ON THE CONVERSE OF THE TROTTER PRODUCT FORMULA

BY

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Suppose that $A(t)$, $0 \leq t < \infty$, is a strongly continuous operator-valued function with $A(0) = I$ and $\|A(t)\| \leq 1$ for all t. Suppose in addition that $A(t)$ is strongly differentiable at $t = 0$, and that the closure X of $A'(0)$ is the generator of a (C_0) semigroup e^{tX}. Then

$$\lim_{n \to \infty} A(t/n)^n = e^{tX}, \tag{1}$$

where the convergence is in the strong operator topology. This result, proved in [1], includes the Trotter product formula [5] as a special case. The converse is false in general; that is, (1) may hold without $A(t)$ being differentiable at $t = 0$. Some explicit counter-examples were given in [2, Section 5.7]. Those examples were set in infinite dimensional Hilbert space. It is a plausible guess that in *finite* dimensions the converse is true. The proof of this conjecture is the main result of this paper. We have also added a few more remarks on the situation in infinite dimensions.

THEOREM. *Let V be a finite dimensional normed space. Let $A(t)$, $0 \leq t < \infty$, be a continuous function whose values are operators on V, and suppose that $A(0) = I$. Assume that there is an operator X on V such that* (1) *holds for all $t \geq 0$. Then the derivative $A'(0)$ exists and equals X.*

Proof. We write $A(t)$ as $e^{B(t)}$ for some continuous operator-valued function $B(t)$ with $B(0) = 0$. (Strictly speaking, $A(t)$ need have this form only for sufficiently small values of t, but the behavior of $A(t)$ outside a neighborhood of $t = 0$ is irrelevant for the theorem.) We have

$$\lim_{n \to \infty} e^{nB(t/n)} = e^{tX}. \tag{2}$$

If we could "take logarithms" in (2) it would be relatively easy to conclude the proof by showing that $B'(0) = X$. The main difficulty is to justify this step. We will use the following lemmas.

LEMMA 1. *Let $\rho(t)$, $0 \leq t < \infty$, be a real-valued continuous function. Assume the following:*

(a) $\rho(0) = 0$.

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(b) *There is a positive number T such that, given any $t \in [0, T]$, if n is a sufficiently large integer then*

$$n\rho(t/n) \in I + 2\pi\mathbf{Z}.$$

Here I is the interval $[-\pi/4, \pi/4]$ and $\mathbf{Z}$ is the set of all integers as usual. Then $\rho(t) = O(t)$ as $t \to 0$.

Proof. It is convenient to reformulate the lemma in terms of the function

$$g(t) = \frac{1}{2\pi}\rho(1/t).$$

The hypotheses say that g is continuous, $g(t) \to 0$ as $t \to \infty$, and if $t \geq R = 1/T$ then, for all sufficiently large integers n,

$$ng(nt) \in I_0 + \mathbf{Z}.$$

Here I_0 is the interval $[-1/8, 1/8]$. We have to prove that $tg(t)$ is bounded as $t \to \infty$.

Consider the set $E_N = \{x \geq R\colon ng(nx) \in I_0 + \mathbf{Z} \text{ for all } n \geq N\}$. Because g is continuous this set is closed, and by hypothesis $\bigcup_{N=1}^{\infty} E_N = [R, \infty)$. The Baire category theorem implies that some E_{N_0} contains a nondegenerate open interval $J = (a, b)$, with $R \leq a < b \leq R + 1$.

Since $g(x) \to 0$ as $x \to \infty$, we choose $N_1 \geq N_0$ so that $|g(nx)| \leq 1/16$ for all $n \geq N_1$, $x \in J$.

Now for $x \in J$ and $n \geq N_0$ let $z_n(x)$ be the (unique) integer closest to $ng(nx)$. We have $|ng(nx) - z_n(x)| \leq 1/8$. Because $ng(nx)$ is a continuous function of x, so is $z_n(x)$. Since J is connected, $z_n(x)$ must be a constant, z_n.

Next fix any $x_0 \in J$. Because J is open, there is an integer $N_2 \geq N_1$ such that if $n \geq N_2$ then $x_n = (1 + 1/n)x_0 \in J$. We claim that if $n \geq N_2$ then $z_{n+1} = z_n$. Indeed,

$$\begin{aligned}(n + 1)g((n + 1)x_0) &= (n + 1)g(nx_n)\\ &= ng(nx_n) + g(nx_n)\\ &= (z_n + \varepsilon) + g(nx_n).\end{aligned}$$

Here $|\varepsilon| \leq 1/8$, while $|g(nx_n)| \leq 1/16$ since $n \geq N_1$. Therefore $(n + 1)\cdot g((n + 1)x_0)$ is within $1/8 + 1/16 = 3/16$ of the integer z_n. Since z_{n+1} is the integer closest to $(n + 1)g((n + 1)x_0)$ we must have $z_{n+1} = z_n$. In other words, z_n is a constant z for $n \geq N_2$. Thus for all $x \in J$ and $n \geq N_2$ we have

$$|ng(nx) - z| \leq 1/8.$$

Actually, z must be 0. If not, suppose that $z > 0$. Fix $x \in J$ and consider $y = 2zx$. Then

$$ng(ny) = \frac{1}{2z}\cdot 2nz\, g(2nz \cdot x),$$

and the right side is within $(1/2z)(1/8) = 1/16z$ of $z/2z = 1/2$ if n is large. On the other hand, the left side must be within $1/8$ of an integer when n is large.

