THE QUESTION OF EQUIVALENCE OF PRINCIPAL AND COPRINCIPAL SOLUTIONS OF SELF-ADJOINT DIFFERENTIAL SYSTEMS

BY

CALVIN D. AHLBRANDT

1. Introduction

A useful result in the study of differential systems of the form

$$U' = AU + BV, \quad V' = CU - A^*V \tag{1.1}$$

is that, in certain cases, the system (1.1) is disconjugate in some neighborhood of ∞ if and only if the obverse system [2, p. 173]

$$U' = -A^*U + CV, \quad V' = BU + AV \tag{1.2}$$

is disconjugate in some neighborhood of ∞. Given certain assumptions of a variational nature, Reid [4] has established that system (1.1) is disconjugate in a neighborhood of ∞ if and only if there exists a solution which is principal at ∞. Therefore, in those cases where both sets of hypotheses hold, system (1.1) has a principal solution at ∞ if and only if system (1.2) has a principal solution at ∞. The question considered in [2] was that of when a principal solution $(U_\infty; V_\infty)$ gives rise to a principal solution $(U_1; V_1) = (V_\infty; U_\infty)$ of (1.2), i.e., in the terminology used there, the question of when a principal solution is also coprincipal. The present paper gives a number of conditions which are equivalent to the condition of a solution being principal at ∞ if and only if it is coprincipal at ∞. These conditions involve limit type behavior for every conjoined basis. The statements of these conditions which are given in Section 4 are unchanged in the two cases of B and C having either the same or opposite signs for their associated quadratic forms. The coefficients B and C may be singular if certain "normality" assumptions are included. Whereas the previous work primarily concerned the case where B and C were of "opposite" signs, the present study is directed at the case where B and C are of the "same" sign, although, as mentioned, much of the work applies to both cases. As an example of the type of results obtained, limit information is obtained for $U^{-1}V^{*-1}$ in either case for $(U; V)$ an antiprincipal solution. This result is analogous to a result on principal solutions obtained by Reid in the case of B and C nonnegative. Combining these results leads to the conclusion that a large class of such equations has the property that principal solutions and coprincipal solutions coincide.

The primary results of this paper are consequences of combining the following observations. First, the distinguished solution W_∞ of the Riccati equation

$$W' = C - A^*W - WA - WBW, \tag{1.3}$$

Received February 7, 1975.

which is defined as $W_\infty = V_\infty U_\infty^{-1}$ whenever a principal solution at ∞ exists, is the limit as $s \to \infty$ of $W_s(x) = V(x, s)U^{-1}(x, s)$ where $(U(\ , s); V(\ , s))$ is the solution of (1.1) such that $(U(s, s); V(s, s)) = (0; E)$. Second, the solution $(U(\ , s); V(\ , s))$ may be obtained in terms of any solution $(U; V)$ with U nonsingular in a neighborhood of ∞ by reduction of order and quadrature. Third, a solution of (1.1) is principal and coprincipal at ∞ if and only if the distinguished solution of (1.3) is the multiplicative inverse, at each point of some neighborhood of ∞, of the distinguished solution of the Riccati equation

$$W' = B + AW + WA^* - WCW. \tag{1.4}$$

Matrices of one column are called vectors, all identity matrices are denoted by E, and 0 is used for the zero matrix of any dimensions. If H and K are $n \times r$ matrices, or functions, the symbol $(H; K)$ is used to denote the $2n \times r$ partitioned matrix with first, [last], n rows agreeing with H, [K]. If H and K are $n \times n$ hermitian matrices, we write $H \geq K$, [$H > K$], to indicate that the quadratic form $\pi^*(H - K)\pi$ is nonnegative [positive] for every unit column vector π. Matrix valued functions will be said to have a property of differentiability, continuity, or integrability if and only if each entry of the matrix has that property.

2. Definitions and hypothesis

We will assume throughout the paper that the following hypotheses **H** and **N** are satisfied.

H. *The coefficients A, B, and C in system* (1.1) *are continuous $n \times n$ complex matrix valued functions on* $(0, \infty)$ *with $B^* = B$ and $C^* = C$ on* $(0, \infty)$.

