensional) 1-*handle* to M^2 by identifying two disjoint disks on the boundary of a 3-cell with two disjoint disks on M^2. We can attach a 2-handle to M^2 along a simple closed curve $J \subset M^2$ by identifying an annulus in the boundary of a 3-cell with an annular regular neighborhood of J in M^2.

A cube-with-handles of genus n is the 3-manifold obtained by attaching n 1-handles to the boundary of a 3-ball. A *set of handle disks* for a cube-with-handles H^3 of genus n is a collection $D_1, \ldots, D_n$ of pair-wise disjoint spanning disks of H^3 so that $\bigcup D_i$ does not separate H^3. Then the closure of the complement of a regular neighborhood of $\bigcup D_i$ in H^3 will be a 3-cell.

2. Degree one mappings from 3-manifolds onto 3-manifolds

THEOREM 2.1. *Let M^3 and N^3 be closed orientable 3-manifolds and let (U, V) be a Heegaard splitting of N^3. Let $f\colon M^3 \to N^3$ be a degree one mapping. Then f is homotopic to a monotone mapping $g\colon M^3 \to N^3$ so that $g \mid g^{-1}(U)$ is a homeomorphism.*

Proof. This is a direct consequence of Theorem 8.3 of [12].

THEOREM 2.2. *Let M^3 and N^3 be orientable 3-manifolds with boundary, and let $U_1, U_2, \ldots, U_n$ be a collection of 1-handles in N^3 attached to ∂N^3 so that $\mathrm{cl}\,(N^3 - \bigcup U_i)$ is a cube-with-handles. Let $f\colon M^3 \to N^3$ be a boundary preserving mapping; then f is homotopic to a boundary preserving mapping $g\colon M^3 \to N^3$ so that $g \mid g^{-1}(\bigcup U_i)$ is a homeomorphism. The homotopy can be chosen to be constant on ∂M^3.*

Proof. This is a direct consequence of Theorem 8.4 of [12].

3. Boundary-retractable 3-manifolds with boundary

Let K^3 be a compact orientable 3-manifold whose boundary is a connected surface of genus n. Then K^3 is said to be *boundary-retractable* if there exists a wedge P of n simple closed curves in ∂K^3 and a retraction $r\colon K^3 \to P$.

THEOREM 3.1. *Let K^3 be a compact orientable 3-manifold whose boundary is a connected surface of genus n. Then the following are equivalent:*

(i) *K^3 is boundary-retractable;*

(ii) *there exist n pairwise disjoint connected orientable spanning surfaces $F_1, \ldots, F_n$ in K^3, each with connected boundary, so that $\bigcup \partial F_i$ does not separate ∂K^3;*

(iii) *there is a boundary preserving mapping from K^3 onto a cube-with-handles of genus n.*

Proof. The equivalence of (i) and (iii) is essentially Theorem 3 of [5]. In [5] it is assumed that K^3 can be embedded in S^3, however this assumption is not necessary for the proof. Condition (ii) is an intermediate step in the proof.

THEOREM 3.2. *Let K^3 be a compact orientable 3-manifold with connected boundary. Let K_1^3 and K_2^3 be submanifolds of K^3 so that $K_1^3 \cup K_2^3 = K^3$ and $K_1^3 \cap K_2^3$ is a spanning disk of D of K^3. Then K^3 is boundary-retractable if and only if K_1^3 and K_2^3 are boundary-retractable.*

Proof. By using Theorem 3.1 it is easy to see that if K_1^3 and K_2^3 are both boundary-retractable, then K^3 is boundary-retractable.

So let us assume that K^3 is boundary-retractable and has genus n. By Theorem 3.1 there is a boundary-preserving mapping $f: K^3 \to H^3$ where H^3 is a cube-with-handles of genus n. By Dehn's Lemma [11], $f(\partial D)$ bounds a spanning separating disk E in H^3. Let $D_1, D_2, \ldots, D_n$ be a set of handle disks for H^3. We will show how to modify $D_1, D_2, \ldots, D_n$ so that $E \cap (\bigcup D_i) = \emptyset$. We suppose $D_1, \ldots, D_n$ are chosen so that $\bigcup D_i$ is in general position with respect to E and so that the number of components of $E \cap (\bigcup D_i)$ is minimal.

