APPROXIMATION WITH INTERPOLATORY CONSTRAINTS

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In this article, we are interested in questions of the existence of approximating functions which have certain approximation properties with respect to a given function f and, at the same time, interpolate the values of f and/or its derivatives. By "approximation properties" we refer to questions of uniform approximation and/or the degree of approximation. A common feature of the results presented here is the method of proof. In each case a set of auxiliary approximating functions which "surround" f is considered. The approximations to these auxiliary functions by the given approximating functions are then found to have the desired approximation properties and to have the given f in their convex hull.

With regard to the questions of uniform approximation, we have the following generalization of the Stone-Weierstrass Theorem:

THEOREM 1. *Let $\mathscr{A}$ be an algebra of real valued continuous functions on a compact set K, and suppose $\mathscr{A}$ separates points. Let $f \in C(K), x_1, x_2, \ldots, x_k \in K$, and $\varepsilon > 0$ be given. Then there is some $p \in \mathscr{A}$ such that*

$$p(x_i) = f(x_i), \quad i = 1, \ldots, k$$

and

$$\sup_{x \in K} |f(x) - p(x)| = \|f - p\| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Let $\mathscr{G}(\varepsilon) = \{g_1, g_2, \ldots, g_{2^k}\}$ be a set of functions in $C(K)$ with the following properties:

(i) $g_j(x_i) = f(x_i) \pm c_{ij}$ with $0 < c_{ij} < \varepsilon/2, i = 1, \ldots, k, j = 1, \ldots, 2^k$.

(ii) $\{g_j(x_i) - f(x_i)\} = \{y_i\}$ takes on the 2^k possible signatures in E^k as j varies from 1 to 2^k. (In other words, for any $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ with $\alpha_1, \alpha_2, \ldots, \alpha_k \neq 0$ there is some j such that $[g_j(x_i) - f(x_i)]\alpha_i > 0, i = 1, \ldots, k$.)

(iii) $\|g_i - f\| < \varepsilon/2, j = 1, \ldots, k$.

(Such a class $\mathscr{G}$ of functions can be seen to exist by perturbing f slightly in a continuous manner in a neighborhood of the points $x_i, i = 1, \ldots, k$.)

By the Stone-Weierstrass Theorem, for each $j, j = 1, \ldots, 2^k$, there is $p_j \in \mathscr{A}$ such that $\|p_j - g_j\| < \eta$ where $\eta = \min c_{ij}$. Clearly, $\{p_j(x_i) - f(x_i)\}$ has the same signature as $\{g_j(x_i) - f(x_i)\}, i = 1, \ldots, k$. To each such p_j we correspond a vector $z_j \in E^k$ by

$$z_j = (p_j(x_1) - f(x_1), \ldots, p_j(x_k) - f(x_k)). \tag{1}$$

Received February 1, 1975.

Since the vectors z_j are in the same orthant (i.e., have the same signature) as the corresponding vectors $w_j = \{g_j(x_i) - f(x_i)\}$, and since the w_j have an element in each orthant, 0 is in the convex hull of the $z_j, j = 1, \ldots, 2^k$. Then there are $b_1, b_2, \ldots, b_{2^k} \geq 0$ with $\sum_{j=1}^{2^k} b_j = 1$ such that

$$\sum_{j=1}^{2^k} b_j z_j = (0, 0, \ldots, 0). \tag{2}$$

Let $p(x) = \sum_{j=1}^{2^k} b_j p_j(x)$. Clearly, $p \in \mathscr{A}$. From (1) and (2),

$$p(x_i) = f(x_i), \quad i = 1, \ldots, k. \tag{3}$$

Also,

$$\begin{aligned} \|f - p\| &= \|\sum b_j(f - p_j)\| \\ &\leq \sum b_j \|f - p_j\| \\ &\leq \sum b_j(\|f - g_j\| + \|g_j - p_j\|) \\ &\leq \sum b_j(\varepsilon/2 + \eta) = \varepsilon/2 + \eta < \varepsilon. \end{aligned} \tag{4}$$

By (3) and (4), p has the desired properties. ∎

By Jackson's classic theorem, if $f(x)$ has modulus of continuity $\omega(f; \delta)$ on $[-1, 1]$, there is a sequence $\{p_n(x)\}$ where p_n is a polynomial of degree $\leq n$ such that $\|f - p_n\| < c\omega(f; 1/n)$ where c is an absolute constant. Paszkowski [4] was the first to prove that this result could be obtained with a sequence of polynomials interpolating f at m prescribed points with the constant c depending only on $(x_1, \ldots, x_m)$. The following result of Teljakovskii [6] is an extension of Jackson's Theorem in another direction:

LEMMA. *If $f \in C^k[-1, 1]$, $k = 0, 1, \ldots,$ then there is a sequence $\{p_n(x)\}$ where $p_n(x)$ is a polynomial of degree $\leq n$ such that*

$$\|f^{(i)} - p_n^{(i)}\| < \frac{c}{n^{k-i}} \omega(f^{(i)}; 1/n),$$

$i = 0, 1, \ldots, k$, and for all n, where c is a constant depending only on k.