N. *Systems* (1.1) *and* (1.2) *are identically normal on* $(0, \infty)$.

The system (1.1) is called *identically normal on* $(0, \infty)$ if the only vector solution $(u; v)$ of the associated vector system

$$u' = Au + Bv, \quad v' = Cu - A^*v \tag{2.1}$$

such that u vanishes on a nondegenerate subinterval of $(0, \infty)$ is the zero solution $(u; v) = (0, 0)$.

Distinct points c and d of $(0, \infty)$ are called *conjugate* relative to (1.1) or (2.1) if there exists a vector solution $(u; v)$ of (2.1) such that $u(c) = 0 = u(d)$ holds and u is not identically zero between c and d. System (2.1) is called *disconjugate* on an interval if that interval contains no conjugate pairs.

It follows from hypothesis **H** that if $(U_1; V_1)$ and $(U_2; V_2)$ are solutions of (1.1), then the function $U_1^*V_2 - V_1^*U_2$ is constant. If $(U; V)$ is a solution such that $U^*V - V^*U$ is 0, then the solution is called *self-conjoined*. If $(U; V)$ is a self-conjoined solution of (1.1) such that the columns of the $2n \times n$ matrix $(U(x); V(x))$ are linearly independent for some value x, and consequently for all x in $(0, \infty)$, then the solution is called a *conjoined basis* [7; p. 306] for (1.1).

We will restrict our discussion to systems of the form (1.1) which satisfy one of the following sets of hypotheses.

H$(B \geq 0;\ C \geq 0)$. *Hypothesis* **H** *holds and* B *and* C *satisfy the conditions* $B \geq 0$ *and* $C \geq 0$.

H$(B \geq 0;\ C \leq 0)$. *Hypothesis* **H** *holds and* B *and* C *satisfy the conditions* $B \geq 0$ *and* $C \leq 0$.

The reason for the study of these cases is that the obverse system (1.2), or the reciprocal system

$$U' = -A^*U - CV, \quad V' = -BU + AV, \tag{2.2}$$

in the respective cases, will also satisfy the Clebsch condition, $B \geq 0$, usually imposed on system (1.1). The following result is basic to our later work.

THEOREM 2.1. *Suppose that condition* **N** *is satisfied and either hypothesis* **H**$(B \geq 0;\ C \geq 0)$ *holds or hypothesis* **H**$(B \geq 0;\ C \leq 0)$ *holds and system* (1.1) *is disconjugate in some neighborhood of* ∞. *If* $(U;\ V)$ *is any conjoined basis for system* (1.1), *then* U *and* V *are nonsingular in some neighborhood of* ∞.

In the first case, the result is a consequence of Theorem 4.1 of Reid [4, p. 155] applied to systems (1.1) and (1.2) together with Lemma 7.1 of Reid [7, p. 357]. In the second case, the result follows from Theorem 3.1 of [1, p. 423] and the above-mentioned Sturmian theorem, Lemma 7.1 of Reid.

3. The distinguished solution

Reid's [4] construction of the principal solution at ∞ of system (1.1) is applicable to either of the cases considered in the hypotheses of Theorem 2.1. For (c, ∞) an interval of disconjugacy for (1.1) and t and s distinct points of (c, ∞), the solution $(U_{st};\ V_{st})$ of (1.1) is defined as the unique solution determined by the boundary conditions $U_{st}(s) = E$, $U_{st}(t) = 0$. Reid established that the limits

$$U_{s,\infty}(x) = \lim_{t\to\infty} U_{st}(x) \quad \text{and} \quad V_{s,\infty}(x) = \lim_{t\to\infty} V_{st}(x)$$

exist and $(U_{s,\infty};\ V_{s,\infty})$ is a principal solution at ∞. In particular, $(U_{s,\infty};\ V_{s,\infty})$ has the properties of being a conjoined basis with $U_{s,\infty}$ nonsingular on (c, ∞) and if $S(x, s;\ U)$ is defined on the class of self-conjoined solutions $(U;\ V)$ with U nonsingular on (c, ∞) by the relation

$$S(x, s;\ U) = \int_s^x U^{-1}BU^{*-1}\, dt, \tag{3.1}$$

for x, s in (c, ∞), then $S(x, s;\ U_{s,\infty})$ is nonsingular for $x \neq s$ and