Suppose $E \cap (\bigcup D_i)$ contains a simple closed curve component. We choose such a component which is innermost on E. We replace the disk this component bounds on $\bigcup D_i$ with the disk it bounds on E and push to one side of E. This will modify $\bigcup D_i$ so as to eliminate at least one component of $E \cap (\bigcup D_i)$, so we can assume $E \cap (\bigcup D_i)$ contains no simple closed curve components.

Thus, each component of $E \cap (\bigcup D_i)$ must be an arc. If $E \cap (\bigcup D_i) \neq \emptyset$, let A be a component of $E \cap (\bigcup D_i)$ so that $E = E_1 \cup E_2$ where $E_1 \cap E_2 = A$ and $E_1 \cap (\bigcup D_i) = A$. Then A is contained in some D_j. Replace a regular neighborhood of A in D_j by two disks, each parallel to E_1 and on opposite sides of E_1. The result will be two disks D_{j1} and D_{j2}. We claim that at least one of

$$\partial D_{j1} \cup \left(\bigcup_{i \neq j} \partial D_i \right) \quad \text{and} \quad \partial D_{j2} \cup \left(\bigcup_{i \neq j} \partial D_i \right)$$

does not separate ∂H^3. Suppose $\partial D_{j1} \cup (\bigcup_{i \neq j} \partial D_i)$ separates ∂H^3 into two components U and V where $\partial D_{j2} \subset U$. Let J be a simple closed curve in ∂H^3 which intersects ∂D_j transversely in exactly one point and which does not intersect $\bigcup_{i \neq j} \partial D_i$. We can suppose that the one point of $\partial D_j \cap J$ is contained in ∂D_{j1}. We also suppose J is in general position with respect to $\partial D_{j1} \cup \partial D_{j2}$, and that each point of $J \cap \partial D_{j2}$ corresponds to a point of $J \cap \partial D_{j1}$, and each point of $J \cap \partial D_{j1}$ except for $J \cap D_j$ corresponds to a point of $J \cap \partial D_{j2}$. Since each point of $J \cap D_{j1}$ corresponds to a crossing from U to V or from V to U, J intersects ∂D_{j1} algebraically trivially. Thus, J intersects ∂D_{j2} algebraically once, and $\partial D_{j2} \cup (\bigcup_{i \neq j} \partial D_i)$ does not separate K^3.

Thus, either $D_1, \ldots, D_{j1}, \ldots, D_n$ or $D_1, \ldots, D_{j2}, \ldots, D_n$ is a collection of spanning disks of H^3 whose union does not separate H^3, and whose union does not separate H^3, and whose union has fewer components of intersection with E than $E \cap (\bigcup D_i)$. This is a contradiction, so we must be able to choose $D_1, \ldots, D_n$ so that $E \cap (\bigcup D_i) = \emptyset$.

Suppose $D_1, \ldots, D_n$ are also chosen so the $\bigcup D_i$ is in general position with respect to a triangulation of H^3 for which f is simplicial. Let $F_i = f^{-1}(D_i)$ for $i = 1, \ldots, n$. Then each F_i is an orientable surface with connected boundary. By another cut and paste argument we can modify $F_1, \ldots, F_n$ so that $D \cap (\bigcup F_i) = \emptyset$. By Theorem 3.1, K_1^3 and K_2^3 are boundary-retractable.

In the following theorem, the homology used has integer coefficients.

THEOREM 3.3. *Let K^3 be a genus 2 cube-with-holes. Let J_1 and J_2 be disjoint nontrivial simple closed curves on ∂K^3 which are each homologous to zero in K^3. Suppose J_1 bounds on orientable surface F_1 in K^3 with a spine P which is a wedge of simple closed curves each of which has linking number zero with J_2. Then J_2 bounds an orientable surface F_2 in K^3 which is disjoint from F_1, and K^3 is boundary-retractable.*

