A COMBINATORIAL ANALOGUE OF A THEOREM OF MYERS

BY

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The title refers to this theorem [5]: Let M be a complete, connected, Riemannian n-manifold whose Ricci curvature is everywhere bounded below by some number $R > 0$. Then M is compact and has diameter at most $\pi\sqrt{(n - 1)/R}$.

The purpose of this note is to prove an analogous theorem for n-dimensional simplicial complexes, using a combinatorial analogue of Myers' proof. I am grateful to H. Gluck for encouragement, and to the National Science Foundation for support. I am indebted to the referee for an unusually painstaking effort to organize this paper more clearly.

Let K be a connected cell complex; let v, w be vertices of K. A *path* with endpoints v and w is a sequence $\alpha = (a_1, \ldots, a_r)$ of 1-cells of K such that the boundary $\partial a_i = \{b_{i-1}, b_i\}$, where the b_i are vertices with $b_0 = v$, $b_r = w$. The *length* $|\alpha|$ of such a path is r, the number of its 1-cells. The *distance* $d(v, w)$ is min $|\alpha|$ for paths with v and w as endpoints. The *diameter* of K is max $d(v, w)$ for v, w vertices of K.

I shall first define combinatorial curvature and present the analogue of Myers' theorem in the 2-dimensional case.

Let K be a cell complex which is a 2-manifold without boundary. Let $|\partial c|$ denote the number of sides of a 2-cell c. If v is a vertex, then the *curvature* at v is defined to be $R^*(v) = 2 - \sum (1 - 2/|\partial c|)$ where the sum is taken over all 2-cells c containing v.

Proposition 1. *Let K be a connected cell complex which is a 2-manifold without boundary. Assume there is a number $R > 0$ such that $R^*(v) \geq R$ for every vertex v of K. Then K has diameter $\leq 1 + 2/R$.*

The proof starts on page 14.

Remark 1. If one visualizes K as being made of regular polygons all of the same side-length, then for any 2-cell c with v as vertex, the angle of c at v is $\pi(1 - 2/|\partial c|)$. Thus the piecewise linear curvature of K at v, defined to be $2\pi - \sum_{v \in c}$ (angle of c at v) (see Aleksandroff and Zalgaller [2]), is just $\pi R^*(v)$. So $R^*(v)$ is a natural analogue of the Gaussian curvature of a smooth surface at a point.

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Remark 2. A connected cell complex K has finite diameter if and only if it has finitely many cells. To prove this, fix a vertex v and let $V_r = \{\text{vertices } w\colon d(v, w) \leq r\}$. Then

$$V_{r+1} = \bigcup_{w \in V_r} \{x\colon x \in \text{star}\,(w)\}.$$

Now V_0 is finite, so assume by induction that V_r is finite. Since star (w) has finitely many vertices for every $w \in K$, V_{r+1} is finite. Thus if the diameter of K is finite, then K has finitely many vertices, and hence finitely many cells.

In the 2-dimensional case, the proof of Myers' theorem may be summarized thus: Let $\alpha\colon [0, L] \to M$ be a geodesic parametrized by arc-length. Let P be a vector field along α so that for each t, the pair $P(t)$, $d\alpha(t)/dt$ forms an orthonormal basis for the tangent space to M at $\alpha(t)$. For 2-manifolds, the Ricci curvature coincides with the Gaussian curvature, and by hypothesis, this is everywhere $\geq R > 0$. The second variational formula now implies that if $L > \pi/\sqrt{R}$, then α can be varied in the direction P so as to obtain a strictly shorter path with the same endpoints. Thus every two points of M are distant at most $\pi/\sqrt{R}$, and the 2-dimensional case of Myers' theorem follows.

Proposition 1 is proved by an analogous variational argument, based upon the following definition of a combinatorial variation of a path in a cell complex.