$$S^{-1}(x, s;\ U_{s,\infty}) \to 0$$

as $x \to \infty$. Since $\lim_{t\to\infty} V_{st}U_{st}^{-1}$ exists and is $V_{s,\infty}U_{s,\infty}^{-1}$, which is defined as the distinguished solution W_∞ of (1.3), we have

$$W_\infty = \lim_{t\to\infty} V_{st}U_{st}^{-1}. \tag{3.2}$$

The distinguished solution is well defined, since any other principal solution is of the form $(U_{s,\infty}C;\ V_{s,\infty}C)$, where C is a nonsingular matrix.

Suppose that $(U;\ V)$ is a conjoined basis for (1.1) and c is such that U is nonsingular on (c, ∞). For t any point of (c, ∞), construct the solution $(U_0(\ \ , t);\ V_0(\ \ , t))$ of (1.1) defined by the equations

$$\begin{aligned} U_0(x, t) &= U(x)\int_x^t U^{-1}(r)B(r)U^{*-1}(r)\,dr, \\ V_0(x, t) &= -U^{*-1}(x) + V(x)\int_x^t U^{-1}(r)B(r)U^{*-1}(r)\,dr. \end{aligned} \tag{3.3}$$

Then $U_0(x, t)$ is nonsingular for $x \neq t$, the solution $(U_0(\ \ , t);\ V_0(\ \ , t))$ is a conjoined basis for (1.1), and the identity

$$U_0^*(x, t)V(x) - V_0^*(x, t)U(x) = E \tag{3.4}$$

holds for all t, x in (c, ∞). For s and t distinct points of (c, ∞), we have

$$(U_{st}(x);\ V_{st}(x)) = (U_0(x, t)U_0^{-1}(s, t);\ V_0(x, t)U_0^{-1}(s, t)).$$

Consequently, if $s < x < t$ holds, then

$$V_{st}(x)U_{st}^{-1}(x) = V_0(x, t)U_0^{-1}(x, t).$$

Substitution in relation (3.2) gives

$$W_\infty(x) = \lim_{t\to\infty} V_0(x, t)U_0^{-1}(x, t) \quad \text{for } x > c. \tag{3.5}$$

Notice that dependence upon s has been removed. Since $U_0(x, t)$ is nonsingular for $t \neq x$, it follows from relation (3.4) that

$$V(x)U^{-1}(x) - U_0^{*-1}(x, t)V_0^*(x, t) = U_0^{*-1}(x, t)U^{-1}(x)$$

for $t \neq x$. Therefore

$$V_0(x, t)U_0^{-1}(x, t) = V(x)U^{-1}(x) - U^{-1}(x)S^{-1}(t, x, U)U^{-1}(x). \tag{3.6}$$

Application of (3.5) gives a characterization [7; Theorem 7.3.4, p. 320] of the distinguished solution as

$$W_\infty(x) = V(x)U^{-1}(x) - U^{*-1}(x)\left(\lim_{t\to\infty} S^{-1}(t, x; U)\right)U^{-1}(x) \tag{3.7}$$

independent of the choice of the conjoined basis $(U;\ V)$ so long as x is chosen from a neighborhood of ∞ where U is nonsingular. In the case where $(U;\ V)$ is principal at ∞, the second term in the right-hand side of (3.7) is 0.

Under the hypotheses of Theorem 2.1, a conjoined basis $(U; V)$ of (1.1) gives rise to a conjoined basis $(U_1; V_1)$ of (1.2) defined by $(U_1; V_1) = (V, U)$. An application of the above remarks to system (1.2) gives the following characterization of the distinguished solution W^0_∞ of equation (1.4). For x in some neighborhood of ∞, the identity

$$(3.8) \qquad W^0_\infty(x) = U(x)V^{-1}(x) - V^{*-1}(x)\left(\lim_{t\to\infty} S_1^{-1}(t, x; V)\right) V^{-1}(x)$$