Proof. Let F_2 be an orientable spanning surface of K^3 bounded by J_2. Since P does not link J_2, we can modify this surface by adding handles so that it does not intersect P. We assume that the resulting surface, still called F_2, is in general position with respect to F_1. It is not difficult to modify F_2 to eliminate any simple closed curves of $F_1 \cap F_2$ which bound a disk on F_1. Any remaining simple closed curves of $F_1 \cap F_2$ must separate J_1 from P on F_1. If $F_1 \cap F_2 \neq \emptyset$, let C be a simple closed curve of $F_1 \cap F_2$ which is innermost on F_1. Then C bounds a surface E in F_1 which contains P and which intersects F_2 only in C. If C separates F_2, we can replace the surface C bounds in F_2 by E, and push the resulting surface off F_1 to eliminate C as a curve of intersection. If C does not separate F_2, we can replace an annulus regular neighborhood of C on F_2 with two copies of E_j one on each side of F_1. Again, the number of components of $F_1 \cap F_2$ is reduced. Proceeding in this fashion, we modify F_2 so that $F_1 \cap F_2 = \emptyset$. A Theorem 3.1 now implies that K^3 is boundary retractable.

4. A uniqueness theorem

In this section we show that a boundary-retractable cube-with-holes K^3 embedded in S^3 uniquely determines a homotopy 3-sphere M^3 and a mapping $f: S^3 \to M^3$ so that $f \mid \mathrm{cl}\,(S^3 - K^3)$ is a homeomorphism and $f(K^3)$ is a cube-with-handles. Theorem 4.2 contains a generalized version of this result.

If G is a group, and A and B are subsets of G, let $[A, B]$ denote the subgroup of G generated by all commutators of the form $a^{-1}b^{-1}ab$ where $a \in A$ and $b \in B$. If we let $G_1 = G$, $G_2 = [G_1, G]$, and in general $G_{m+1} = [G_m, G]$, then the sequence $G_1, G_2, G_3, \ldots$ is called the *lower central series of G*. Each G_i is a normal subgroup of G, and $G_\omega = \bigcap_{i=1}^{\infty} G_i$ is also normal. Theorem 1 of [5] asserts that if h is a homomorphism from G onto a free group F which induces an isomorphism of G/G_2 onto F/F_2, then $\ker h = G_\omega$.

LEMMA 4.1. *Let K^3 be a compact orientable boundary-retractable 3-manifold with connected boundary of genus n. We also suppose that $H_1(K^3, Z)$ is isomorphic to the direct sum of n copies of the integers. Let $f_1\colon K^3 \to H_1^3$ and $f_2\colon K^3 \to H_2^3$ be boundary preserving mappings of K^3 onto cubes-with-handles H_1^3 and H_2^3. Let J be a simple closed curve in ∂K^3. Then $f_1(J)$ bounds a disk in H_1^3 if and only if $f_2(J)$ bounds a disk in H_2^3.*

Proof. Let $x \in J$, and let

$$f_{1*}\colon \Pi_1(K^3, x) \to \Pi_1(H^3, f_1(x))$$

and

$$f_{2*}\colon \Pi_1(K^3, x) \to \Pi_1(H^3, f_2(x))$$

be the induced maps on fundamental groups. By Theorem 1 of [5], $\ker f_{1*} = G_\omega = \ker f_{2*}$ where G_ω is the intersection of the lower central series of $G = \Pi_1(K^3, x)$. Using Dehn's lemma, we see that $f_i(J)$ bounds a disk in H_i^3 if and only if J represents an element of $\ker f_{i*} = G_\omega$ for $i = 1, 2$.

THEOREM 4.2. *Let M^3 be a compact orientable 3-manifold, possibly with boundary. Let K^3 be a boundary-retractable submanifold with connected boundary. Let $f_1\colon M^3 \to N_1^3$ and $f_2\colon M^3 \to N_2^3$ be mappings onto orientable 3-manifolds N_1^3 and N_2^3 so that for $i = 1, 2$,*

(1) *$f_i \mid \mathrm{cl}\,(M^3 - K^3)$ is a homeomorphism and*

(2) *$f_i \mid K^3$ is a boundary preserving mapping onto a cube-with-handles H_i^3. Then N_1^3 is homeomorphic to N_2^3.*