That is, $1/2$ must be within distance $(1/8 + 1/16z)$ of an integer, which is absurd. We get a similar contradiction if $z < 0$.

Now we have established that for all $x \in J$ and all $n \geq N_2$, $|ng(nx)| \leq 1/8$. If $y \in k \cdot J$ for $k \geq N_2$, write $y = kx$, $x \in J$; then we have

$$\begin{aligned} |yg(y)| &= |kxg(kx)| \\ &\leq (R + 1)\,|kg(kx)| \\ &\leq (R + 1)/8 \\ &= C. \end{aligned}$$

That is, $|yg(y)| \leq C$ if $y \in \bigcup_{k=N_2}^{\infty} k \cdot J$. But since J is a nontrivial interval, the latter union contains all sufficiently large real numbers. ■

LEMMA 2. *Let x be an element of a Banach algebra with unit* 1. *Suppose*

(a) $\|e^x - 1\| < 1$

(b) $\rho(x) < 1/2$ (ρ = *spectral radius*).

Define $y = e^x - 1$. Then $x = \log(1 + y)$, where the logarithm is defined by the series

$$\log(1 + y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} y^n.$$

Proof. By hypothesis (a) we have $\|y\| < 1$, so the series for $\log(1 + y)$ is convergent. We have

$$\begin{aligned} \log(1 + y) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (e^x - 1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left[\sum_{j=1}^{\infty} \frac{x^j}{j!}\right]^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=n}^{\infty} A_{nk} x^k \end{aligned} \tag{3}$$

where the A_{nk} are certain nonnegative coefficients which do not depend on x.

It is evident *a priori* that the right side of (3) must reduce to x if $\|x\|$ is sufficiently small (by the inverse function theorem applied to $f(x) = e^x - 1$ near $x = 0$). Therefore, *whenever* we can interchange the order of summation in (3), the right side must be x by virtue of identities which the A_{nk} must satisfy.

To justify the interchange when $\rho(x) < 1/2$ we simply show that the series is absolutely convergent; that is, we assert that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} A_{nk} \|x^k\| < \infty. \tag{4}$$

To prove (4) we have to estimate A_{nk}. Now by definition

$$A_{nk} = \sum_{j_1 + j_2 + \cdots + j_n = k,\; j_i \geq 1} \frac{1}{j_1!\, j_2! \cdots j_n!}$$

and this is certainly less than B_{nk}, the number of n-tuples $(j_1, j_2, \dots, j_n)$ of positive integers with $\sum j_i = k$. But B_{nk} is obviously the coefficient of λ^k in the expansion of

$$\left(\sum_1^\infty \lambda^j\right)^n = \lambda^n(1 - \lambda)^{-n}.$$

Thus the left side of (4) is majorized by the series

$$\sum_{n=1}^\infty \frac{1}{n} \sum_{k=n}^\infty B_{nk}\|x^k\|. \tag{5}$$

Write $\rho(x) = \rho < 1/2$. Then we have $\|x^k\| \leq M\rho^k$ for some constant M. Therefore (5) is majorized by the series

$$\sum_{n=1}^\infty \frac{1}{n} \sum_{k=n}^\infty B_{nk}\rho^k = \sum_{n=1}^\infty \frac{1}{n} \rho^n(1 - \rho)^{-n}. \tag{6}$$

Since $\rho < 1/2$, the quantity $\rho(1 - \rho)^{-1} < 1$, and series (6) is convergent. This establishes (4). ∎

LEMMA 3. *Let $B(t)$, $0 \leq t < \infty$, be a continuous function with values in a Banach space. Suppose that $B(0) = 0$, and that for all $t \geq 0$, $\lim_{n\to\infty} nB(t/n) = tX$, where X is independent of t. Then B is differentiable at* 0, *and $B'(0) = X$.*

Proof. Let $f(t) = \|B(t) - tX\|$. Then f is continuous, $f(0) = 0$, and $\lim_{n\to\infty} nf(t/n) = 0$. We must prove that $\lim_{t\to 0} f(t)/t = 0$. This can be shown by a standard Baire category argument along the lines of the proof of Lemma 1. In greater detail, consider the function $g(t) = tf(1/t)$. The function g is continuous, and $g(nx) \to 0$ as $n \to \infty$, for any fixed $x > 0$. We claim that $g(x) \to 0$ as $x \to \infty$. Given $\varepsilon > 0$, let $E_N = \{x: g(nx) \leq \varepsilon \text{ for all } n \geq N\}$. As in the proof of Lemma 1, some E_{N_0} contains a nondegenerate interval J. It is immediate that $g(y) \leq \varepsilon$ for all y in $\bigcup_{k=N_0}^\infty k \cdot J$, and this set contains all sufficiently large real numbers. ∎

Completion of the proof of the theorem. Let Δ be the disk in the complex plane with center at 0 and radius 1/8. There is a positive number ε such that if $|e^y - 1| < \varepsilon$ then z must be in Δ, modulo an integer multiple of $2\pi i$. Let D be the disk with center at 1 and radius ε.

Choose $T_0 > 0$ small enough so that $\|e^{tX} - I\| < \varepsilon$ for $0 \leq t \leq T_0$. Since $A(t/n)^n \to e^{tX}$, we have $\|A(t/n)^n - I\| < \varepsilon$ for n sufficiently large, and so the spectrum $\sigma(A(t/n)^n)$ is contained in the disk D. Now

$$\sigma(A(t/n)^n) = \{e^{n\lambda}: \lambda \in \sigma(B(t/n))\}.$$

Hence we conclude that, if $0 \leq t \leq T_0$ and n is large,

$$\{n\lambda: \lambda \in \sigma(B(t/n))\} \subseteq \Delta + 2\pi i\mathbf{Z}. \tag{7}$$

Write $\rho(B(t)) = \rho(t)$, the spectral radius. From (7) we have, if $0 \leq t \leq T_0$ and n is large,

$$n\rho(t/n) \subseteq I + 2\pi\mathbf{Z} \tag{8}$$

where I is the interval $[-1/8, 1/8]$. Also, since we are dealing with finite dimensional operators, $\rho(B(t)) = \rho(t)$ is a continuous function of t (cf. [3, p. 213]). Therefore we may apply Lemma 1 to deduce that $\rho(t) = O(t)$ as $t \to 0$. That is, there is a constant M such that $\rho(t) \leq Mt$ for small t. Pick $T_1 \leq T_0$ such that $\rho(t) \leq Mt$ for $0 \leq t \leq T_1$ and also $MT_1 < 1/2$.

Then for $0 \leq t \leq T_1$ and $n = 1, 2, \ldots$ we have

$$\rho(nB(t/n)) = n\rho(t/n) \leq n \cdot M \cdot t/n \leq MT_1 < 1/2. \tag{9}$$

Also, when n is large we have $\|e^{nB(t/n)} - I\| = \|A(t/n)^n - I\| < 1$. Accordingly we may apply Lemma 2 with $x = nB(t/n)$ to deduce that

$$nB(t/n) = \log(I + (A(t/n)^n - I)). \tag{10}$$

The right side of (10) converges, as $n \to \infty$, to

$$\log(I + (e^{tX} - I)) = tX,$$

since we are dealing with the principal branch of the logarithm. Thus the hypotheses of Lemma 3 are satisfied, and therefore $B'(0)$ exists and equals X. It is immediate from this that $A'(0)$ exists and equals X as well. ■

This theorem could be expressed more abstractly in terms of curves in a Lie group G. The product formula (1) would read $A(t/n)^n \to \exp(tX)$ for some X in the Lie algebra of G, and we ask when this implies that $A'(0)$ exists and equals X. Since any Lie group is locally topologically isomorphic to a finite dimensional matrix group, the preceding theorem may be applied and we get the same conclusion.

Finally, what about possible infinite dimensional generalizations? The counter-examples in [2] would seem to rule out any such possibilities when we are dealing with the strong operator topology, but we can say something if we insist upon norm convergence in formula (1). Accordingly, suppose that V is a Banach space, and $t \to A(t)$ is a norm-continuous function from $[0, \infty)$ to $B(V)$, the bounded operators on V. Suppose that there is an operator $X \in B(V)$ such that $A(t/n)^n \to e^{tX}$ in norm. Can we conclude that $A'(0)$ (norm derivative) exists and equals X? In the proof of the preceding theorem the finite dimensionality of V was needed just once, to allow us to conclude that $\rho(B(t))$ was a continuous function. Unfortunately, in the infinite dimensional situation the best we can say in general is that $\rho(B(t))$ is upper semicontinuous (cf. [4, p. 282] for an example of the failure of continuity). And it is not hard to see that Lemma 1 fails in general for upper semicontinuous functions ρ. On the other hand, there are large classes of operators on which the spectral radius is continuous relative to the operator norm topology. Thus, for example, the theorem can certainly be extended to the case of functions $A(t)$ whose values are normal operators in Hilbert space.

References

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