holds in some neighborhood of ∞, where S_1 is defined by

$$S_1(t, x; V) = \int_x^t V^{-1}CV^{*-1}\, ds.$$

4. A characterization of the equivalence of principal and coprincipal solutions

Given the hypotheses of Theorem 2.1 and a self-conjoined solution $(U; V)$ of system (1.1), we know that $(U; V)$ is principal at ∞ if and only if $W_\infty = VU^{-1}$ holds in some neighborhood of ∞ and $(U; V)$ is coprincipal at ∞ if and only if $W^0_\infty = UV^{-1}$ holds in some neighborhood of ∞. Consequently, in those situations, a self-conjoined solution $(U; V)$ is principal and coprincipal if and only if the relation $W_\infty W^0_\infty = E$ holds in some neighborhood of ∞. However, relations (3.7) and (3.8) are valid for any conjoined basis $(U; V)$. These observations lead to the following theorem.

THEOREM 4.1. *Suppose that the hypotheses of Theorem* 2.1 *hold. Then the following conditions are equivalent.*

(i) *A solution of* (1.1) *is principal at* ∞ *if and only if it is coprincipal at* ∞.

(ii) *The distinguished solutions at* ∞ *of the Riccati equations* (1.3) *and* (1.4) *are multiplicative inverses in some neighborhood of* ∞.

(iii) *There exists a conjoined basis* $(U; V)$ *for* (1.1) *with the property that the equation*

$$(4.1) \qquad MU^{-1}V^{*-1}N = M + N$$

holds in some neighborhood of ∞, *where* M *and* N *are the hermitian functions defined by*

$$M(x) = \lim_{s\to\infty} S^{-1}(s, x, U), \quad N(x) = \lim_{s\to\infty} S_1^{-1}(s, x, V)$$

where S *and* S_1 *are defined for large* x *and* s *by*

$$S(s, x, U) = \int_x^s U^{-1}BU^{*-1}\, dt, \; S_1(s, x, V) = \int_x^s V^{-1}CV^{*-1}\, dt.$$

(iv) *Every conjoined basis* $(U; V)$ *for* (1.1) *has the property given in condition* (iii).

(v) *There exists a conjoined basis* $(U; V)$ *for* (1.1) *such that the relation*

$$(4.2) \qquad \lim_{s\to\infty} S^{-1}(s, x; U)U^{-1}(s)V^{*-1}(s)S_1^{-1}(s, x; V) = 0$$

holds for large x.

(vi) *The limit relation* (4.2) *holds for every conjoined basis* $(U;\ V)$ *of* (1.1).

(vii) *There exists a conjoined basis* $(U;\ V)$ *of* (1.1) *for which* $\lim_\infty U^{-1}V^{*-1}$ *exists and is* 0.

(viii) *Every solution* $(U;\ V)$ *of* (1.1) *which is antiprincipal at* ∞ *is also anticoprincipal at* ∞ *and has the properties*

(α) $\lim_\infty U^{-1}V^{*-1}$ *exists and is* 0, *and*

(β) *the equation* $U^{-1}(x)V^{*-1}(x) = \int_x^\infty U^{-1}BU^{*-1}\,dt + \int_x^\infty V^{-1}CV^{*-1}\,dt$ *is satisfied for large* x.

Conditions (i) through (iv) are equivalent as a consequence of prior remarks. Relation (4.1) of Lemma 4.1 in [2] provides the identity

$$S(s, x, U) + S_1(s, x, V) + U^{-1}(s)V^{*-1}(s) = U^{-1}(x)V^{*-1}(x) \tag{4.3}$$

for x and s in some neighborhood of ∞ dependent upon which conjoined basis $(U;\ V)$ is used. Premultiplication by S^{-1} and postmultiplication by S_1^{-1} of equation (4.3) followed by letting s become unbounded gives the identity

$$\begin{aligned}\lim_{s\to\infty} & S^{-1}(s, x, U)U^{-1}(s)V^{*-1}(s)S_1^{-1}(s, x, V)\\ &= M(x)U^{-1}(x)V^{*-1}(x)N(x) - M(x) - N(x)\end{aligned} \tag{4.4}$$

for large x. Therefore, condition (iii) is equivalent to (v) and (iv) is equivalent to (vi). Equation (4.1) implies that M is nonsingular if and only if N is nonsingular, i.e., condition (iv) implies that a conjoined basis is antiprincipal at ∞ if and only if it is anticoprincipal at ∞. But if $M(x)$ and $N(x)$ are nonsingular, then relation (4.2) implies that $U^{-1}(s)V^{*-1}(s)$ has limit 0; furthermore, S and S_1 have limits and condition (β) follows from relation (4.3). Hence condition (viii) follows from the combined conditions (iv) and (vi). Condition (v) follows from condition (vii) since S^{-1} and S_1^{-1} always have limits.