Proof. Let $Q = \mathrm{cl}\,(M^3 - K^3) \cup \partial K^3$. Then N_1^3 is homeomorphic to the identification space formed by identifying Q and H_1^3 using the homeomorphism $f_1 \mid \partial K^3$. Let $D_1, \ldots, D_n$ be a set of handle disks for H_1^3. The above identification space can also be constructed in two stages as follows: First attach 2-handles to Q along the curves $f_1^{-1}(\partial D_i) \subset \partial K^3$ for $i = 1, \ldots, n$. Then attach a 3-handle to the result so that the 3-handle and the 2-handles form a cube-with-handles which is attached to Q in the same way as H_1^3.

By Lemma 4.1, the simple closed curves $f_2 f_1^{-1}(\partial D_1), \ldots, f_2 f_1^{-1}(\partial D_n)$ bound disks in H_2^3. By a standard cut and past argument, these disks can be chosen to be disjoint. Hence, they will be a set of handle disks for H_2^3. Thus N_2^3 is also homeomorphic to the manifold obtained by attaching 2-handles to Q along the curves $f_1^{-1}(\partial D_1), \ldots, f_1^{-1}(\partial D_n)$ and attaching a 3-handle to the result.

5. Mappings from S^3 onto homotopy 3-spheres

By a *homotopy* 3-*sphere* we will mean a closed 3-manifold with the same homotopy type as the 3-sphere S^3. A *fake* 3-*sphere* is a homotopy 3-sphere which is not homeomorphic to S^3. A *homotopy* 3-*cell* is a compact contractible 3-manifold with 2-sphere boundary.

Let M^3 be a homotopy 3-sphere. It is not difficult to construct a degree one mapping from S^3 onto M^3. Let $M^3 = B_3^3 \cup B_4^3$ where B_3^3 is a 3-cell, B_4^3 is a homotopy 3-cell, and $B_3^3 \cap B_4^3 = \partial B_3^3 = \partial B_4^3$. Similarly, let S^3 be the union of two 3-cells B_1^3 and B_2^3. First map B_1^3 homeomorphically onto B_3^3. Since $\Pi_2(B_4^3) = 0$, this map can be extended to take B_2^3 onto B_4^3.

Let (U, V) be a Heegaard splitting for M^3. Applying Theorem 2.1, we see that there is a monotone mapping $g\colon S^3 \to M^3$ so that $g \mid g^{-1}(U)$ is a homeomorphism. Then $f^{-1}(V) = K^3$ is a cube-with-holes in S^3 which is the closure of the complement of the handlebody $g^{-1}(U)$. (This result is also Theorem 8 of [3] and can be deduced from either [2] or [9].)

Conversely, let U be a genus n cube-with-handles in S^3, and let $K^3 = \text{cl}\,(S^3 - U)$. If K^3 is boundary-retractable, there is a boundary-preserving mapping f_1 from K^3 onto a genus n cube-with-handles V. If we identify U and V along ∂U and ∂V using the homeomorphism $f_1 \mid \partial U$, we will obtain a 3-manifold M^3 with Heegaard splitting (U, V). A degree one mapping $f\colon S^3 \to M^3$ can be defined by letting $f \mid U = \text{id}$ and $f \mid K^3 = f_1$. Since f has degree one, $f_*\colon \Pi_1(S^3) \to \Pi_1(M^3)$ is an epimorphism by 3.9 (b) of [10], and thus M^3 is a homotopy 3-sphere. By Theorem 4.2 the homeomorphism type of M^3 does not depend on the choice of the map f_1. We will call M^3 the *homotopy* 3-*sphere associated with the cube-with-holes* $K^3 \subset S^3$.

THEOREM 5.1. *Let n be a number so that there are no fake* 3-*spheres of Heegaard genus less than n. Let* K^3 *be a boundary-retractable cube-with-holes in* S^3, *and let* M^3 *be its associated homotopy* 3-*sphere. Suppose* $K^3 = K_1^3 \cup K_2^3$ *where* $K_1^3 \cap K_2^3$ *is a spanning disk D of* K^3, *and where* $H_i^3 = \text{cl}\,(S^3 - K_i^3)$ *is a cube-with-handles for* $i = 1, 2$. *If* K_1^3 *and* K_2^3 *have genus less than n, then* M^3 *is homeomorphic to* S^3.

