A RELATIONSHIP BETWEEN GROUP COHOMOLOGY CHARACTERISTIC CLASSES

BY

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1. Suppose that G, H, and Π are groups with G nonAbelian and that $E: 1 \to G \to H \to \Pi \to 1$ is an extension of G by Π. Then E induces a "semi-action" of Π on G, $\phi: \Pi \to \text{Out } G$ (where $\text{Out } G = \text{Aut } G/\text{In } G$). The classical question is: given a $\phi: \Pi \to \text{Out } G$, is there an extension E which induces ϕ? The classical answer by Eilenberg and MacLane [5] is: there is an extension if and only if a certain group cohomology obstruction $k \in H^3(\Pi; C)$ is zero, where C is the center of G.

This obstruction arises in a number of ways in topology: in classifying homotopy equivalences, in classifying certain kinds of manifolds and actions on them, and as the k-invariant of certain classifying spaces, [2], [3], [9]. In this paper we study methods, other than the direct definition, of computing this obstruction.

Consider the special case of above where $\Pi = \text{Out } G$ and $\phi = \text{id}$ and call the resulting obstruction $U \in H^3$ (Out G; C). Then U is the universal example for this obstruction (see [9, 4.8]). In the spirit that it is sufficient to compute the universal example, we prove:

THEOREM 1. *Let $c \in H^2$ (In G; C) be the characteristic class for the extension $0 \to C \to G \to \text{In } G \to 1$. Then c transgresses to U in the Lyndon spectral sequence for the extension*

$$1 \to \text{In } G \to \text{Aut } G \to \text{Out } G \to 1.$$

The proof of the corresponding topological fact is given in Section 4. To complete the proof of Theorem 1, in Section 5 we show:

PROPOSITION 2. *Let Γ be a normal subgroup of Π and $F \to E \to B$ be a Serre fibration in which F, E, and B are Eilenberg-MacLane spaces of type $(\Gamma, 1)$, $(\Pi, 1)$, and $(\Pi/\Gamma, 1)$, respectively. Then under the natural isomorphism between group cohomology and (singular) cohomology of $K(\ \ , 1)$'s, the Lyndon spectral sequence corresponds to the Serre spectral sequence.*

This fact seems to be well known, but to the author's knowledge it is not in the literature.

There is a classical family of examples of Eilenberg and MacLane which lead to some interesting consequences of Theorem 1. In [6], they ask: given a group Π, an Abelian group C which is a Π-module, and an element $0 \neq$

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$k \in H^3(\Pi; C)$, is there a group G with center C and for which k is the obstruction? They answer affirmatively by constructing a certain free group F and showing $G = C \times F$ works (when $\Pi \neq Z_2$). For these groups we then have:

(a) The class $U \in H^3$ (Out G; C) is nonzero (since it pulls back to $k \neq 0$).

(b) The class $c \in H^2$ (In G; C) is zero, since $C \to G \to F = \text{In}\, G$ is the trivial extension.

By Theorem 1, therefore, in these cases the zero class transgresses to a nonzero class, so it must be that U is in the ideterminacy of transgression for these groups, i.e., here transgression is a differential $d^3\colon E_3^{0,2} \to E_3^{3,0}$, and it must be that $U \in \text{image}\, d^2\colon E_2^{1,1} \to E_2^{3,0}$.

This does not only happen for trivial extensions. In Section 3 we give an example of a group for which the classes c and U are both nonzero, but still $U \in \text{image}\, d^2$.

There are two interesting consequences to U being in the image of d^2. First, U is then in the image of a homomorphism which we potentially can compute and thus find U directly. To demonstrate computability for the above cases (trivial and nontrivial), in Section 3 we find an element $e \in E_2^{1,1}$ and show $d^2(e) = U$, giving sufficient (definitely not necessary) conditions for this.

Second, if $U \in \text{image}\, d^2$, then $d^3(c) = 0$ so c lives to E_∞. Hence, there is a class $c' \in H^2$ (Aut G: C) which restricts to c. The class c' represents an extension E' which gives us the following:

COROLLARY 3. *If $U \in \text{image}\, d^2$, then we can complete the following diagram*

$$\begin{array}{ccccccccc} E'\colon 0 & \to & C & \to & & \to & \text{Aut}\, G & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ E\colon 0 & \to & C & \to & G & \to & \text{In}\, G & \to & 1 \end{array}$$

where the extension E' induces the natural action of Aut G *on C.*

Not surprisingly, this does not always happen. For example, let Z_5 be generated by a, let Z_8 be generated by r which acts on a by $a^r = a^{-1}$, and let $G = Z_5 \times_\phi Z_8$ be the resulting semidirect product. Then $C \cong Z_4$ generated by $c = r^2$. Let $\phi\colon G \to G$ by $\phi(a^i r^j) = a^{2i} r^{-j}$. Easily, $\phi \in \text{Aut}\, G$, $\phi(c) = c^{-1}$, and $\phi^2 = \theta(r)$, the innerautomorphism determined by r. Suppose there were a group H which satisfied the conclusion of Corollary 3. Then there would be $s, \Psi \in H$ such that $\Psi \to \phi$ under $H \to \text{Aut}\, G$ and $r \to s$ under $G \to H$. Note that $s^2 = c$. Since $s \to \theta(r)$ under $H \to \text{Aut}\, G$, $\Psi^2 s^{-1} \to \phi^2\theta(r)^{-1} = 1 \in \text{Aut}\, G$. So $\Psi^2 = c^i s$, for some i. Hence $\Psi^4 = c^{2i}s^2 = c^j$, j odd. Therefore c commutes with Ψ, as $c = \Psi^4$ or Ψ^8. However, since $\phi(c) = c^{-1}$ and E' induces the natural action of Aut G on C, we must have $\Psi c \Psi^{-1} = c^{-1} \neq c$. Therefore c cannot commute with Ψ; hence no such H exists. This example developed from a discussion with George Bergman.

Since it is not always true that $U \in \operatorname{im} d^2$, it would be very interesting to determine for which groups it is or is not. Hopefully, this would lead to a method for computing U given G.

Finally, if $U = 0$, then Corollary 3 applies, but it is easy to find the extension E' directly. For $U = 0$ implies there is an extension

$$E'': G \xrightarrow{i} H \xrightarrow{j} \text{Out } G$$

which induces the identity on Out G. In other words, E'' induces a homomorphism $\Psi: H \to \text{Aut } G$ (by $\Psi(h)(x) = y$ if $hi(x)h^{-1} = i(y)$) and $j'\Psi = j$ (where j': Aut $G \to$ Out G is the quotient homomorphism). If $i': C \to H$ is the composite

$$C \longrightarrow G \xrightarrow{i} H,$$

then it is easy to check that

$$E': C \xrightarrow{i'} H \xrightarrow{\Psi} \text{Aut } G$$

works.

I would like to thank Professor Leonard Evens for his helpful discussions.

2. We first develop some basic algebra we need.

Let G be a group and C be its center. Then restriction Aut $G \to$ Aut C naturally induces a homomorphism ρ: Out $G \to$ Aut C. Thus ρ makes C into an Out G-module.

Recall the bar construction on Π, $B = B(\Pi) = \{B_n\}$. For $n < 0$, $B_n = 0$; for $n \geq 0$, B_n is the free Abelian group generated by all symbols of the form $x_0[x_1 \mid \cdots \mid x_n]$, where $x_i \in \Pi$, $x_i \neq 1$ if $i \geq 1$. Set $x_0[x_1 \mid \cdots \mid x_n] = 0$ if some $x_i = 1$, $x_i \geq 1$. The $Z(\Pi)$ (= group ring) module structure is

$$z(x_0[x_1 \mid \cdots \mid x_n]) = zx_0[x_1 \mid \cdots \mid x_n].$$

For $n \geq 1$, define ∂_n: $B_n \to B_{n-1}$ by

$$\partial_n x_0[x_1|\cdots|x_n] = x_0x_1[x_2|\cdots|x_n] + (-1)^n x_0[x_1|\cdots|x_n] + \sum_{i=1}^{n-1} (-1)^i x_0[x_1|\cdots|x_ix_{i+1}|\cdots|x_n].$$

For ϕ: $\Pi \to$ Aut C making the Abelian group C into a Π-module, define

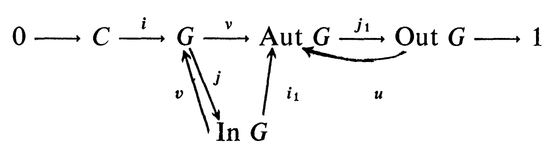
$$H_\phi^n(\Pi; C) = \ker \partial_{n+1}^*/\operatorname{im} \partial_n^*,$$

where

$$\partial_n^*: \operatorname{Hom}_{Z(\Pi)}(B_{n-1}, C) \to \operatorname{Hom}_{Z(\Pi)}(B_n, C).$$

Consider the diagram

(2.1)



where i and i_1 are the inclusions, j and j_1 are the projections, and $v = i_1 j$ so that the row is exact. For every $\gamma \in \text{Out } G$, pick $u(\gamma) \in \text{Aut } G$ such that $j_1 u = 1$, but pick $u(1) = 1$. (u is just a function.) Since $j_1(u(\gamma)u(\gamma_1)) = j_1(u(\gamma\gamma_1))$, by exactness there are elements $f_1(\gamma, \gamma_1) \in G$ such that

$$(2.2) \qquad u(\gamma)u(\gamma_1) = vf_1(\gamma, \gamma_1)u(\gamma\gamma_1), \quad \gamma, \gamma_1 \in \text{Out } G.$$

By the associative law in Aut G, it follows that

$$v[\gamma(f_1(\gamma_1, \gamma_2))f_1(\gamma, \gamma_1, \gamma_2)] = v[f_1(\gamma, \gamma_1)f_1(\gamma\gamma_1, \gamma_2)]$$

so that there are elements $k(\gamma, \gamma_1, \gamma_2) \in C$ such that

$$(2.3) \quad \gamma f_1(\gamma_1, \gamma_2)f_1(\gamma, \gamma_1\gamma_2) = ik(\gamma, \gamma_1, \gamma_2)f_1(\gamma, \gamma_1)f_1(\gamma\gamma_1, \gamma_2), \ \gamma, \gamma_1, \gamma_2 \in \text{Out } G.$$

PROPOSITION 2.4. *The cochain* B_3 (Out G) $\to C$ *by*

$$\gamma[\gamma_1 \mid \gamma_2 \mid \gamma_3] \to \gamma k(\gamma_1, \gamma_2, \gamma_3)$$

is a cocycle. Its cohomology class $U \in H^3_\rho$ (Out G, C) *is independent of the choices made in the construction of k. There is an extension of G by* Out G *which induces* $1\colon \text{Out } G \to \text{Out } G$ *if and only if* $U = 0$.

Proof. This is a direct consequence of Eilenberg and MacLane [6] (or see MacLane [10, pp. 124–128]). Define

$$(2.5) \qquad f(\gamma, \gamma_1) = jf_1(\gamma, \gamma_1) \in \text{In } G, \quad \gamma, \gamma_1 \in \text{Out } G.$$

Then

$$(2.6) \qquad u(\gamma)u(\gamma_1) = i_1 f(\gamma, \gamma_1)u(\gamma, \gamma_1), \quad \gamma, \gamma_1 \in \text{Out } G,$$

and

$$(2.7) \qquad \gamma f(\gamma_1, \gamma_2)f(\gamma, \gamma_1\gamma_2) = f(\gamma, \gamma_1)f(\gamma\gamma_1, \gamma_2), \quad \gamma, \gamma_1, \gamma_2 \in \text{Out } G.$$

Compare with (2.2) and (2.3).

Of course there is a "semi-action" of Out G on In G given by

$$(2.8) \qquad \gamma(x) = u(\gamma)(x) = u(\gamma)xu(\gamma)^{-1}, \quad \gamma \in \text{Out } G, x \in \text{In } G.$$

Then

$$(2.9) \qquad \gamma(\gamma_1(x)) = \theta f(\gamma, \gamma_1)(\gamma\gamma_1(x)), \quad \gamma, \gamma_1 \in \text{Out } G, x \in \text{In } G,$$

where $\theta(y)(x) = yxy^{-1}$, $x, y \in \text{In } G$. (Compare with [10, p. 125, 8.6].)

For every $x \in \text{In } G$, pick $v(x) \in G$ such that $jv = 1$, but pick $v(1) = 1$. (v is just a function.) Since $j(v(x)v(y)) = jv(xy)$, by exactness there are elements $g(x, y) \in C$ such that

$$(2.10) \qquad v(x)v(y) = ig(x, y)v(xy), \quad x, y \in \text{In } G.$$

Of course g is the factor set of the extension

$$C \xrightarrow{\ i\ } G \xrightarrow{\ j\ } \text{In } G,$$

it determines the characteristic class $c \in H^2$ (In G; C), and it is thought of as the obstruction to v being a homomorphism.

Let Aut G act on G naturally and let Aut G act on In G by innerautomorphisms (in Aut G). Note that j commutes with these actions. Thus $j\alpha v(x) = jv(\alpha x)$ so that again by exactness there are elements $M(\alpha, x) \in C$ such that

$$\alpha v(x) = iM(\alpha, x)v(\alpha x), \quad \alpha \in \text{Aut } G, x \in \text{In } G. \tag{2.11}$$

It is easy to verify

$$M(\alpha, xy) = M(\alpha, x)M(\alpha, y)g(\alpha x, \alpha y)\alpha g(x, y)^{-1}, \quad \alpha \in \text{Aut } G, x, y \in \text{In } G \tag{2.12}$$

$$M(\alpha\beta, x) = \alpha M(\beta, x)M(\alpha, \beta x), \quad \alpha, \beta \in \text{Aut } G, x \in \text{In } G. \tag{2.13}$$

Suppose $a, b, x, y \in \text{In } G$ and $ab = xy$. Then $v(ab) = v(xy)$ so

$$g(a, b)^{-1}v(a)v(b) = g(x, y)^{-1}v(x)v(y).$$

If $\alpha \in \text{Aut } G$ and $\alpha(a)b = xy$, we get

$$g(\alpha a, b)^{-1}v(\alpha a)\alpha v(a)^{-1}\alpha v(a)v(b) = g(x, y)^{-1}v(x)v(y)$$

or

$$\alpha v(a)v(b) = M(\alpha, a)g(\alpha a, b)g(x, y)^{-1}v(x)v(y). \tag{2.14}$$

Now redefine f and f_1 by:

DEFINITION 2.15. Let f be given by (2.6) and let $f_1 = vf$. Then all other equations hold, in particular (2.3).

PROPOSITION 2.16. *For* $\gamma, \gamma_1, \gamma_2 \in \text{Out } G$,
$k(\gamma, \gamma_1, \gamma_2) = M(u\gamma, f(\gamma_1, \gamma_2))g(\gamma f(\gamma_1, \gamma_2), f(\gamma, \gamma_1\gamma_2))g(f(\gamma, \gamma_1), f(\gamma\gamma_1, \gamma_2))^{-1}$.

Proof. Use (2.7) to make the obvious substitution in (2.14) and compare with (2.3).

Since $j_1(u(\gamma)\alpha) = \gamma j_1(\alpha)$ by exactness there are elements $F(\gamma, \alpha) \in \text{In } G$ such that

$$u(\gamma)\alpha = i_1F(\gamma, \alpha)u(\gamma j_1(\alpha)), \quad \gamma \in \text{Out } G, \alpha \in \text{Aut } G. \tag{2.17}$$

It is easy to verify

$$F(\gamma, \alpha\beta) = F(\gamma, \alpha)F(\gamma j_1\alpha, \beta), \quad \gamma \in \text{Out } G, \alpha, \beta \in \text{Aut } G, \tag{2.18}$$

using the associative law for $u(\gamma)\alpha\beta$, and

$$f(\gamma, \gamma_1)F(\gamma\gamma_1, \alpha) = \gamma F(\gamma_1, \alpha)f(\gamma, \gamma_1 j_1\alpha), \quad \gamma, \gamma_1 \in \text{Out } G, \alpha \in \text{Aut } G, \tag{2.19}$$

using the associative law for $u(\gamma)u(\gamma_1)\alpha$, and

$$F(\gamma, i_1x) = \gamma(x), \quad \gamma \in \text{Out } G, x \in \text{In } G \tag{2.20}$$

by (2.8) and (2.17).

3. We give sufficient conditions that $U \in \operatorname{im} d_2\colon E_2^{1,1} \to E_2^{3,0}$ and identify d_2 in these cases.

Assume that

$$\alpha g(x, y) = g(\alpha x, \alpha y), \quad \alpha \in \text{Aut } G,\ x, y \in \text{In } G, \tag{3.1}$$

where g was given in (2.10).

LEMMA 3.2. *If g satisfies* (3.1), *then for fixed* $\alpha \in$ Aut G, M_α: In $G \to C$ *is a homomorphism, where* $M_\alpha(x) = M(\alpha, x)$.

Proof. This is immediate from (2.12).

We abuse notation and define $M(\gamma, x) = M(u(\gamma), x)$, $\gamma \in$ Out G, $x \in$ In G and similarly for M_γ.

LEMMA 3.3. *If g satisfies* (3.1), *then the function*

$$T\colon \text{Out } G \to \text{Hom } (\text{In } G; C)$$

by $T(\gamma) = M_\beta \beta^{-1}$ *is a crossed homomorphism.*

Proof. Recall the action of Out G on Hom is

$$(\beta \cdot f)(x) = \beta(f(\beta^{-1}x))$$

(see [10, p. 348]). Then

$$\begin{aligned} M_{\alpha\beta}((\alpha\beta)^{-1}(x)) &= \alpha\beta v(\beta^{-1}\alpha^{-1}x)v(x)^{-1} \\ &= \alpha M_\beta(\beta^{-1}\alpha^{-1}x)\alpha v(\alpha^{-1}x)v(x)^{-1} \\ &= \alpha(M_\beta \circ \beta^{-1})(x)(M_\alpha \circ \alpha)(x). \end{aligned}$$

Let $\{E_r^{p,q}\}$ be the Lyndon Spectral sequence for the extension In $G \to$ Aut $G \to$ Out G with coefficients in C and all actions are natural. By [10, p. 351], $E_2^{1,1} \cong H^1(\text{Out } G; H^1(\text{In } G; C))$. By [10, p. 106], $H^1(\Pi; A) =$ crossed homomorphisms/principal crossed homomorphisms. Since In G acts trivially on C, $H^1(\text{In } G; C) = \text{Hom } (\text{In } G; C)$. Thus, elements of $E_2^{1,1}$ are represented by elements of

$$\text{Hom}_\phi\, (\text{Out } G; \text{Hom } (\text{In } G, C))$$

(where Hom_ϕ denotes crossed homomorphisms).

Assume

$$g(\gamma f(\gamma_1, \gamma_2), f(\gamma, \gamma_1\gamma_2)) = 0 = g(f(\gamma, \gamma_1), f(\gamma\gamma_1, \gamma_2)), \quad \gamma, \gamma_1, \gamma_2 \in \text{Out } G. \tag{3.4}$$

Assume that $M(\gamma, x)$ satisfies (2.13) for $\gamma \in$ Out G, $x \in$ In G.

PROPOSITION 3.5. *Suppose g satisfies* (3.1) *so that* $[T] \in E_2^{1,1}$. *If G satisfies Lemma* 3.3 *and* (3.4), $d_2[T] = U \in E_2^{3,0}$.

Before proving (3.4) we discuss (3.1) and Lemma 3.3. If $C \to G \to$ In G is the trivial extension, then we can choose $h = 0$ so that easily it satisfies (3.1) and Lemma 3.3. For $G = C \times H$, where $H = \text{In } G$ so the center of $H = 1$, Aut G = Aut $C \times$ Hom $(H, C) \times$ Aut H *as a set*, and multiplication is given by

$$(a, b, c) \cdot (a_1, b_1, c_1) = (aa_1, ab_1 + bc_1, cc_1).$$

Similarly for Out G = Aut $C \times$ Hom $(H, C) \times$ Out H. Using this, it is not hard to see (3.4) is satisfied.

We give a nontrivial example with details left to the reader. Let Z_5 be generated by a, Z_4 be generated by r, and let Z_4 act on Z_5 by $rar^{-1} = a^{-1}$ (which is *not* the natural action). Let $G = Z_5 \times_\phi Z_4$, the semidirect product. Then $C \cong Z_2$ generated by c, where $c \to r^2$, In $G \cong Z_5 \times_\phi Z_2$ generated by $\bar{a}$ and $\bar{r}$ and the extension $C \to G \to$ In G is easily nontrivial.

Let $v(\bar{a}^i\bar{r}^j) = a^i r^j$. Then $g(\bar{a}^i\bar{r}, \bar{a}^j\bar{r}) = c$ and $g(x, y) = 0$ for all other (x, y).

To compute Aut G, observe both Z_5 and C are characteristic in G. Let $\alpha, \beta \in$ Aut G by $\alpha(r) = r$, $\alpha(a) = a^2$, $\beta(r) = r^{-1}$, $\beta(a) = a$. Then $\langle\alpha\rangle \cong Z_4$, $\alpha^2 = \bar{r}$, $\langle\beta\rangle \cong Z_2$, and $\langle\bar{a}, \alpha, \beta\rangle$ = Aut G (where $\langle\ \rangle$ denotes "subgroup generated by"). In fact, Aut $G \cong Z_5 \times_\phi (Z_4 \times Z_2)$.

Finally, by above it follows that Out $G \cong Z_2 \times Z_2$ generated by $\bar{\alpha}, \bar{\beta}$. Let $u(\bar{\alpha}^i\bar{\beta}^j) = \alpha^i\beta^j$. Then $f(\bar{\alpha}\bar{\beta}^i, \bar{\alpha}\bar{\beta}^j) = \bar{r}$ and $f(x, y) = 0$ otherwise. By using these definitions it is not hard to see that this g satisfies (3.1) and Lemma 3.3, that the corresponding M satisfies (3.4), and that the universal example for this G is not zero by identifying it in $H^3(Z_2 \times Z_2; Z_2)$.

We note in passing that for the dihedral group of order 8, no g satisfies (3.1) (but that its $U = 0$).

Proof of Proposition 3.5. Following MacLane [10, p. 351] let

$$K^{p,q} = \text{Hom}_{\text{Aut } G}(B_p(\text{Out } G) \otimes B_q(\text{Aut } G); C)$$

(which is essentially the E_0 term of the Lyndon spectral sequence).

$$(\delta' f)(b' \otimes b'') = (-1)^{p+q+1} f(\partial b' \otimes b''), \quad b' \in B_{p+1}, b'' \in B_q$$

$$(\delta'' f)(b' \otimes b'') = (-1)^{q+1} f(b' \otimes \partial b''), \quad b' \in B_p, b'' \in B_{q+1}.$$

Note that to define elements of $K^{p,q}$ it is sufficient, by using the action of Aut G, to give their values on elements of the form

$$[\gamma_1 \mid \cdots \mid \gamma_p] \otimes \alpha_0[\alpha_1 \mid \cdots \mid \alpha_q], \quad \gamma_i \in \text{Out } G, \alpha_i \in \text{Aut } G.$$

Recall the functions f and F were defined by (2.5) and (2.14). Define $A \in K^{3,0}$, $B, C \in K^{2,0}$, $D \in K^{1,1}$ by

$$A([\gamma_1 \mid \gamma_2 \mid \gamma_3] \otimes \alpha_0[\ ]) = k(\gamma_1, \gamma_2, \gamma_3), \tag{3.6}$$

$$B([\gamma_1 \mid \gamma_2] \otimes \alpha_0[\ ]) = k(\gamma_1, \gamma_2, \gamma_2^{-1}\gamma_1^{-1}\alpha_0), \tag{3.7}$$

$$C([\gamma_1 \mid \gamma_2] \otimes \alpha_0[\ ]) = M(\gamma_1, f(\gamma_2, \gamma_2^{-1}\gamma_1^{-1}\alpha_0)), \tag{3.8}$$

$$D([\gamma_1] \otimes \alpha_0[\alpha_1]) = M(\gamma_1, F(\gamma_1^{-1}\alpha_0, \alpha_1)). \tag{3.9}$$

It is not hard to verify the following:

(3.10) $\delta' B = A$, by the relation $\delta k = 0$ (see [10, IV.8.4]),

(3.11) $B = C$, by Lemma 3.3 and by Proposition 2.16,

(3.12) $\delta'' C = \delta' D$, by Lemma 3.2, (2.7), (2.19), and (3.4),

(3.13) $\delta'' D = 0$, by (2.18).

(3.14)
$$\begin{aligned} D([\gamma_1] \otimes \alpha_0[\alpha_1]) &= M(\gamma_1, F(\gamma_1^{-1}, \alpha_1)) \\ &= M(\gamma_1, \gamma_1^{-1}(\alpha_1)) \\ &= T(\gamma_1)(\alpha_1) \quad \text{for } \alpha_0, \alpha_1 \in \text{In } G, \end{aligned}$$

by (2.20) and Lemma 3.3.

Therefore, $[A]$ represents U in $E_2^{3,0}$ by (3.6) and Proposition 2.4; $[D] \in E_2^{1,1}$ by (3.13) and represents T by (3.14) and by MacLane [10, pp. 348, 352]; and $d_2([D]) = [A]$ by (3.10)–(3.12). This completes the proof.

4. Under the natural isomorphism $H^*(\Pi, C) \to H^*(K(\Pi, 1); \{C\})$, (where $\{C\}$ denotes local coefficients if necessary when C is not a trivial Π-module), we abuse notation and let $c \in H^2(K(\text{In } G, 1); C)$ correspond to the characteristic class of the (central) extension $C \to G \to \text{In } G$ and let $U \in H^3(K(\text{Out } G, 1); \{C\})$ correspond to the universal obstruction in $H^3(\text{Out } G; C)$. In this section we prove:

PROPOSITION 4.1. *In the Serre spectral sequence for the fibration*

$$K(\text{In } G, 1) \xrightarrow{j} K(\text{Aut } G, 1) \xrightarrow{p} K(\text{Out } G, 1),$$

c transgresses to U.

First, construct the space B which has U as its only (twisted) k-invariant, so we obtain a fibration

$$K(C, 2) \xrightarrow{i} B \xrightarrow{q} K(\text{Out } G, 1).$$

Let $\iota \in H^2(K(C, 2); C)$ be the fundamental class.

PROPOSITION 4.2. (i) *In the fibration q, ι transgresses to U. Also, $\pi_1(B) =$ Out G, $\pi_2(B) = C$, and $\pi_i(B) = 0$, otherwise.*

(ii) *There is a fibration lying over B*

$$K(G, 1) \longrightarrow K(\text{Aut } G, 1) \xrightarrow{\pi} B$$

which is the universal fibration for $K(G, 1)$ fibrations. The homotopy sequence for π is all zero except for $\pi_2(B) \to \quad \to \quad \to \pi_1(B)$ and this is the natural sequence $C \to G \to \text{Aut } G \to \text{Out } G$ (of (2.1)).

Part (i) follows immediately from the construction of B. Part (ii) is one of the main results of [9]. (That π_i of the classifying space for $K(G, 1)$ bundles is as described in (i) is a result of Gottlieb [7].)

LEMMA 4.3. *There is a map of fibrations*

$$\begin{array}{ccccc} K(\mathrm{In}\ G, 1) & \xrightarrow{j} & K(\mathrm{Aut}\ G, 1) & \xrightarrow{p} & K(\mathrm{Out}\ G, 1) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ K(C, 2) & \xrightarrow{i} & B & \xrightarrow{q} & K(\mathrm{Out}\ G, 1) \end{array}$$

such that $h =$ *identity and* $f^*(\iota) = c$.

Proof of Proposition 4.1. Proposition 4.1 follows immediately from Lemma 4.3 by naturality, since transgression commutes with maps of fibrations and since i transgresses to U in q by Proposition 4.2 (i).

Proof of Lemma 4.3. Consider the following diagram:

$$\begin{array}{ccccccc} K(G, 1) & \searrow^{j_1} & & & & & \\ \| & & & & & & \\ K(G, 1) & & T & \xrightarrow{\pi_1} & K(C, 2) & & \\ & \searrow^{j} & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i} & & \\ & & K(\mathrm{Aut}\ G, 1) & \xrightarrow{\pi} & B & \xrightarrow{q} & K(\mathrm{Out}\ G, 1) \end{array}$$

where q and π are the fibrations above and

$$K(G, 1) \xrightarrow{j_1} T \xrightarrow{\pi_1} K(C, 2)$$

is the fibration pulled-back from π by i. By the usual argument,

$$T \xrightarrow{i_1} K(\mathrm{Aut}\ G, 1) \xrightarrow{q\pi} K(\mathrm{Out}\ G, 1)$$

is a fibration. Easily $q\pi$ is (homotopic to) p so that $T = K(\mathrm{In}\ G, 1)$, $i_1 = j$. Letting $g = \pi$ and $f = \pi_1$, we are done when we show $\pi_1^*(\iota) = c$.

Since (i, i_1) is a map of fibrations, the induced maps commute with the homotopy sequences of π and π_1. Hence, using Proposition 4.2 (ii) it follows that the homotopy sequence for π_1 reduces to the natural $C \to G \to \mathrm{In}\ G$. Making j_1 into a fiber map, we then get the fibration

$$K(C, 1) \longrightarrow K(G, 1) \xrightarrow{j_1} K(\mathrm{In}\ G, 1) = T$$

which is classified by π_1. But by [8, Theorem 1], $\pi_1^*(\iota)$ which is the characteristic class for this fibration corresponds to the characteristic class of the group extension

$$\pi_1(K(C, 1)) \to \pi_1(K(G, 1)) \to \pi_1(K(\mathrm{In}\ G, 1))$$

under the natural isomorphism $H^*(\mathrm{In}\ G; C) \cong H^*(K(\mathrm{In}\ G, 1); C)$, so we are done.

5. In this section we will prove Proposition 2. Let $K = \{K^{p,q}\}$, δ' and δ'' be as in the proof of Proposition 3.5. The filtration on K is the usual filtration for bicomplexes, $F_p = \sum_{h=0}^{p} B_h(\Pi/\Gamma) \otimes B(\Pi)$. (See [10, XI, Section 10].) We realize this geometrically.

Let $W = W(\Pi)$ be the standard free acyclic CW complex corresponding to the group Π and let $C_* = C_*(W)$ be its CW chains. This can be constructed by making the bar resolution into a semisimplicial complex and then using Milnor's geometric realization [11]. See for example [8, Section 3]. Denote the n-cells of W by $\langle a_0, \ldots, a_n \rangle$ and the corresponding generators of C_n by $(\langle a_0, \ldots, a_n \rangle)$, $a_i \in \Pi$, no $a_i = 1$, $i > 0$. There is a free cellular action of Π on W,

$$a\langle a_0, a_1, \ldots, a_n \rangle = \langle aa_0, a_1, \ldots, a_n \rangle.$$

Denote by $K = K(\Pi)$ the quotient space $W(\Pi)/\Pi$ which is a $K(\Pi, 1)$.

LEMMA 5.1. *Let $B_* = B_*(\Pi)$ be the bar resolution of* Π.

(a) *There is an isomorphism $B_* \cong C_*(W)$ induced by*

$$a_0[a_1 | \cdots | a_n] \to \langle a_0, \ldots, a_n \rangle.$$

(b) *This isomorphism induces the natural isomorphism*

$$\Phi\colon H^*(\Pi\colon C) \to H^*(K(\Pi, 1); \{C\})$$

for the case $K(\Pi, 1)$ *is* $K(\Pi)$.

Proof. This is proven in [8, 3.4]. The isomorphism Φ is just the composite of the one induced by (a) and the isomorphism from equivariant cohomology in the universal cover of a space X to the local cohomology of X given by Eilenberg [4].

Now form the fibration

$$W(\Pi) \xrightarrow{p} W(\Pi/\Gamma) \times W(\Pi) \xrightarrow{j} W(\Pi/\Gamma).$$

Let Π/Γ act on $W(\Pi/\Gamma)$ as above, with $\Gamma \subset \Pi$, let Γ act on $W(\Pi)$ naturally, and let Π act on $W(\Pi/\Gamma) \times W(\Pi)$ diagonally. Then the maps i and p are equivariant, $W(\Pi)/\Gamma = K(\Gamma, 1)$, $W(\Pi/\Gamma) \times W(\Pi)/\Pi = K(\Pi, 1)$, and we have induced a fibration

$$K(\Gamma, 1) \xrightarrow{j} K(\Pi, 1) \xrightarrow{q} K(\Pi/\Gamma),$$

where $K(\Pi/\Gamma)$ is as above (and the homotopy sequence is $\Gamma \to \Pi \to \Pi/\Gamma$).

Proof of Proposition 2. For any space X let $S(X)$ denote the total singular complex. For each of the following, we give a graded cohomology group, a filtration on it, and the resulting spectral sequence in the sense of Serre [12] and argue as we go along that each is isomorphic to the preceding.

(1) $K^{p,q} = \mathrm{Hom}_\pi (B_p(\Pi/\Gamma) \otimes B_q(\Pi); A)$, $F_p = \sum_{h=0}^{p} \mathrm{Hom}_\pi (B_h \otimes B; A)$, $E^{p,q}$. This is the Lyndon spectral sequence for the extension $\Gamma \to \Pi \to \Pi/\Gamma$ as defined by MacLane [10, XI.10].

(2) ${}_2K$, ${}_2F$, ${}_2E$ are obtained from (1) by substituting $C_pW(\Pi/\Gamma)$ for $B_p(\Pi/\Gamma)$ and $C_qW(\Pi)$ for $B_q(\Pi)$. Note that these are equivariant CW cochains. The isomorphisms result easily from Lemma 5.1 (a).

(3) ${}_3K$, ${}_3F$, and ${}_3E$ are obtained from (2) by substituting equivariant singular cochains for equivariant CW cochains. The isomorphisms result from the usual (and easy for this case) chain equivalence from CW chains to singular. (See the κ of Eilenberg and MacLane in [4, Section 7].)

(4) ${}_4K = \text{Hom}\,(SK(\Pi/\Gamma) \otimes SK(\Gamma, 1), \{A\})$ the (local) cochains on the twisted tensor product, constructed from the fibration q. See Brown [1, p. 236, 229]. ${}_4E$ arises from the filtration ${}_4F$ which we use to correspond to the above filtrations, and is the same as given by Brown [1, Section 3, p. 229]. Eilenberg has shown that the equivariant cochains on the universal cover of X is isomorphic, induced by projection, to the local (or twisted) cochains of X [4, Section 25, p. 212]. Hence, comparing p and q, the cochains and filtrations correspond, so the resulting spectral sequences are isomorphic.

(5) Let ${}_5E$ be the usual spectral sequence of q, so that ${}_5K = SK(\Pi, 1)$ and F is defined by real dimension (as in Brown [1, Section 7, p. 236]). By Brown [1, Corollary 7.2, p. 237], ${}_4E$ is isomorphic to ${}_5E$. We are now done.

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