THEOREM 4.2. *Suppose that hypotheses* $\mathbf{H}(B \geq 0;\ C \leq 0)$ *and* $\mathbf{N}$ *hold. Assume that system* (1.1) *is disconjugate in some neighborhood of* ∞ *and for* D *a fundamental solution of* $D' = AD$, *we have the condition*

$$\pi^* \left(\int_a^x D^{-1}BD^{*-1}\,dt\right) \pi \to \infty \quad \text{as } x \to \infty$$

for every constant unit vector π. *Then all of the conditions* (i) *through* (viii) *of Theorem* 4.1 *hold.*

The statement of Theorem 4.1 of [2] gives the result that principal implies coprincipal. Now suppose that $(U_1;\ V_1)$ is a coprincipal solution. Then for $(U;\ V)$ a principal solution, $(U;\ V)$ is also coprincipal and there exists a nonsingular $n \times n$ constant matrix C such that $(U_1;\ V_1) = (U;\ V)C$ holds. Thus $(U_1;\ V_1)$ is a principal solution. Therefore, condition (i) and the hypotheses of Theorem 4.1 above hold.

5. The case of B and C nonnegative

Suppose that hypotheses $\mathbf{H}(B \geq 0;\ C \geq 0)$ and $\mathbf{N}$ hold. Then the relation

$$U^*(b)V(b) - U^*(a)V(a) = \int_a^b (V^*BV + U^*CU)\, dt \tag{5.1}$$

implies that if $(U;\ V)$ is any self-conjoined basis, then U^*V is increasing.

In the case of $(U_\infty;\ V_\infty)$ principal or coprincipal at ∞, it follows from Theorem 8.1 of [4] and a duality argument that the equation

$$-U_\infty^*(x)V_\infty(x) = \int_x^\infty (V_\infty^*BV_\infty + U_\infty^*CU_\infty)\, dt \tag{5.2}$$

holds for all positive x. Therefore, $U_\infty^*V_\infty$ is negative definite on $(0, \infty)$ and increasing to 0. In comparison, notice that $U^*(x, s)V(x, s)$ is positive definite for x in (s, ∞) and increasing, hence the function $U^{-1}(\ \ , s)V^{*-1}(\ \ , s)$ has a limit at ∞. Relation (4.3) then yields the result that $(U(\ \ , s);\ V(\ \ , s))$ is antiprincipal and anticoprincipal at ∞. If the assumption of principal if and only if coprincipal is added, then the antiprincipal and anticoprincipal solutions satisfy conditions (α) and (β) of Theorem 4.1 (viii). In particular, in that case $U^{-1}(\ \ , s)V^{*-1}(\ \ , s)$ has limit 0 at ∞ and any antiprincipal solution $(U;\ V)$ has $U^{-1}V^{*-1}$ positive definite and decreasing to 0. It is to be noted that whereas condition (5.2) holds for principal solutions, examples exist where condition (α) does not hold for antiprincipal or anticoprincipal solutions. Indeed, in the case where a solution $(U;\ V)$ is both principal and anticoprincipal at ∞, both V^*U and $U^{-1}V^{*-1}$ must go to 0, contrary to their product being the identity matrix. A specific example is given by the solution $\sinh(x^{-1})$ of $(y'/x^{-2})' - x^{-2}y = 0$, which is a transformation of $y'' - y = 0$ at 0. In the case of s an interior point, $(U(\ \ , s);\ V(\ \ , s))$ is principal and anticoprincipal at s and none of the conditions of Theorem 4.1 hold.