Proof. By Theorem 3.2, both K_1^3 and K_2^3 are boundary-retractable. Let N^3 be the homotopy 3-sphere associated with $K_1^3 \subset S^3$, and let $f\colon S^3 \to N^3$ be a mapping so that $f \mid H_1^3$ is a homeomorphism and $f(K_1^3)$ is a cube-with-handles. Then $(f(H_1^3), f(K_1^3))$ is a Heegaard splitting of genus less than n, so by assumption N^3 is homeomorphic to S^3. Note that f induces a boundary-preserving mapping from H_2^3 onto $f(H_2^3)$. If $E_1, \dots, E_m$ is a set of handle disks for H_2^3, by Dehn's Lemma and a cut and paste argument, the simple closed curves $f(\partial E_1), \dots, f(\partial E_m)$ bound pairwise disjoint disks in $f(H_2^3)$. Since $N^3 \cong S^3$ is irreducible, $f(H_2^3)$ is a cube-with-handles.

Since $K_2^3 \subset H_1^3$, f embeds K_2^3 in N^3. Let M_1^3 be the homotopy 3-sphere associated with $f(K_2^3) \subset N^3$. Again, M_1^3 has Heegaard genus less than n, so

M_1^3 is homeomorphic to S^3. But $gf \mid \text{cl}\,(S^3 - K^3)$ is a homeomorphism, and $gf(K^3)$ is a cube-with-handles, so by Theorem 4.2 M^3 is homeomorphic to M_1^3.

THEOREM 5.2. *Let K^3 be a genus 2 boundary-retractable cube-with-holes in S^3 so that $H^3 = \text{cl}\,(S^3 - K^3)$ is a cube-with-handles. Let M^3 be the associated homotopy 3-sphere. If K^3 contains a spanning disk D such that ∂D does not bound a disk on ∂K^3, then M^3 is homeomorphic to S^3.*

Proof. Let $f\colon S^3 \to M^3$ be a mapping so that $f \mid H^3$ is a homeomorphism and $f(K^3)$ is a cube-with-handles. Let $N(D)$ be a regular neighborhood of D in K^3.

Case 1. The disk D does not separate K^3 and $H^3 \cup N(D)$ is a cube with a knotted hole. Then $\text{cl}\,(K^3 - N(D))$ is a solid torus, so K^3 is a cube-with-handles. A homeomorphism from S^3 onto itself satisfies the conditions of Theorem 4.2, so M^3 is homeomorphic to S^3.

Case 2. The disk D does not separate K^3 and $H^3 \cup N(D)$ is a solid torus. By Dehn's Lemma, $f(\partial D)$ bounds a disk F in $f(K^3)$. Let $N(F)$ be a regular neighborhood of F in $f(K^3)$, and let J be a simple closed curve in ∂K^3 which intersects ∂F transversely in one point and which intersects $N(F)$ in an arc. Let B^3 be a 3-cell in $\text{cl}\,(f(K^3) - N(F))$ so that $B^3 \cap \partial F(K^3)$ is a 2-cell containing $J - (N(F) \cap J)$ and $B^3 \cap N(F)$ is two 2-cells. Then $N(F) \cup B^3$ is a solid torus, and there is a spanning disk E of $f(K^3)$ so that $N(F) \cup B^3$ is the closure of one of the components of $f(K^3) - E$. Then the argument given in the proof of Theorem 3.2 shows that there exists a set of handle disks D_1, D_2 for $f(K^3)$ which are disjoint from E. Thus, $\text{cl}\,(f(K^3) - N(F))$ is a solid torus, and

$$(f(H^3) \cup N(F), \text{cl}\,(f(K^3) - N(F)))$$

is a Heegaard splitting for M^3 of genus 1. It is well known that any homotopy 3-sphere of Heegaard genus 1 is homeomorphic to S^3.

Case 3. The disk D separates K^3. Let $K^3 = K_1^3 \cup K_2^3$ where $K_1^3 \cap K_2^3 = D$. If either K_1^3 and K_2^3 is a solid torus, Case 1 or Case 2 applies. If K_1^3 and K_2^3 are both cubes with knotted holes, their complements are solid tori, and Theorem 5.1 applies.