We prove the following theorem which is an extension of the original Jackson result simultaneously in both directions:

THEOREM 2. *Let $f \in \mathscr{C}^k[-1, 1]$, $k = 0, 1, \ldots,$ and let $x_1, x_2, \ldots, x_m$ be any prescribed points of $[-1, 1]$. Then there is a constant $d > 0$ (d depends on $x_1, \ldots, x_m$ and on k, but not on f) and a sequence $\{p_n(x)\}$ where $p_n(x)$ is a polynomial of degree $\leq n$ such that for all sufficiently large n,*

$$p_n^{(i)}(x_j) = f^{(i)}(x_j), \quad j = 1, 2, \ldots, m; \; i = 0, 1, \ldots, k \tag{5}$$

and

$$\|f^{(i)} - p_n^{(i)}\| < \frac{d}{n^{k-i}} \omega(f^{(i)}; 1/n), \quad i = 0, 1, \ldots, k. \tag{6}$$

Our proof will make use of the lemma, but not of Paszkowski's Theorem. (Then, taking $k = 0$ in Theorem 2, we will have a new proof of Paszkowski's result.)

Proof. Let $\mathscr{S}$ be the set of all $(k + 1) \times m$ matrices each of whose elements are 1 or -1. $\mathscr{S}$ has $N = 2^{mk+m}$ elements. For each sufficiently large integer n we define a set $\mathscr{G}_n$ of N functions corresponding to the N elements of $\mathscr{S}$:

Let $S \in \mathscr{S}$ be given. First, in a small neighborhood of each x_j, $g(x)$ is defined as a polynomial interpolating the data

$$g^{(i)}(x_j) = f^{(i)}(x_j) \pm \frac{2c}{n^{k-i}}\,\omega(f^{(i)};\, 1/n) \tag{7}$$

where c is the constant in the lemma, with the signature $\pm$ taken according to whether $a_{i+1,j} = \pm 1$ in S, $i = 0, \ldots, k$; $j = 1, \ldots, m$. Then, if the neighborhoods about x_j are taken to be sufficiently small,

$$|g^{(i)}(x) - f^{(i)}(x)| \le \frac{3c}{n^{k-i}}\,\omega(f^{(i)};\, 1/n) \tag{8}$$

holds for the subset of $[-1, 1]$ for which g has been defined. We need an estimate for $\omega(g^{(i)}; \delta)$. On each separate neighborhood for which g has been defined, by (8),

$$\omega(g^{(i)};\, \delta) = \max_{|h| \le \delta} \Delta g^{(i)} \le \max_{|h| \le \delta} \left[\Delta f^{(i)} + \frac{6c}{n^{k-1}}\,\omega(f^{(i)};\, 1/n)\right],$$

where Δf denotes $f(x + h) - f(x)$. In estimating $\omega(g^{(i)}; \delta)$ for the entire subset of $[-1, 1]$ on which g has been defined, we must consider the possibility that the maximum in the above inequality is taken on for x, $x + h$ in separate neighborhoods, which may come about if $\delta \ge \eta = \min_{j \ne j'} |x_j - x_{j'}|$. In this case

$$\begin{aligned}\omega(g^{(i)};\, \delta) &< \max_{h \le \delta} \left[\Delta f^{(i)} + \frac{6c}{n^{k-i}}\,\frac{\delta}{\eta}\,\omega(f^{(i)};\, 1/n)\right] \\ &= \omega(f^{(i)};\, \delta) + \frac{6c}{n^{k-i}}\,\frac{\delta}{\eta}\,\omega(f^{(i)};\, 1/n).\end{aligned}$$

Hence, for $\delta = 1/n$ and $n > 6c/\eta$,

$$\omega(g^{(i)};\, \delta) \le 2\omega(f^{(i)};\, \delta). \tag{9}$$

The segments of polynomials comprising g are then connected "smoothly" in such a way that $g \in C^k[-1, 1]$, (8) remains valid throughout $[-1, 1]$, and (9) remains valid for δ sufficiently small. For the ensuing discussion it is assumed that n is sufficiently large for (9) to hold.

By the lemma, we may define $\mathscr{P}_n$ to be a corresponding set of N polynomials of degree $\le n$ such that the polynomial p corresponding to $g \in \mathscr{G}_n$ satisfies

$$\|g^{(i)} - p^{(i)}\| < \frac{c}{n^{k-i}}\,\omega(g^{(i)};\, 1/n) \le \frac{2c}{n^{k-i}}\,\omega(f^{(i)};\, 1/n), \quad i = 0, \ldots, k. \tag{10}$$

From (8) and (10) we have

$$\|f^{(i)} - p^{(i)}\| \leq \frac{5c}{n^{k-i}} \omega(f^{(i)}; 1/n). \tag{11}$$

Now, to each $S \in \mathscr{S}_n$ there corresponds $g \in \mathscr{G}_n$ to which there corresponds $p \in \mathscr{P}_n$. Furthermore, by (7) and (10) this corresponding polynomial satisfies

$$\begin{aligned} p^{(i)}(x_j) - f^{(i)}(x_j) > 0 &\Leftrightarrow a_{i+1,j} = 1 \\ p^{(i)}(x_j) - f^{(i)}(x_j) < 0 &\Leftrightarrow a_{i+1,j} = -1. \end{aligned} \tag{12}$$

We denote the elements of $\mathscr{P}_n$ by $p_1, p_2, \ldots, p_N$. To each $p_i \in \mathscr{P}_n$ we correspond a point $\alpha_i \in E^{mk+m}$:

$$\begin{aligned} \alpha_i = \{p_i(x_1) - f(x_1), p_i'(x_1) - f'(x_1), \ldots, p_i^{(k)}(x_1) - f^{(k)}(x_1), \\ p_i(x_2) - f(x_2), \qquad \ldots, p_i^{(k)}(x_2) - f^{(k)}(x_2), \ldots, \\ p_i(x_m) - f(x_m), \qquad \ldots, p_i^{(k)}(x_m) - f^{(k)}(x_m)\}, \\ i = 0, 1, \ldots, n. \end{aligned} \tag{13}$$

It follows from (12) and the definition of $\mathscr{S}_n$ that the points α_i take on the N possible signatures in E^{km+m}—i.e., there is exactly one point α_i in each of the N orthants in E^{km+m}. Hence the origin in E^{km+m} lies in the convex hull of $\alpha_1, \alpha_2, \ldots, \alpha_N$ and there are $b_1, b_2, \ldots, b_N \geq 0$ with

$$\sum_{i=1}^{N} b_i = 1$$

such that

$$\sum_{i=1}^{N} b_i \alpha_i = (0, 0, \ldots, 0). \tag{14}$$

Let $p(x) = \sum_{i=1}^{N} b_i p_i(x)$. Then $p(x)$ is a polynomial of degree $\leq n$. It follows from (14) and (13) that

$$\{p^{(i)}(x_i) - f^{(i)}(x_j)\}_{i=0,\ldots,k\ j=1,\ldots,m} = \{0, 0, \ldots, 0\}. \tag{15}$$

But (15) is equivalent to (5). Also,

$$\|f^{(i)} - p^{(i)}\| = \left\| \sum_{\nu=1}^{N} b_\nu [f^{(i)} - p^{(i)}] \right\| \leq \sum_{\nu=1}^{N} b_\nu \|f^{(i)} - p^{(i)}\|.$$

From (11) this is

$$\begin{aligned} &\leq \sum b_\nu \frac{5c}{n^{k-i}} \omega(f^{(i)}; 1/n) \quad \text{for all } n \geq 4c/\eta \\ &\leq \frac{d}{n^{k-i}} \omega(f^{(i)}; 1/n) \qquad \text{for all } n, \end{aligned}$$

establishing (6) and hence the Theorem.

Remark. If S^k is the class of all functions $f \in C^k$ with $f^{(k)} \in \text{Lip } 1$, it is possible, by Theorem 2, to find a sequence of polynomials $\{p_n\}$ that interpolates any element f of S^k and its first k derivatives at prescribed points while simultaneously approximating f at least as well as $O(1/n^{k+1})$. In a classwide sense this error is best possible even for approximation without interpolation. More specifically, there is a function $f \in S^k$ and $a > 0$ such that for all n,

$$E_n(f) = \inf_{p \in \mathscr{P}_n} \|f - p\| \geq \frac{a}{n^{k+1}}$$

where $\mathscr{P}_n$ is the class of all polynomials of degree $\leq n$. However, it might be thought that for each particular $f \in S^k$ it is possible to simultaneously interpolate f and its derivatives and approximate to within $O(E_n(f))$, i.e., to within the order of the degree of best approximation to the particular function f. Interestingly, Platte [5] has shown that this cannot, in general, be the case. (Note that the method of proof of Theorem 2 fails for the class A of functions analytic on $[-1, 1]$—it is impossible to construct the class $\mathscr{G}$ of auxiliary functions with the necessary smoothness condition of analyticity.)

A function $f(x)$ on $[a, b]$ is said to be *piecewise monotone* if $[a, b]$ may be partitioned into a finite number of subintervals on which f is alternately nondecreasing and nonincreasing. $f(x)$ and $g(x)$ are said to be *comonotone* on $[a, b]$ if they are piecewise monotone and are alternately nondecreasing and nonincreasing on the same subintervals. If f is piecewise monotone on $[a, b]$ we denote by $\mathscr{P}_n^*(f)$ the set of all polynomials of degree $\leq n$ comonotone with f on $[a, b]$. The *degree of comonotone approximation* of f, $E_n^*(f)$ is defined by

$$E_n^*(f) = \min_{p \in \mathscr{P}_n^*} \|f - p\|.$$

If S is a set of comonotone functions, the *degree of comonotone approximation to the set* S is given by

$$E_n^*(S) = \sup_{f \in S} E_n^*(f).$$

If f is monotone on $[a, b]$ then $E_n^*(f)$ is called the *degree of monotone approximation* to f. Lorentz and Zeller [2] have shown that for a monotone function f

$$E_n^*(f) = O[\omega(f; 1/n)], \tag{16}$$

while Passow and Raymon [3] have shown that for a piecewise monotone function f and for any $\varepsilon > 0$,

$$E_n^*(f) = O[\omega(f; 1/n^{1-\varepsilon})]. \tag{17}$$

We present a theorem on the degree of approximation to a piecewise monotone function f subject to constraints of both comonotonicity and interpolation:

THEOREM 3. *Let $f(x)$ be continuous and piecewise monotone on $[a, b]$, and let $a \leq x_1 < x_2 < \cdots < x_m \leq b$. Let $\mathscr{P}_n^{**}(f; x_1, \ldots, x_m)$ be the set of polynomials p of degree $\leq n$ comonotone with f and satisfying $p(x_i) = f(x_i)$, $i = 1, \ldots, m$. Then*

$$E_n^{**}(f; x_1, \ldots, x_m) = \min_{p \in \mathscr{P}_n^{**}} \|f - p\| = O[E_n^*(S(\omega_f))]$$

where $S(\omega_f)$ is the set of all functions g such that $\omega(g; \delta) \leq \omega(f; \delta)$ for all $\delta > 0$.

This theorem is proved in the same manner as Theorems 1 and 2, by taking the class of auxiliary functions to be comonotone with f such that their modulus of continuity is of the same order of magnitude as that of the given function f. The desired comonotone interpolating polynomial is then in the convex hull of the comonotone approximations to the auxiliary functions.

Applying (16) and (17) to Theorem 3, we obtain the following:

COROLLARY. (a) *If f is monotone on $[a, b]$, E_n^{**}*

$$E_n^{**}(f; x_1, \ldots, x_m) = O[\omega f; 1/n)];$$

(b) *If f is piecewise monotone on $[a, b]$,*

$$E_n^{**}(f; x_1, \ldots, x_m) = O[\omega(f; 1/n^{1-\varepsilon}] \quad \textit{for any } \varepsilon > 0.$$

The following theorem is corollary to these results:

THEOREM 4. *Let $f(x)$ be a continuous piecewise monotone function with a finite number of zeros on $[a, b]$ (i.e., piecewise positive). Then there is a sequence $\{p_n(x)\}$ with p_n a polynomial of degree $\leq n$ such that:*

(i) *for n sufficiently large p_n and f are comonotone and copositive (i.e., $p_n f \geq 0$) on $[a, b]$, and*

(ii) *$p_n \to f$ uniformly on $[a, b]$.*

Estimates for the degree of approximation to f are the same as those in the above corollary.

Proof. Let $x_1, \ldots, x_m$ be the zeros of f and apply the above corollary. For n sufficiently large the result follows.

Finally, we state a theorem on the simultaneous approximation and interpolation of a function in E^k. We do not include the proof because it can be proved by a method very similar to the methods in the proofs of Theorems 1 and 2; also, it is an immediate corollary of a recent result of D. J. Johnson [1, Theorem 1]:

THEOREM. *Let X be a compact subset of E^k. Let $x_1, x_2, \ldots, x_m \in X$ and let $f \in C(X)$. There is a constant $d > 0$ (d depends on $x_1, \ldots, x_m$ and f, but not on n) and a sequence $\{p_n(x)\}$ with p_n a polynomial of degree $\leq n$ such that for all sufficiently large n:*

(i) *$p_n(x_i) = f(x_i)$, $i = 1, 2, \ldots, m$; and*

(ii) *$\|p_n - f\| \leq d\omega(f; 1/n)$ where $\omega(f; \delta)$ is the modulus of continuity of f.*

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