THEOREM 5.1. *Suppose that hypotheses* $\mathbf{H}(B \geq 0;\ C \geq 0)$ *and* $\mathbf{N}$ *hold. If* $(U;\ V)$ *is a self-conjoined basis for system* (1.1), *then* U^*V *is increasing on* $(0, \infty)$. *For* s *any point of* $(0, \infty)$, *the solution* $(U(\ \ , s);\ V(\ \ , s))$ *is antiprincipal and anticoprincipal at* ∞ *and* $U^*(\ \ , s)V(\ \ , s)$ *is positive definite on* (s, ∞). *If* $(U;\ V)$ *is principal or coprincipal at* ∞, *then relation* (5.2) *holds for all* x *in* $(0, \infty)$ *and the function* U^*V *is negative definite on* $(0, \infty)$ *and has limit* 0.

THEOREM 5.2. *Suppose that hypotheses* $\mathbf{H}(B \geq 0;\ C \geq 0)$ *and* $\mathbf{N}$ *hold and the classes of principal and coprincipal solution of* (1.1) *coincide. If* $(U;\ V)$ *is antiprincipal or anticoprincipal at* ∞, *then condition* (β) *of Theorem* 4.1 (viii) *holds,* $U^{-1}V^{*-1}$ *is positive definite on its maximal domain of existence and decreasing to* 0.

The analogy between these results leads to the following interconnection and the added conclusion that there exists a class of equations which satisfy hypothesis $\mathbf{H}(B \geq 0;\ C \geq 0)$ and $\mathbf{N}$ and have the property of the classes of principal and coprincipal solutions coinciding.

THEOREM 5.3. *If hypotheses* $\mathbf{H}(B \geq 0;\ C \geq 0)$ *and* $\mathbf{N}$ *hold and* $(U_\infty;\ V_\infty)$ *is a principal solution at* ∞ *of equation* (1.1), *then* $(U_1;\ V_1)$ *defined as*

$$(-U_\infty^{*-1},\ V_\infty^{*-1})$$

is a solution of the system

$$U' = A_1 U + B_1 V, \quad V' = C_1 U - A_1^* V, \tag{5.3}$$

where $A_1 = -A^*$, $B_1 = WBW$, $C_1 = W_1 C W_1$, *for* $W = V_\infty U_\infty^{-1}$, $W_1 = U_\infty V_\infty^{-1}$, *which has the properties that the integrals*

$$\int_1^\infty U_1^{-1} B_1 U_1^{*-1}\, dt \quad \text{and} \quad \int_1^\infty V_1^{-1} C_1 V_1^{*-1}$$

are positive definite and convergent and $\lim_\infty U_1^{-1} V_1^{*-1}$ *exists and is* 0. *If, in addition, system* (5.3) *satisfies hypothesis* $\mathbf{N}$, *then system* (5.3) *has the property of a solution being principal at* ∞ *if and only if it is coprincipal at* ∞. *In particular, this result holds if B and C are positive definite on* $(0, \infty)$.

Relation (5.2) gives the result

$$U_1^{-1}(x) V_1^{*-1}(x) = \int_1^\infty U_1^{-1} B_1 U_1^{*-1}\, dt + \int_x^\infty V_1^{-1} C_1 V_1^{*-1}\, dt.$$

For D a fundamental solution of $D' = AD$, define functions G and F by $G = D^{-1} B D^{*-1}$ and $F = D^* C D$. For $(Y;\ Z)$ and $(U;\ V)$ related by $Y = D^{-1} U$ and $Z = D^* V$, hypothesis $\mathbf{H}$ implies that $(U;\ V)$ is a self-conjoined solution of (1.1) if and only if $(Y;\ Z)$ is a self-conjoined solution of the system

$$Y' = GZ, \quad Z' = FY. \tag{5.4}$$

THEOREM 5.4. *Suppose that hypotheses* $\mathbf{H}(G \geq 0,\ F \geq 0)$ *and* $\mathbf{N}$ *hold and we have the condition*

$$\pi^* \left(\int_a^x G(t)\, dt \right) \pi \to \infty \quad \text{as } x \to \infty$$

for every constant unit vector π. *If* $(Y;\ Z)$ *is a principal or coprincipal solution of* (5.4) *at* ∞, *then*

$$\pi^* \left(\int_a^x Y Z^{-1} F Z^{*-1} Y^*\, dt \right) \pi \to \infty \quad \text{as } x \to \infty$$

for every constant unit vector π.