LEMMA 5.3. *Let U^3 be a genus n cube-with-handles. If a 2-handle P^3 is attached to U^3 so that $\Pi_1(U^3 \cup P^3)$ is free on $n - 1$ generators, then $U^3 \cup P^3$ is also a cube-with-handles.*

Proof. Let C be the simple closed curve on ∂U^3 along which P^3 is attached, and let $x \in C$. The group $\Pi_1(U^3 \cup P^3, x)$ has a natural presentation with n generators and one relation given by C. By Theorem N3, p. 167 of [8], C must represent a primitive element in $\Pi_1(U^3, x)$. By [13] or [4], there exists a set of handle disks $E_1, \dots, E_n$ for U^3 so that $C \cap \partial E_1$ is a single transverse point of intersection, and $C \cap \partial E_i = \emptyset$ for $i = 2, \dots, n$. Thus $U^3 \cup P^3$ is homeomorphic to the closure of U^3 minus a regular neighborhood of E_1.

THEOREM 5.4. *Let n be an integer so there is no fake* 3*-sphere of Heegaard genus less than n. Let* K^3 *be a genus n boundary-retractable cube-with-holes in* S^3 *so that* cl $(S^3 - K^3) = H^3$ *is a cube-with-handles. Let* M^3 *be the associated homotopy* 3*-sphere. Let D be a spanning nonseparating disk of* H^3, *and let* $N(D)$ *be a regular neighborhood of D in* H^3. *If* $K^3 \cup N(D)$ *is a cube-with-handles, then* M^3 *is homeomorphic to* S^3.

Proof. Let f: $S^3 \to M^3$ be a mapping so that $f \mid H^3$ is a homeomorphism, and $f(K^3)$ is a cube-with-handles. Let $T^3 = K^3 \cup N(D)$ and let $D_1, \ldots, D_{n-1}$ be a set of handle disks for T^3. By Dehn's Lemma, each simple closed curve $f(\partial D_i)$ bounds a disk in $f(T^3)$, and by a standard cut and paste argument, these disks can be assumed to be pairwise disjoint. Thus, the fundamental group of $f(T^3)$ is free on $n - 1$ generators. But $f(T^3)$ is also homeomorphic to the 3-manifold obtained by attaching a 2-handle to the cube-with-handles $f(K^3)$. By Lemma 5.3, $f(T^3)$ is a cube-with-handles. Then $(f(T^3), f(\text{cl}\ (H^3 - n(D))))$ is a genus $n - 1$ Heegaard splitting for M^3, and M^3 is homeomorphic to S^3.

THEOREM 5.5. *Let* K^3 *be a genus* 2 *boundary-retractable cube-with-holes in* S^3, *where* cl $(S^3 - K^3) = H^3$ *is a cube-with-handles. If there exists a nontrivial unknotted simple closed curve J in* $S^3 - K^3$, *then the associated homotopy* 3-sphere M^3 *is homeomorphic to* S^3.

Proof. Let D be a disk bounded by J which is in general position with respect to ∂K^3. Then each component of $D \cap \partial K^3$ is a simple closed curve. If one of these simple closed curves bounds a disk on ∂K^3, using a standard cut and paste argument, we can modify D to eliminate all such components of $D \cap \partial K^3$. We must have $D \cap \partial K^3 \neq \emptyset$ by our assumption on the nontriviality of J. Let E be a subdisk of D so that $E \cap \partial K^3 = \partial E$. If $E \subset K^3$, then Theorem 5.2 implies that M^3 is homeomorphic to S^3. So we suppose $E \subset H^3$. Let $N(E)$ be a regular neighborhood of E in H^3. Then $T^3 = \text{cl}\ (H^3 - N(E))$ is a solid torus, and $J \subset T^3$. Since J is unknotted and nontrivial in T^3, it is not hard to see that cl $(S^3 - T^3) = K^3 \cup N(E)$ is also a solid torus. Then it follows from Theorem 5.4 that M^3 is homeomorphic to S^3.

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