Let K be any cell complex. A *ribbon* in K is a sequence $C = (c_1, \ldots, c_r)$ of 2-cells such that $c_i \cap c_{i+1}$ is a 1-cell d_i for $i = 1, \ldots, r - 1$. Let v be a vertex of c_1 and w a vertex of c_r; then C determines two paths α and β from v to w, as I shall show in a moment. C is called a *variational field*, and β a *variation*, of α.

This is how to construct α and β from C. There are just two paths, say $\partial_\alpha c_1$ and $\partial_\beta c_1$ in ∂c_1 such that:

(i) neither uses the 1-cell d_1;
(ii) each has v as one endpoint;
(iii) the other endpoints, $e_1(\alpha)$ and $e_1(\beta)$ respectively, are vertices of d_1.

It is possible, if $v \in d_1$, for either $\partial_\alpha c_1$ or $\partial_\beta c_1$ (but not both) to be empty; if, say, $\partial_\alpha c_1 = \emptyset$, then $e_1(\alpha)$ is defined to be v. Now there are paths $\partial_\alpha c_2$ and $\partial_\beta c_2$ in ∂c_2 uniquely determined by the rules:

(i) neither uses d_1 or d_2;
(ii) $e_1(\alpha)$ is one endpoint of $\partial_\alpha c_2$ and $e_1(\beta)$ one endpoint of $\partial_\beta c_2$;
(iii) the other endpoints, $e_2(\alpha)$ and $e_2(\beta)$, are vertices of d_2. Again, if, say, $\partial_\alpha c_2 = \emptyset$, then $e_2(\alpha)$ is defined to be $e_1(\alpha)$. Continuing in this way I obtain paths $\partial_\alpha c_j$ and $\partial_\beta c_j$ for $j = 1, \ldots, r$; $\partial_\alpha c_r$ and $\partial_\beta c_r$ are required to have w as one endpoint. Then α and β are defined by juxtaposing the sequences $\partial_\alpha c_1, \ldots, \partial_\alpha c_r$ and $\partial_\beta c_1, \ldots, \partial_\beta c_r$.

If K is a 2-manifold without boundary, then each path α has two special variational fields, one on either side of α. They may be described explicitly by using the dual complex K^* of K (see Hudson [4]). Say $\alpha = (a_1, \ldots, a_r)$ with $a_i \cap a_{i+1} = b_i$ for $i = 1, \ldots, r - 1$, let b_0 and b_r be the endpoints of α, and let K^* be the dual complex to K. Then the sequence $(b_0^*, \ldots, b_r^*)$ is a ribbon α^*, called the dual of α. Pick vertices v of b_0^* and w of b_r^*. Then α^* determines paths β and γ from v to w constructed as juxtapositions $\partial_\beta(b_0^*), \ldots, \partial_\beta(b_r^*)$ and $\partial_\gamma(b_0^*), \ldots, \partial_\gamma(b_r^*)$. Let β'' be the juxtaposition $\partial_\beta(b_1^*), \ldots, \partial_\beta(b_{r-1}^*)$, and let γ'' be defined similarly. Then their dual ribbons in K, $(\beta'')^*$ and $(\gamma'')^*$, are the required variational fields of α.

Proof of Proposition 1. Let α be a path in K; let C_1 and C_2 be its special variational fields and β_1 and β_2 the corresponding variations. Say $\alpha = (a_1, \ldots, a_r)$ with $b_i = a_i \cap a_{i+1}$ for $i = 1, \ldots, r - 1$; so $|\alpha| = r$. Applying the curvature hypothesis to each b_i and adding we get

$$\sum_{i=1}^{r-1} R^*(b_i) \geq (|\alpha| - 1)R. \tag{1}$$

Substituting the definition of $R^*(b_i)$ gives

$$\sum_{i=1}^{r-1} \Big(\sum_{c \in C_1 \cup C_2,\, b_i \in c} (1 - 2/|\partial c|)\Big) \leq (|\alpha| - 1)(2 - R). \tag{2}$$

So for either C_i or C_2—let us say, for C_1—

$$\sum_{i=1}^{r-1} \Big(\sum_{c \in C_1,\, b_i \in c} (1 - 2/|\partial c|)\Big) \leq (|\alpha| - 1)(1 - R/2). \tag{3}$$

Let L denote the left-hand side of (3), and say $C_1 = (c_1, \ldots, c_s)$. Then

$$L = \sum_{j=1}^{s} ((1 - 2/|\partial c_j|) \times (\text{number of } b_i \text{ which are} \in c_j)).$$

The number of b_i which are $\in c_j$ is $|\partial_\alpha c_j|$ if $j = 1$ or s, and $|\partial_\alpha c_j| + 1$ otherwise. Hence

$$\begin{aligned} L &= \sum_{j=1}^{s} (1 - 2/|\partial c_j|)(|\partial_\alpha c_j| + 1) - \sum_{k=1,s} (1 - 2/|\partial c_k|) \\ &= \sum_{j=1}^{s} |\partial_\alpha c_j| + \sum_{j=1}^{s} (1 - 2(|\partial_\alpha c_j| + 1)/|\partial c_j|) - 2 \\ &\quad + 2(1/|\partial c_1| + 1/|\partial c_s|). \end{aligned}$$

Now

$$|\partial c_j| = |\partial_\alpha c_j| + |\partial_{\beta_1} c_j| + 1 \quad \text{if } j = 1 \text{ or } s,$$

while

$$|\partial c_j| = |\partial_\alpha c_j| + |\partial_{\beta_1} c_j| + 2 \quad \text{otherwise.}$$

So

$$\begin{aligned} L &= |\alpha| + \sum_{j=1}^{s} (|\partial_{\beta_1} c_j| - |\partial_\alpha c_j|)/|\partial c_j| + (1/|\partial c_1| + 1/|\partial c_s|) - 2 \\ &> |\alpha| + (|\beta_1| - |\alpha|)/3 - 2, \end{aligned}$$

since $|\partial c| \geq 3$ for any 2-cell. Substituting into (3) gives

$$(|\beta_1| - |\alpha|)/3 - 2 < -|\alpha|R/2 - (1 - R/2).$$

Hence $|\beta_1| < |\alpha|$ provided that $|\alpha|R/2 + 1 - R/2 - 2 \geq 0$; that is, provided that $|\alpha| \geq 1 + 2/R$. It follows that every pair of vertices can be connected by a path no longer than $1 + 2/R$. This proves the proposition.

Remark 1. A simplicial complex satisfying the hypotheses of Proposition 1 must have five or fewer 2-simplexes at each vertex. However, if the dual cell complex K^* of a simplicial complex K satisfies these hypotheses, then the corresponding assumption on K is weaker. Let $\langle u, v, w\rangle$ be a 2-simplex of K, and let a, b, and c be the number of 2-simplexes in star (u), star (v), and star (w), respectively. Then K^* satisfies the curvature hypothesis of Proposition 1 provided that

$$(1/a + 1/b + 1/c) \geq (1 + R)/2 \quad \text{for every } \langle u, v, w\rangle \in K.$$

Thus vertices of order 5, 6, and 7 will do, but 5, 6, and 8 will not.

Remark 2. If K satisfies the hypotheses of Proposition 1 and is simplicial, then K has at most twenty 2-simplexes. For at each vertex there are at most five faces, so $3F \leq 5V$; also Euler's formula gives $F - 3F/2 + V = 2$, so $V = 2 + F/2 \leq 2 + 5V/6$. Hence $V \leq 12$ and so $F \leq 20$. However, if the dual complex to K satisfies these hypotheses, then K can be arbitrarily large. For example let P_r be a simple closed curve made of r segments and let K_r be its suspension $S^0 * P_r$. Then K_r^* satisfies the hypotheses of Proposition 1, though K_r does not (for $r \geq 6$). Of course K_r has diameter 2, but K_r^* has diameter $[r/2] + 1$. In the proposition we may use $R = 2/r$ and it then estimates the diameter of K_r^* as $r + 1$.

Remark 3. The proof of Proposition 1 only requires that the curvature of K at v be "on the average" bounded below away from 0. The next corollary is an example of what can be done in this direction. It is not apparent what the analogous result is for smooth surfaces.

COROLLARY 2. *Let K be a connected cell complex which is a 2-manifold without boundary. Assume there is a number $R > 0$ such that whenever $\langle v, w\rangle$ is a 1-simplex of K, then*

$$\left(2 - \sum_{v \in c} (1 - 2/|\partial c|)\right) + \left(2 - \sum_{w \in c''} (1 - 2/|\partial c''|)\right) \geq 2R.$$

Then K is finite and has diameter $\leq 3/R$.

Proof. For every 2-cell c'', $|\partial c''| \geq 3$; also every vertex w belongs to at least three 2-cells. Hence $2 - \sum_{w \in c''} (1 - 2/|\partial c''|) \leq 1$.

The curvature hypothesis now implies that for any vertex v,

$$2 - \sum_{v \in c} (1 - 2/|\partial c|) \geq 2R - 1;$$

if this is positive, then Proposition 1 shows K is finite with diameter $\leq 1 + 2/(2R - 1)$, which is $<3/R$ since $R \geq 1/2$ and $2R - 1 \leq 2$. If $2R - 1 \leq 0$, apply the curvature hypothesis of this corollary to the 1-cells $a_2, \ldots, a_{r-1}$ and add; using the notation of the proof of Proposition 1, this gives

$$\sum_{i=1}^{r-2}\left(\sum_{b_i \in c}(1 - 2/|\partial c|) + \sum_{b_{i+1} \in c''}(1 - 2/|\partial c''|)\right) \leq (|\alpha| - 2)(4 - 2R).$$

The previous calculation shows that

$$\sum_{b_1 \in c}(1 - 2/|\partial c|) + \sum_{b_{r-1} \in c''}(1 - 2/|\partial c''|) \leq 2(3 - 2R).$$

Adding these inequalities gives

$$2\sum_{i=1}^{r-1}\sum\nolimits_{c \in C_1 \cup C_2,\, b_i \in c}(1 - 2/|\partial c|) \leq (|\alpha| - 2)(4 - 2R) + 2(3 - 2R).$$

The corollary now follows from the proof of Proposition 1, using this inequality instead of (2).

I now turn to the n-dimensional generalization of Proposition 1. I shall first outline the proof of Myers' theorem in dimension n, and then give the combinatorial definitions needed to state the analogous combinatorial theorem, Theorem 3.

The proof of Myers' theorem is a generalization of the argument given for dimension 2. Let $\alpha\colon [0, L] \to M$ be a geodesic in M parametrized by arc-length. Let $P_1, \ldots, P_{n-1}$ be parallel vector fields along α such that for each $t \in [0, L]$, $P_1(\alpha(t)), \ldots, P_{n-1}(\alpha(t))$, $d\alpha(t)/dt$ form an orthonormal basis of the tangent space to M at $\alpha(t)$. The curvature hypothesis implies that for fixed t the average over i of the sectional curvatures $R(d\alpha(t)/dt, P_i(\alpha(t)))$ is $\geq R/(n-1)$. Hence for some i'' the average over t of this sectional curvature is also $\geq R/(n-1)$. The second variational formula now implies that if

$$L > \pi\sqrt{(n-1)/R},$$

then α can be varied in the direction $P_{i''}$ so as to obtain paths with the same end-points which are strictly shorter than α. Thus every two points of M are distant at most $\pi\sqrt{(n-1)/R}$, which proves the theorem.

Now let K be a simplicial complex which is an n-manifold and let K^* be its dual cell complex. I shall describe the appropriate notion of Ricci curvature first in terms of K^* and then in terms of K. Let $\alpha = (a_1, \ldots, a_r)$ be a path in K^* and let c_1 be a 2-cell which has a_1 as face. I now describe how to *parallel translate* c_1 along α to obtain a variational field C_1 of α. Let $a_1, \ldots, a_i$ be the portion of α such that $a_1 < c_1, \ldots, a_i < c_1$ but $a_{i+1} \not< c_1$. Let $d_1 < c_1$ be the 1-cell such that $a_i \cap d_1 = b_i$. Then there is a unique 2-cell $c_2 \in K^*$ such that $d_1 < c_2$ and $a_{i+1} < c_2$. (For the duals d_1^* and a_{i+1}^* are $(n-1)$-simplexes of K, both faces of the n-simplex b_i^*; then $d_1^* \cap a_{i+1}^*$ is an $(n-2)$-simplex whose dual is c_2). Continuing this procedure defines the required ribbon C_1. There are n 2-cells which have a_1 as face; hence there are n distinguished variational fields $C_1, \ldots, C_n$ and corresponding variations $\beta_1, \ldots, \beta_n$ of α.

To see how $C_1, \dots, C_n$ fit together, let us examine the case $|\alpha| = 2$. Then a_1 and a_2 are each faces of n 2-cells of K^*, but one of these 2-cells has both a_1 and a_2 as faces. Hence at the vertex b_1 there are $2n - 1$ distinguished 2-cells distributed among n variational fields, each 2-cell appearing in exactly one field. (And this situation occurs at any internal vertex of any path in K^*.) If one extracts from K^* its 2-skeleton and makes it out of regular polygons of the same side-length, then the total angle of $C_1 \cup \cdots \cup C_n$ at b_1 is

$$\sum\nolimits_{c \in C_1 \cup \cdots \cup C_n,\, b_1 \in c} (1 - 2/|\partial c|)\pi,$$

a $(2n - 1)$-fold sum. To decide whether this represents positive or negative curvature I compare it to the case in which, on the average, the curvature is zero; that is, in which each C_i has angle π at b_1, on the average. So the *Ricci curvature of K^* at b_1 in the direction a_1–a_2* is defined to be

$$R^*(b_1, a_1\text{–}a_2) = n - \sum_{j=1}^{2n-1} {}_{c_j \in C_1 \cup \cdots \cup C_n \text{ and } b_1 \in c_j} (1 - 2/|\partial c_j|).$$

In terms of K, b_1 corresponds to an n-simplex b_1^*, a_1 and a_2 to $(n - 1)$-faces of b_1^*, and $c_1, \dots, c_{2n-1}$ to the $(n - 2)$-faces of $a_1^* \cup a_2^*$. For each j, $|\partial c_j|$ can be measured as the number $N(c_j^*)$ of n-simplexes of which c_j^* is a face. The *Ricci curvature* of K, a simplicial n-manifold without boundary, is defined at an n-simplex s in a "direction" t–t'', where t and t'' are $(n - 1)$-faces of s. $R(s, t\text{–}t'')$ is defined to be

$$2 \sum_{j=1}^{2n-1} {}_{u_j < t \text{ or } < t'',\, \dim u_j = n-2} 1/N(u_j) \quad - n + 1.$$

THEOREM 3. *Let K be a connected simplicial n-manifold without boundary. Assume there is a number $R > 0$ such that $R(s. t\text{–}t'') \geq R$ for all s and t–t''. Then K is finite and has diameter $\leq 2 + 1/R$.*

Proof. Let K^* be the cell complex dual to K. I adopt the notation used to define curvature in K^*. Let $\alpha = (a_1, \dots, a_r)$ be a path in K^* with $a_i \cap a_{i+1} = b_i$ for $i = 1, \dots, r - 1$. Then for each b_i the curvature hypothesis implies that

$$\sum\nolimits_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (1 - 2/|\partial c|) \leq n(1 - R''/2) \quad \text{where } R'' = 2R/n.$$

Adding these inequalities for $i = 1, \dots, r - 1$, we get

$$\sum_{i=1}^{r-1} \sum\nolimits_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (1 - 2/|\partial c|) \leq n(|\alpha| - 1)(1 - R''/2),$$

that is,

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{r-1} {}_{c \in C_j,\, b_i \in c} (1 - 2/|\partial c|) \right) \leq n(|\alpha| - 1)(1 - R''/2).$$

Hence for some j, say for $j = 1$,

$$\sum_{i=1}^{r-1} {}_{c \in C_1,\, b_i \in c} (1 - 2/|\partial c|) \leq (|\alpha| - 1)(1 - R''/2).$$

This is just inequality (3) of the proof of Proposition 1, with R replaced by R''. It follows that K^* is finite and has diameter $\leq 1 + n/R$. I have yet to infer a bound for the diameter of K.

Let v and v'' be vertices of K and let s and s'' be n-simplexes having v and v'' as vertices respectively. There is then a sequence of n-simplexes

$$S = (s_1, \dots, s_{k+1})$$

such that $s = s_1$, $s'' = s_{k+1}$, $k \leq 1 + n/R$, and $s_i \cap s_{i+1}$ is an $(n - 1)$-simplex t_i for $i = 1, \dots, k$. Let $v_1(1), \dots, v_n(1)$ be the vertices of t_1. All but one of these, say $v_j(1)$, are vertices of t_2. Set $v_h(2) = v_h(1)$ for $h \neq j$, and let $v_j(2)$ be the vertex of t_2 which is not a vertex of t_1. Continuing in this way, the vertices of each t_i are labeled $v_1(i), \dots, v_n(i)$. For each $j = 1, \dots, n$ the sequence $v_j(1), \dots, v_j(k)$ determines a path U_j. Now each s_i, for $i = 2, \dots, k$, has just one 1-simplex which is not a face of t_{i-1} or t_i. These $k - 1$ 1-simplexes are distributed among the n paths U_j. Hence some $U_{j''}$ must have length $\leq (k - 1)/n$. This is a path from some vertex of s to some vertex of s''. So there is a path from v to v'' of length $\leq 2 + (k - 1)/n$. The theorem now follows.

Remark. When $n = 3$ the curvature hypothesis reduces to: $\sum_{j=1}^{5} 1/N(u_j) \geq 1 + R/2$ whenever $u_1, \dots, u_5$ are edges of a tetrahedron. The question arises: what happens if $\sum_{j=1}^{5} 1/N(u_j) = 1$ sometimes? Using noncombinatorial methods I have shown [6]:

THEOREM. *If $N(u_j) = 5$ for every 1-simplex, then K is finite.*

One might suppose that an equality of the form $\sum 1/N(u_j) = 1$ represents positive Ricci curvature too small to be measured combinatorially, so that a finiteness theorem would still be true; but this is not so, as I shall show.

Consider a 3-manifold K such that every tetrahedron has three edges u_1, u_2, and u_3 with $N(u_i) = 4$ and the other three edges with $N(u_j) = 6$. Then $\sum_{k=1}^{5} 1/N(u_k)$ equals 1 or 13/12 according as the omitted edge is a u_i or u_j. If in every tetrahedron the u_i meet in a vertex, then one can show that a path in K^* along the Ricci curvature is always zero must run around the boundary of a 2-cell; so any long, direct path accumulates nonzero curvature, and the method of this paper shows that K is finite. In fact it can be shown that the universal cover of K has to be a certain subdivision of the regular polytope with Schlaefli symbol $\{3, 4, 3\}$ (regular octahedra, three about each edge; see Coxeter [3]). However, if in every tetrahedron $\bigcap u_i = \emptyset = \bigcap u_j$ then one can have infinitely long paths along which the Ricci curvature is always zero, as the next example shows, and such a complex can be infinite.

Example. Here is an example for each n to show that the theorem is false unless the inequality $R > 0$ is strict. It is a triangulation of $S^{n-1} \times \mathbf{R}$. Let $x_1, \dots, x_n$ be axes in $\mathbf{R}^n$ and let $+k$ and $-k$ be points on the positive and negative x_k-axis. The boundary of the convex linear hull of the $\{\pm k\}$ is a simplicial triangulation L of S^{n-1}. The $(n-1)$-simplexes of L are in one-to-one correspondence with sequences $\{\pm 1, \ldots, \pm n\}$; the integers all occur in increasing order, and the signs are arbitrary. Every $(n-3)$-simplex of L is a face of four $(n-1)$-simplexes. Triangulate $\mathbf{R}$ so that the integers are the vertices. $L \times \mathbf{R}$ is a cell complex; by a "prism" I shall mean the product of an $(n-1)$-simplex of L and a unit interval of $\mathbf{R}$. The required simplicial complex K is a subdivision of $L \times \mathbf{R}$. The vertices of K are just those of $L \times \mathbf{R}$, which may be denoted $\{(\pm k, p)\}$. Then the n-simplexes of K are by definition in one-to-one correspondence with sequences

$$\{(\pm 1, p), \ldots, (\pm k, p), (\pm k, p+1), \ldots, (\pm n, p+1)\},$$

in which the first integer takes on all values from 1 to n in increasing order, the signs are arbitrary except that those of $(\pm k, p)$ and $(\pm k, p+1)$ must be the same, and k is arbitrary between 1 and n.

To simplify notation let us consider first the prism P in which all the signs are $+$ and $p = 0$. Let u be an $(n-2)$-simplex of P; I shall calculate $N(u)$.

Notation. If H is an n-dimensional cell complex and $a \in H$, then $N(a, H)$ is the number of n-cells of H to which a is a face; H might be a subcomplex of some larger complex.

To find $N(u)$ I need to know $N(u, P)$ and how many prisms u belongs to; this number depends on the projections $L(u)$ and $\mathbf{R}(u)$ of u into L and $\mathbf{R}$. First let us assume that $\mathbf{R}(u) = [0, 1]$. Write the vertices of u in lexicographical order. Then p must change from 0 to 1, say from $(h, 0)$ to $(k, 1)$. There are three cases:

(I) $k = h$; then $N(u, P) = 1$;
(II) $k = h + 1$; then $N(u, P) = 2$;
(III) $k = h + 2$; then $N(u, P) = 3$.

The number of prisms to which u belongs equals $N(L(u), L)$. In Case I, $\dim L(u) = n - 3$, so $N(L(u), L) = 4$. In Cases II and III, $\dim L(u) = n - 2$, so $N(L(u), L) = 2$. It follows that $N(u, K)$ is 4 in Cases I and II, and 6 in Case III.

Now let us assume that $\mathbf{R}(u) = 0$ (the case $\mathbf{R}(u) = 1$ is similar). Then u is one of these types:

(IV) $\{(1, 0), \ldots, (n, 0)\}$ but where a single term is missing;
(V) $\{(1, 0), \ldots, (n-1, 0)\}$;
(VI) $\{(2, 0), \ldots, (n, 0)\}$.

It is not hard to see that $N(u, K)$ is 4 in Case IV and 6 in Cases V and VI. In Case VI, for example, the possible n-simplexes are $\{(\pm 1, 0), (2, 0), \ldots, (n, 0), (n, 1)\}$, $\{(\pm 1, -1), (2, -1), (2, 0), \ldots, (n, 0)\}$, and $\{(\pm 1, -1), (\pm 1, 0), (2, 0), \ldots, (n, 0)\}$ provided the first two terms have the same sign.

Now let $s = \{(1, 0), \ldots, (k, 0), (k, 1), \ldots, (n, 1)\}$ be an n-simplex of P. I count the number of $(n-2)$-faces u of s such that $N(u, K) = 6$. If $3 \leq k \leq n-2$, then s has three faces of type III. If $k = n-1$ (or if $k = 2$), then s has one face of type III and one of type VI (or of type V, respectively)—but if $n = 3$, there is also a face of type V. If $k = n$ (or 1), then s has one face each of types III, V and VI. Thus every n-simplex of K has at most three $(n-2)$-faces u for which $N(u) = 6$ and $N(u'') = 4$ for its other $(n-2)$-faces. Consequently

$$R(s, t\text{–}t'') \geq 2((2n-4)/4 + 3/6) - n + 1 = 0$$

everywhere; the equality definitely holds for certain choices of s, t and t''. And the conclusion of Theorem 3 is of course false. In fact if t is an $(n-1)$-simplex of L and $\hat{t}$ its barycenter, then $\hat{t} \times \mathbf{R}$ is the underlying space of an infinite path in K^*, no finite portion of which has shorter variations.

I shall conclude with two generalizations of Theorem 3. The first just combines the theorem with Corollary 2.

COROLLARY 4. *Let K be a simplicial complex which is an n-manifold without boundary. Assume there is a number $R > 0$ such that $R(s, t\text{–}t') + R(s'', t'\text{–}t'') \geq 2R$ whenever s and s'' are n-simplexes intersecting in an $(n-1)$-simplex t', and t and t'' are $(n-1)$-faces of s and s'' respectively, different from t'. Then K is finite and has diameter $\leq 2 + (4n + 1 - 3R)/3Rn$.*

Proof. I use the notation of the proof of Theorem 3. The curvature hypothesis says that for each $i = 1, \ldots, r-2$,

$$\sum_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (1 - 2/|\partial c|) + \sum_{c'' \in C_1 \cup \cdots \cup C_n,\, b_{i+1} \in c''} (1 - 2/|\partial c''|) \leq 2n(1 - R''/2),$$

where $R'' = 2R/n$. Now each of the two sums on the left-hand side is a $(2n-1)$-fold sum, each of whose terms is $\geq 1/3$, since $|\partial c|$ is always ≥ 3. It follows that

$$\sum_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (1 - 2/|\partial c|) \leq 2n(1 - R''/2) - (2n-1)/3.$$

As in the proof of Corollary 2 it follows that

$$2 \sum_{i=1}^{r-1} \sum_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (1 - 2/|\partial c|) \leq (|\alpha| - 2)2n(1 - R''/2) + 2(2n(1 - R''/2) - (2n-1)/3).$$

The proof of Proposition 1 shows that K^* has diameter $\leq (4n+1)/3R$. The corollary now follows from the proof of Theorem 3.

The other generalization of Theorem 3 has to do with not requiring K to be a manifold. If K is a simplicial complex, not necessarily a manifold, then it has a dual cone complex K^* (see Akin [1]). All that is needed to define Ricci curvature and to prove Theorem 3 is that the 2-skeleton of K^* be a connected cell complex. In terms of K this means:

(1) K is a *geometrical n-circuit*, that is, every $(n-1)$-simplex of K is a face of exactly two n-simplexes of K, and the complement of its $(n-2)$-skeleton is connected and

(2) K has no $(n-2)$-dimensional singularities—the link of every $(n-2)$-simplex is a simple closed curve.

In fact the second condition can be evaded. For if K is a geometrical n-circuit, then the dual of an $(n-2)$-simplex is the cone on a number of simple closed curves, and in the definitions and arguments I can restrict attention to the cone on just one of these curves. Thus if c is a 2-cone, then ∂c refers to one of the components of bdy c singled out by the context, and $|\partial c|$ refers to the number of 1-faces of that component. In terms of K the adjustments to be made are these: if s is an n-simplex and u an $(n-2)$-face of s, then $N(u; s)$ denotes the number of 1-simplexes in that component of link (u, K) to which link (u, s) belongs; and the Ricci curvature at s in the direction t–t'' is redefined to be

$$2 \sum_{j=1}^{2n-1} {}_{u_j < t \text{ or } t'', \dim u = n-2}\ 1/N(u; s) \quad - n + 1.$$

If these changes are made, the proof of Theorem 3 will also prove:

THEOREM 5. *Let K be a geometrical n-circuit. Assume there is a number $R > 0$ such that $R(s, t\text{–}t'') \geq R$ for all s and t–t''. Then K is finite and has diameter $\leq 2 + 1/R$.*

For example K might be a triangulation of a complex analytic variety. I do not know if Myers' theorem has been generalized to smooth manifolds with singularities.

Corollary 4 can also be generalized to geometrical n-circuits.

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