For W_1 defined as the hermitian function YZ^{-1}, we have

$$W_1' = G - W_1 F W_1$$

and

$$\int_a^x W_1 F W_1\, dt = W_1(a) - W_1(x) + \int_a^x G(t)\, dt.$$

Since $W_1 = Z^{-1*}(Y^* Z) Z^{-1}$, we conclude from Theorem 5.1 that W_1 is negative definite on $(0, \infty)$ and the conclusion of the theorem follows.

COROLLARY. *Suppose that r and p are positive continuous real valued functions on* $(0, \infty)$ *and* $\int_1^\infty r^{-1}(t)\,dt$ *diverges. Then a solution of*

$$(ry')' - py = 0 \tag{5.5}$$

is principal at ∞ *if and only if it is coprincipal at* ∞. *Consequently, we have the following results for any pair of real linearly independent solutions of* (5.5):

(i) $\lim_\infty y_1/y_2$ *exists and is* 0 *if and only if* $\lim_\infty y_1'/y_2'$ *exists and is* 0.

(ii) $\lim_\infty y_1/y_2$ *exists and is nonzero if and only if* $\lim_\infty y_1'/y_2'$ *exists and is nonzero.*

The same conclusions for the equation $(ry')' + py = 0$ with r and p positive were given in Section 5 of [2].

Notice that for the case of $n = 1$, we have the result

$$(Y^*Y)' = Z^*GY + Y^*GZ = 2r^{-1}Y^*Z$$

in which the last member is negative by Theorem 5.1. Hence y^2 is decreasing, $\int_1^\infty [p/(ry')^2]\,dt$ diverges, and y is coprincipal.

THEOREM 5.5. *Suppose that G is the identity matrix E and there exists a continuous real valued function f such that* $F \geq fE$ *holds on* $(0, \infty)$ *and* $\int_1^\infty f(t)\,dt$ *diverges. Then the functions* $Y^{-1}(\ \ , s)$ *and* $Y'^{-1}(\ \ , s)Y^{*-1}(\ \ , s)$ *have limit* 0 *at* ∞ *and a solution is principal at* ∞ *if and only if it is coprincipal at* ∞.

It follows from the result

$$(Y^*(\ \ , s)Y(\ \ , s))' = 2Y^*(\ \ , s)Z(\ \ , s)$$

and Theorem 5.1 that $Y^*(\ \ , s)Y(\ \ , s)$ is increasing and the integral

$$\int_{s+1}^\infty (Y^*(t, s)Y(t, s))^{-1}\,dt$$

converges. Hence $Y^{-1}(t, s) \to 0$ as $t \to \infty$. Relation (5.1) provides the result that

$$\pi^*Y^*(x, s)Y'(x, s)\pi > \int_s^x \pi^*Y^*(t, s)Y(t, s)\pi f(t)\,dt$$

for every constant unit vector π. The conclusion concerning

$$(Y^*(x, s)Y'(x, s))^{-1}$$

follows since all proper values of $Y^*(t, s)Y(t, s)$ tend to ∞ as t becomes infinite.

REFERENCES

1. CALVIN D. AHLBRANDT, *Equivalent boundary value problems for self-adjoint differential systems*, J. Differential Equations, vol. 9 (1971), pp. 420–435.

2. ———, *Principal and antiprincipal solutions of self-adjoint differential systems and their reciprocals*, Rocky Mountain J. Math., vol. 2 (1972), pp. 169–182.

3. PHILIP HARTMAN, *Self-adjoint, non-oscillatory systems of ordinary, second order, linear differential equations*, Duke Math. J., vol. 24 (1957), pp. 25–35.
4. WILLIAM T. REID, *Principal solutions of non-oscillatory self-adjoint linear differential systems*, Pacific J. Math., vol. 8 (1958), pp. 147–169.
5. ———, *Oscillation criteria for self-adjoint differential systems*, Trans. Amer. Math. Soc., vol. 101 (1961), pp. 91–106.
6. ———, *Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems*, Pacific J. Math., vol. 13 (1963), pp. 665–685.
7. ———, *Ordinary differential equations*, Wiley-Interscience, New York, 1971.

UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI