SPLITTING CERTAIN SUSPENSIONS VIA SELF-MAPS

BY

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The essential content of this paper is a potpourri of examples concerning the splitting of certain suspensions into wedges. Our first example is an improvement of Snaith's [12] stable decomposition of $\Omega^n\Sigma^n X$, $1 < n < \infty$. Here we must work harder than Snaith, but we get a finer splitting (sometimes!). Our second example is provided by the suspension of an H-space. An application is made to the suspension of $SF(2)$ which is in the same spirit as Snaith's decomposition. As a final example we show that a geometric analogue of the classical algebraic transfer for finite covers may, in nice cases, be defined after a single suspension. As an application of these results we use Flynn's calculations [5] to give relatively instant generalizations of Kamata's calculations [8] in [4].

The motivation for describing these three examples in the same paper is that we use the same method throughout. This method, which is completely elementary and often easy to apply, is undoubtedly well-known. However, we know of no published account of the method with the exception of an example due to Holzsager [6].

Throughout this paper all spaces are tacitly assumed to be connected, of finite type, and of the homotopy type of a CW-complex of finite type. $X_{(p)}$ denotes the localization of the space X at p and homology is taken with Z_p-coefficients for p prime unless otherwise stated. Σ^n denotes the n-fold suspension functor (reduced).

0. A general observation about self-maps

Evidently $\Sigma^n X$ splits into a wedge of the form $A_1 \vee \cdots \vee A_k$ provided $\Sigma^n X$ is equipped with self-maps which yield an orthogonal decomposition of $\tilde{H}_*\Sigma^n X$. We make this precise:

DEFINITION 0.0. $(\Sigma^n X)_{(p)}$ is said to be equipped with splitting maps $f_1, \ldots, f_k$ if the f_i are self-maps of $(\Sigma^n X)_{(p)}$ such that if $M_i = f_{i*}\tilde{H}_*\Sigma^n X$, then $f_{i*}(M_j) = 0$ for $i \neq j$, $f_{i*}(M_i) = M_i$ (and $f_{i*}\colon M_i \to M_i$ is an isomorphism), and $\tilde{H}_*\Sigma^n X \cong M_1 \oplus \cdots \oplus M_k$.

PROPOSITION 0.1. *$(\Sigma^n X)_{(p)}$ has the homotopy type of $A_1 \vee \cdots \vee A_k$ where $\tilde{H}_*A_i \cong M_i$, $i = 1, \ldots, k$, if and only if $(\Sigma^n X)_{(p)}$ is equipped with splitting maps $f_1, \ldots, f_k$.*

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Proof. To prove the least obvious part of the equivalence, we consider the mapping telescope $\mathrm{tel}_{f_i}(\Sigma^n X)_{(p)}$ defined to be the quotient of

$$\bigcup_{\mathrm{disj},\, m \geq 0} (\Sigma^n X)_{(p)} \times [2m, 2m + 1]$$

where $(y, 2m + 1)$ is identified to $(f_i(y), 2m + 2)$. Set $A_i = \mathrm{tel}_{f_i}(\Sigma^n X)_{(p)}$. Let θ_i denote the natural map of $(\Sigma^n X)_{(p)}$ in A_i given by $x \to (x, 0)$ and define

$$\theta: (\Sigma^n X)_{(p)} \to A_1 \vee \cdots \vee A_k$$

to be the composite

$$(\Sigma^n X)_{(p)} \xrightarrow{\text{pinch}} (\Sigma^n X)_{(p)} \vee \cdots \vee (\Sigma^n X)_{(p)} \xrightarrow{\theta_1 \vee \cdots \vee \theta_k} A_1 \vee \cdots \vee A_k.$$

To show that θ is a homotopy equivalence, it suffices to show that θ is a mod-p homology isomorphism. This follows directly from 0.0 and the fact that homology commutes with direct limits.

1. A remark on the stable decomposition of $\Omega^n\Sigma^n X$, $1 < n < \infty$

V. Snaith [12] has shown that if X is connected and of the homotopy type of a CW-complex, then $\Omega^n\Sigma^n X$ splits stably into a wedge of spaces $e[\mathscr{C}_n(j), \Sigma_j, X]$ which are defined below. The results of this section, which should be regarded as an addendum to Snaith's work, show that the single suspensions of $\Omega^n\Sigma^n X$ and $e[\mathscr{C}_n(j), \Sigma_j, X]$ split into wedges of nontrivial spaces provided $\tilde{H}_* X$ consists entirely of odd torsion. Throughout Sections 1–3 we assume that $1 < n < \infty$.

Conceptually, our results state that $\Sigma\Omega^n\Sigma^n X$ and $\Sigma e[\mathscr{C}_n(j), \Sigma_j, X]$ each split into two pieces, one of which supports "half" of the "unstable" homology; the other piece supports the remaining "half" of the "unstable" homology together with the "stable" homology. Furthermore, this splitting is best possible in the sense that $\Omega^n\Sigma^n X$ and $e[\mathscr{C}_n(j), \Sigma_j, X]$ cannot be stably split into pieces which segregate the "stable" and "unstable" parts of homology. Of course, we must rigorously define the terms "stable" and "unstable," which we do after reviewing the requisite homological facts concerning $H_*\Omega^n\Sigma^n X$.

By the work of May [10], there is an approximation, $C_n X$, of $\Omega^n\Sigma^n X$ together with a weak equivalence $\alpha_n: C_n X \to \Omega^n\Sigma^n X$. The homologies of $C_n X$ and $\Omega^n\Sigma^n X$ have been described in [2] and [3] as a certain free functor of $H_* X$ called $W_{n-1}H_* X$, which we include here for convenience.

For simplicity, we define $W_{n-1}H_* X$ for those X which are connected and take all homology with Z_p-coefficients for p an odd prime. Here, the definition of basic λ_{n-1}-products is first required.

Let $\{x_\alpha \mid x_\alpha \in \tilde{H}_* X\}$ run over a totally ordered basis for $\tilde{H}_* X$. We define the x_α to be the basic λ_{n-1}-products of weight one. Assume that the basic λ_{n-1}-

products of weight j, $j < k$, have been defined and totally ordered amongst themselves. We define a basic λ_{n-1}-product of weight k to be $\lambda_{n-1}(x, y)$ where

(1) $\lambda_{n-1}(\ ,\ )$ is the Browder operation described in [2],
(2) x and y are basic λ_{n-1}-products with weight(x) + weight$(y) = k$,
(3) $x < y$ and if $y = \lambda_{n-1}(z, w)$ for z and w basic λ_{n-1}-products, then $x \leq z$,
or
(4) $x = y$ where x is a basic λ_{n-1}-product of weight one and n + degree(x) is even.

If x is a basic λ_{n-1}-product of weight k, we write $w(x) = k$.
To complete the definition of $W_{n-1}H_*X$, we consider sequences

$$J = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$$

such that $\varepsilon_j = 0$ or 1 and $s_j \geq \varepsilon_j$. Define $b(J) = \varepsilon_1$, and the excess of J, $e(J)$, by the formula

$$e(J) = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j).$$

J is said to be admissible if $ps_j - \varepsilon_j \geq s_{j-1}$ for $2 \leq j \leq k$. We write Q^J for the sequence of iterated Dyer-Lashof operations given by $\beta^{\varepsilon_1}Q^{s_1}\cdots\beta^{\varepsilon_k}Q^{s_k}$. We define $W_{n-1}H_*X$ (as an algebra) as the free commutative algebra on

$$\{Q^J\lambda_I \mid \lambda_I \text{ is a basic } \lambda_{n-1}\text{-product, } J \text{ is admissible, } e(J) + b(\lambda_I) > \text{degree}(\lambda_I), \text{ and } 2s_k \leq n - 1 + \text{degree}(\lambda_I).\}$$

Remark. $W_{n-1}H_*X$ is naturally equipped with additional structure which we deliberately choose to ignore here. This additional structure is described in [2, Sections 1–2].

Let

$$(*) \qquad Q^{J_1}\lambda_{I_1} * \cdots * Q^{J_m}\lambda_{I_m}$$

be an arbitrary monomial in $W_{n-1}H_*X$.

Define S to be the vector subspace of $W_{n-1}H_*X$ spanned by those monomials of the form (*) where $w(\lambda_{I_s}) = 1$, $s = 1, \ldots, m$. Let U denote the vector space spanned by those monomials of the form (*) where $w(\lambda_{I_s}) > 1$ for some s. Then by [3], the homomorphism j_{n*} induced by the inclusion

$$j_n\colon \Omega^n\Sigma^n X \to \lim \Omega^n\Sigma^n X = QX$$

has kernel U and restricts to a monomorphism on S. To make the statements of the second paragraph in this section precise, we define S as the stable part of $W_{n-1}H_*X$ and U as the unstable part of $W_{n-1}H_*X$.

We decompose U further. Let U_{ev} (U_{od}) be the vector subspace of U spanned by elements of the form (*) where $w(\lambda_{I_i}) + \cdots + w(\lambda_{I_n})$ is even (odd). Finally if M is a graded vector space, we let sM denote the vector space given by raising all elements of M one degree. With these preliminaries we have

THEOREM 1.1. *There exist simply-connected spaces A and B together with a natural map*

$$\phi\colon \Sigma\Omega^n\Sigma^n X \to A \vee B$$

all of which depend functorially on X such that

(1) ϕ_* *is a* mod-p *homology isomorphism for all odd primes* p,
(2) $\phi_*(sS \oplus sU_{\mathrm{od}}) = \tilde{H}_* A$, *and*
(3) $\phi_*(sU_{\mathrm{ev}}) = \tilde{H}_* B$.

Furthermore, if $\beta x \neq 0$ or $P_*^r x \neq 0$ for some $x \in H_q X$ where $n + q$ is odd, then there does not exist a stable map, ψ^s, of $\Omega^n\Sigma^n X$ into $C \vee D$ such that $\psi_*^s(S) = \tilde{H}_*^s C$ and $\psi_*^s(U) = \tilde{H}_*^s D$. —

We have analogous results for $e[\mathscr{C}_n(j), \Sigma_j, X]$. To conveniently state them we first recall from [3] that $H_* e[\mathscr{C}_n(j), \Sigma_j, X] = E_j$ is spanned by elements of the form (*) which satisfy

$$\sum_{s=1}^{m} p^{l(Q^{J_s})} \cdot w(\lambda_{I_s}) = j.$$

We define

$$S_j = S \cap E_j, \quad U_{\mathrm{ev},j} = U_{\mathrm{ev}} \cap E_j, \quad U_{\mathrm{od},j} = U_{\mathrm{od}} \cap E_j, \quad U_j = U \cap E_j.$$

THEOREM 1.2. *There exist simply-connected spaces A_j and B_j together with a natural map*

$$\phi_j\colon \Sigma e[\mathscr{C}_n(j), \Sigma_j, X] \to A_j \vee B_j$$

all of which depend functorially on X such that

(1) ϕ_{j*} *is a* mod-p *homology isomorphism for all odd primes* p,
(2) $\phi_{j*}(sS_j \oplus sU_{\mathrm{od},j}) = \tilde{H}_* A_j$, *and*
(3) $\phi_{j*}(sU_{\mathrm{ev},j}) = \tilde{H}_* B_j$.

Furthermore if $\beta x \neq 0$ or $P_*^r x \neq 0$ for some $x \in H_q X$, where $n + q$ is odd, then there does not exist a stable map, ψ_j^s of $e[\mathscr{C}_n(j), \Sigma_j, X]$ into $C_j \vee D_j$ such that $\psi_{j*}^s(S_j) = \tilde{H}_*^s(C_j)$ and $\psi_{j*}^s(U_j) = \tilde{H}_*^s(D_j)$.

The following corollary of 1.2 improves Snaith's result at odd primes.

COROLLARY 1.3. *If $\tilde{H}_* X$ consists entirely of odd torsion then $\Sigma e[\mathscr{C}_n(j), \Sigma_j, X]$ splits into a wedge of two spaces whose homologies are given by Theorem* 1.2.

Proof. By the construction of $W_{n-1}H_* X$ and the hypothesis on $\tilde{H}_* X$, we see that

$$H_*(e[\mathscr{C}_n(j), \Sigma_j, X]; Z)$$

has no two-torsion [3, Section 4]. Consequently, ϕ_{j*} is an isomorphism on integral homology. Since $A_j \vee B_j$ is simply connected, the result follows directly from the Whitehead theorem.

Remark. If $\tilde{H}_* X$ has 2-torsion or a Z-summand, then it is easy to see that $H_* e[\mathscr{C}_n(j), \Sigma_j, X]$ has 2-torsion. In this case the map ϕ_j is only an equivalence away from 2.

2. Geometry of these splittings

We assume that the reader is familiar with the space of j "little n-cubes," $\mathscr{C}_n(j)$ [10]. Let $g\colon I^n \to I^n$ be given on points by

$$g(t_1, \ldots, t_n) = (1 - t_1, t_2, \ldots, t_n).$$

Observe that g is *not* contained in $\mathscr{C}_n(1)$ and that g has order 2 (under composition). Further define $\bar{c}\colon \mathscr{C}_n(j) \to \mathscr{C}_n(j)$ by setting $\bar{c}\langle c_1, \ldots, c_j\rangle$ equal to the composite

$$\overbrace{I^n \cup \cdots \cup I^n}^{j\text{ cubes}} \xrightarrow{g \cup \cdots \cup g} I^n \cup \cdots \cup I^n \xrightarrow{c_1 \cup \cdots \cup c_j} I^n \xrightarrow{\quad g \quad} I^n.$$

It is trivial to check that $\bar{c}\langle c_1, \ldots, c_j\rangle$ is indeed an element of $\mathscr{C}_n(j)$. Observe that $\bar{c}$ has been studied in [3].

We recall from [10] that

$$e[\mathscr{C}_n(j), \Sigma_j, X] = \frac{\mathscr{C}_n(j) \times_{\Sigma_j} \overbrace{X \wedge \cdots \wedge X}^{j\text{ times}}}{\mathscr{C}_n(j) \times_{\Sigma_j} (\text{basepoint})}$$

where Σ_j is the symmetric group on j letters and

$$C_n X = \frac{\Sigma \mathscr{C}_n(j) \times X^j}{\approx}$$

where $\approx$ is a certain equivalence relation. We define

$$\bar{c}\colon \mathscr{C}_n(j) \times X^j \to \mathscr{C}_n(j) \times X^j$$

by $\bar{c}(d, x) = (\bar{c}d, x)$. It is trivial to check that $\bar{c}$ induces a continuous self-map of $e[\mathscr{C}_n(j), \Sigma_j, X]$ and of $C_n X$. By abuse of notation, we label all of these maps by $\bar{c}$.

We require the following lemma whose proof is left to the reader.

LEMMA 2.1. *The following diagram Σ_j-equivariantly commutes:*

$$\begin{tikzcd}
\mathscr{C}_n(j) \times C_n X^j \arrow[r, "\theta_n"] \arrow[d, "\bar{c} \times \bar{c}^j"] & C_n X \arrow[d, "\bar{c}"] \\
\mathscr{C}_n(j) \times C_n X^j \arrow[r, "\theta_n"] & C_n X.
\end{tikzcd}$$

We next require a computation of the action of $\bar{c}_*$ in homology. Recall from [1] that $\mathscr{C}_n(2)$ has the equivariant homotopy type of S^{n-1}.

LEMMA 2.2. (1) $\bar{c}_*\iota = -\iota$ *where* ι *is the fundamental class of* $\mathscr{C}_n(2)$.
(2) $\bar{c}_*$ *acts trivially on* $H_*\mathscr{C}_\infty(p)/Z_p$.

Proof. These follow immediately from the results in [3].

3. Proofs of 1.1 and 1.2

To be consistent with [2] and [3], we write $\bar{c}_* = \bar{\chi}$ and $c_* = \chi$ where c is the standard first coordinate inverse in $\Omega^n X$. The idea behind the proofs of 1.1 and 1.2 is that $1_* + \chi$ and $1_* - \chi$ [or $1_* + \bar{\chi}$ and $1 - \bar{\chi}$] yield an orthogonal decomposition of $H_*\Omega^n\Sigma^n X$ [or $H_*e[\mathscr{C}_n(j), \Sigma_j, X]$] and we may apply 0.1.

As an algebraic preliminary we compute the action of χ and $\bar{\chi}$.

LEMMA 3.1. *Let* $M \in W_{n-1}H_*X$,

$$M = Q^{J_1}\lambda_{I_1} * \cdots * Q^{J_k}\lambda_{I_k} \quad \textit{and} \quad s = k + \sum_{i=1}^{k} w(\lambda_{I_i}).$$

(1) $\chi M = (-1)^s M$.
(2) *If* M *is identified as an element of* $H_*e[\mathscr{C}_n(j), \Sigma_j, X]$, *then* $\bar{\chi}M = (-1)^s M$.

Remark. It follows directly from the definitions of S, U_{od}, and U_{ev} that
(1) $\tilde{H}_*\Omega^n\Sigma^n X = (1_* + \chi)(\tilde{H}_*\Omega^n\Sigma^n X) \oplus (1_* - \chi)(\tilde{H}_*\Omega^n\Sigma^n X)$,
(2) $(1_* + \chi)(H_*\Omega^n\Sigma^n X) = S \oplus U_{\mathrm{od}}$, and
(3) $(1_* - \chi)(H_*\Omega^n\Sigma^n X) = U_{\mathrm{ev}}$.
Furthermore, by 3.1(2) the analogous formulas hold if one replaces $\Omega^n\Sigma^n X$ by $e[\mathscr{C}_n(j), \Sigma_j, X]$, χ by $\bar{\chi}$, S by S_j, U_{od} by $U_{\mathrm{od},j}$, and U_{ev} by $U_{\mathrm{ev},j}$.

Proof of 1.1 *and* 1.2. We consider the self maps of $\Sigma\Omega^n\Sigma^n X$ given by

(1) $f_1 = \Sigma(1) + \Sigma(c)$, and
(2) $f_2 = \Sigma(1) - \Sigma(c)$.

Clearly f_1 and f_2 satisfy 0.1 and hence $\Sigma\Omega^n\Sigma^n X$ splits as promised.

To check the second assertion of 1.1, it suffices to exhibit a non-zero Steenrod operation defined on an element $x \in S$ and which does not take values in S. We do this in case $P^r_* x \neq 0$, $x \in H_q X$, $q + n$ is odd. The case $\beta x \neq 0$ is similar and easier.

By the form of the unstable analogues of the Nishida relations [3, Theorem 1.3], we see that $P^r_* Q^{(n+q-1)/2}x$ has a non-trivial summand of the form $ad^{p-1}_{n-1}(x)(P^r_* x)$. Clearly

$$ad^{p-1}_{n-1}(x)(P^r_* x) \in U$$

and we are done.

The proof of 1.2 goes through in an analogous fashion.

Proof of 3.1. (1) By [2, Proposition 1.4] we have

(1) $\chi\lambda_{n-1}(y, z) = -\lambda_{n-1}(\chi y, \chi z)$, and
(2) $\chi Q^s y = Q^s \chi y$.

Further recall that $\chi(a * b) = (-1)^{|a|\,|b|}\chi(b) * \chi(a)$. Since the multiplication for an n-fold loop space is homotopy commutative (if $n > 1$), we have the additional formula

(3) $\chi(Q^{I_1}\lambda_{I_1} * \cdots * Q^{I_k}\lambda_{I_k}) = \chi(Q^{I_1}\lambda_{I_1}) * \cdots * \chi(Q^{I_k}\lambda_{I_k})$.

Combining equations (1)–(3) with the definition of $w(\lambda_I)$, we observe that $\chi M = (-1)^s M$. (The reader should observe that the reason we must add k to $\sum_{i=1}^k w(\lambda_{I_i})$ is that each element of $\tilde{H}_*X$ is regarded as a λ_{n-1}-product of weight one.)

(2). Because we do not have a natural map of $e[\mathscr{C}_n(j), \Sigma_j, X]$ into $\Omega^n\Sigma^n X$ which behaves appropriately in homology, we must modify the argument given above.

Recall from [10] that C_nX is filtered with filtration denoted by F_jC_nX. Furthermore there are cofibrations

$$F_{j-1}C_nX \to F_jC_nX \to e[\mathscr{C}_n(j), \Sigma_j, X].$$

By the results in [3, Section 4] the long exact sequence in homology for this cofibration breaks up into short exact sequences

$$0 \to \tilde{H}_*F_{j-1}C_nX \to \tilde{H}_*F_jC_nX \to \tilde{H}_*e[\mathscr{C}_n(j), \Sigma_j, X] \to 0.$$

Identify $\tilde{H}_ie[\mathscr{C}_n(j), \Sigma_j, X]$ as $\tilde{H}_iF_jC_nX/\tilde{H}_iF_{j-1}C_nX$ and observe that it suffices to show that $\bar{\chi}M = (-1)^s M$ in $\tilde{H}_*F_jC_nX$.

By the definition of λ_{n-1}, Q^s, and lemmas 2.1–2.2, we obtain similar formulas to those in the proof of 3.1(1);

(1) $\bar{\chi}\lambda_n(y, z) = -\lambda_n(\bar{\chi}y, \bar{\chi}z)$,
(2) $\bar{\chi}Q^s y = Q^s\bar{\chi}y$, and
(3) $\bar{\chi}(Q^{I_1}\lambda_{I_1} * \cdots * Q^{I_k}\lambda_{I_k}) = \bar{\chi}Q^{I_1}\lambda_{I_1} * \cdots * \bar{\chi}Q^{I_k}\lambda_{I_k}$.

The result follows.

Remark. The similarities between the proofs of 3.1(1) and 3.1(2) suggest that we ought to be able to obtain a unified proof using a commutative diagram relating χ, $\bar{\chi}$, and α_n. However because π_0C_nX is not a group for nonconnected X while $\pi_0\Omega^n\Sigma^nX$ is a group, we see that this approach is unreasonable. We see that this thought suggests the following.

Remark. Let $\Omega^n_\phi\Sigma^nX$ denote the component of the base point in $\Omega^n\Sigma^nX$. (We do *not* assume that X is connected here.) Then c restricts to a map

$$c\,|\,_{\Omega^n_\phi\,\Sigma^nX}\colon \Omega^n_\phi\Sigma^nX \to \Omega^n_\phi\Sigma^nX.$$

Clearly the same methods as in the previous section will serve to split $\Sigma\Omega_\phi^n\Sigma^n X$. Of course the action of χ is more complex, but it is obvious that

$$\tilde{H}_*\Omega_\phi^n\Sigma^n X = (1 + \chi_*)\tilde{H}_*\Omega_\phi^n X \oplus (1 - \chi_*)\tilde{H}_*\Omega_\phi^n\Sigma^n X,$$

with Z_p-coefficients, $p > 2$. An easy calculation shows that each of these summands is non-zero, and consequently $\Sigma\Omega_\phi^n\Sigma^n X$ splits non-trivially. We have the following evident proposition.

PROPOSITION 3.2. *$\Sigma SF(2n)$ localized away from 2 splits as $A \vee B$ where H_iA and H_jB are nonzero for arbitrarily large i and j.*

It appears likely that one could find stable decompositions of $\Omega_\phi^n\Sigma^n X$ analogous to Snaith's. We intend to return to this question (with different methods) in case $X = S^0$.

4. An application to H-spaces

Let X be a connected H-space which satisfies one of the following two hypotheses:

(1) H^*X is primitively generated and of finite type;
(2) H_*X is primitively generated and of finite type.

In either case, we ambiguously let $\{u_i\}$ run over algebra generators for one of these two algebras. Define the *length* of a monomial in the u_i to be the number of factors (not necessarily distinct) occurring in that monomial. With these preliminaries we obtain the following where X is assumed to be homotopy commutative and associative.

THEOREM 4.1. *Let X be a connected H-space which satisfies* (1) (*or* (2)). *Then $(\Sigma X)_{(p)}$ has the homotopy type of $A_1 \vee \cdots \vee A_{p-1}$ where $\tilde{H}_*A_i$ is spanned by monomials in H^*X (or H_*X) of length j raised one degree, where $j \equiv i\,(p-1)$.*

Observe that [11, Theorem 9.3] is a special case of 4.1. As a corollary we obtain another result of [11] which has been found independently by Holzsager [6] in the case $n = 1$.

PROPOSITION 4.2. *$\Sigma K(Z_{p^r}, n)$ has the homotopy type of $X_1 \vee \cdots \vee X_{p-1}$.*

We also have

PROPOSITION 4.3. *If $n \geq 2$ and each element of $\tilde{H}_*Y$ is p-torsion and primitive then $\Sigma\Omega^n\Sigma^n Y$ has the homotopy type of $A_1 \vee \cdots \vee A_{p-1}$ where H_*A_i is given by* 4.1.

Along these same lines we obtain a splitting of $\Sigma SF(2)$ at p.

PROPOSITION 4.4. *$\Sigma SF(2)_{(p)}$ has the homotopy type of $A_1 \vee \cdots \vee A_{p-1}$. Furthermore H_*A_i is nonzero in arbitrarily high dimensions.*

Evidently one may use 3.2 and 4.1 to obtain a finer splitting of $\Sigma^2 SF(2)$ localized at p. We do not know if a similar result is true for $\Sigma SF(n)$, $n > 2$.

Proof of 4.2. By [9], $H^*K(Z_{p^r}, n)$ is primitively generated.

Remark. It seems likely that H^*X_i is indecomposable over the Steenrod algebra, but we have only checked this for the first few n.

Proof of 4.3. By [2], if $n > 1$, $H_*\Omega^n\Sigma^n Y$ is the free commutative algebra on certain generators described in Section 1. The diagonal Cartan formula for these operations is defined in terms of the coproduct for H_*Y [2]. We deduce immediately that $\Omega^n\Sigma^n Y$ satisfies (ii) provided $n \geq 2$.

Proof of 4.4. By [2], it is easily seen that $H_*\Omega^2_\phi\Sigma^2 S^0$ is primitively generated. Since $\Omega^2_\phi\Sigma^2 S^0$ is homotopy commutative and $\Omega^2_\phi\Sigma^2 S^0$ has the homotopy type of $SF(2)$ the result follows. We remark that $H_*\Omega^n_\phi\Sigma^n S^0$, $n > 2$, is not primitively generated, and so we do not know if an analogous splitting exists for $SF(n)$, $n > 2$.

Proof of 4.1. Let μ denote the multiplication for X and define θ_k to be the composite

$$X \xrightarrow{\Delta} X^k \xrightarrow{\lambda} X$$

where $\Delta(x) = (x, \ldots, x)$ and

$$\lambda(x_1, \ldots, x_k) = \mu(x_1, \ldots, \mu(x_1, \ldots, \mu(x_{k-2}, \mu(x_{k-1}, x_k)) \cdots).$$

We require the following lemma which is stated without proof.

LEMMA 4.4. *If $u_1, \ldots, u_j$ are primitive in H_*X [in H^*X], then*

$$\theta_{k*}(u_1 \cdots u_j) = k^j u_1 \cdots u_j \qquad [\theta_k^*(u_1 \cdots u_j) = k^j u_1 \cdots u_j].$$

To continue with the proof of 4.1, we set $v_i = \Sigma(i) - \theta_k$, $i > 0$, where k is a fixed unit mod p. We define $f_i = v_1 \circ \cdots \circ \hat{v}_i \circ \cdots \circ v_{p-1}$ where $\hat{v}_i$ means that v_i is deleted and $\circ$ means composition of maps.

In case X satisfies (1), let $(u_{i_1} \cdots u_{i_j})_*$ denote the dual basis to the monomials $u_{i_1} \cdots u_{i_j}$.

Let $1 \leq t \leq p - 1$ and M_t be the submodule of $\tilde{H}_*\Sigma X$ spanned by the suspensions of the $(u_{i_1} \cdots u_{i_j})_*$ [or $u_{i_1} \cdots u_{i_j}$ if X satisfies (2)] where $k^j \equiv t(p)$. Let $x \in M_t$. Obviously $f_{i*}x = cx$ where $c \neq 0$ if and only if $i = t$. Evidently Definition 0.0 is satisfied and we are done.

5. Some unstable analogues of the transfer

The results of this section, which are parenthetical to those of the previous four sections, will be used in [4].

Let π denote the p-Sylow subgroup of a finite group G. Suppose that X supports a free G-action and that r denotes the natural projection $X/\pi \to X/G$. The transfer

$$T\colon H_*X/G \to H_*X/\pi$$

enjoys the property that $r_* \circ T$ is multiplication by the index of π in G. Most applications of T usually seem only to require that $r_* \circ T$ induces a mod-p

homology isomorphism of H_*X/G. In favorable cases, we construct a map $T'\colon \Sigma X/G \to \Sigma X/\pi$ such that $(\Sigma r)_* \circ T'_*$ is a mod-p homology isomorphism of $H_*\Sigma X/G$. Our motivation for studying such a map T' is related to the Kahn-Priddy Theorem [7]. That is, we want to find unstable analogues of this theorem and consequently need unstable analogues of the transfer. We hope to return to this question later. That at least a single suspension is required to define T' is evident due to the incompatibility of the cup product structures for H^*BG and $H^*B\pi$. A specific example is provided by letting G be the symmetric group on p letters for p an odd prime.

THEOREM 5.1. *If π is normal in G, then there is a map $T'\colon \Sigma X/G \to \Sigma X/\pi$ such that $\Sigma r \circ T'$ is a p-equivalence. Furthermore $(\Sigma X/\pi)_{(p)}$ has the homotopy type of $(\Sigma X/G)_{(p)} \vee Z$ (where Z will be specified in the proof).*

THEOREM 5.2. *If π is abelian, then there is a map $T'\colon \Sigma BG \to \Sigma B\pi$ such that $\Sigma r \circ T'$ is a p-equivalence. Furthermore $\Sigma B\pi$ has the homotopy type of $(\Sigma BG)_{(p)} \vee Z$ (where Z will be specified in the proof).*

Remark. We do not know of any examples in which a map T' does *not* exist which satisfies the property that $\Sigma r \circ T'$ is a p-equivalence. In relation to this question, D. S. Kahn has pointed out that the Kahn-Priddy transfer cannot be desuspended at all when $\pi = Z_2 \int Z_2$ and $G = \Sigma_4$.

Proof of 5.1. Let $g \in G$. Conjugation by g induces an automorphism of π and of X/π, say θg. Let $\theta_*\colon H_*X/\pi \to H_*X/G$ be given by $\sum_{\alpha \in S} \theta g_{\alpha *}$ where $\{g_\alpha \pi\}_{\alpha \in S}$ runs over a complete set of left cosets for π in G. If T denotes the classical (algebraic) transfer, then we observe that $T \circ r_* = \theta_*$.

We geometrically mimic some standard homological facts.

By abuse of notation we let θg denote the self-map of $\Sigma X/\pi$ which is given by suspending the map θg defined above. Define a map $N\colon \Sigma X/\pi \to \Sigma X/\pi$ by setting $N = \sum_{\alpha \in S} \theta g_\alpha$. Define N for X/G in a similar fashion to obtain a commutative diagram

$$\begin{tikzcd} \Sigma X/\pi \arrow[r, "\Sigma r"] \arrow[d, "N"] & \Sigma X/G \arrow[d, "N"] \\ \Sigma X/\pi \arrow[r, "\Sigma r"] & \Sigma X/G. \end{tikzcd}$$

Form the telescope with respect to N and observe that we have an additional commutative diagram

$$\begin{tikzcd} \Sigma X/\pi \arrow[r] \arrow[d] & \Sigma X/G \arrow[d] \\ \mathrm{tel}_N\,(\Sigma X/\pi)_{(p)} \arrow[r, "\mathrm{tel}\,\Sigma(r)"] & \mathrm{tel}_N\,(\Sigma X/G)_{(p)}. \end{tikzcd}$$

To verify 6.1, we let $f_1 = N$ and $f_2 = k - N$ where $k = |G/\pi|$; we define $M_1 = N_* \tilde{H}_*(\Sigma X/\pi)_{(p)}$ and $M_2 = (k - N)_* \tilde{H}_*(\Sigma X/\pi)_{(p)}$. Since k is relatively prime to p, it suffices to show that

(i) $(\Sigma X/\pi)_{(p)}$ is of the homotopy type of $\mathrm{tel}_N\, (\Sigma X/\pi)_{(p)} \vee \mathrm{tel}_{k-N}\, (\Sigma X/\pi)_{(p)}$, and

(ii) $\mathrm{tel}_N\, (\Sigma X/\pi)_{(p)}$ is of the homotopy type of $(\Sigma X/G)_{(p)}$.

Part (i) follows immediately from 0.1. To verify part (ii), first observe that the natural map $\Sigma X/G \to \mathrm{tel}_N\, (\Sigma X/G)_{(p)}$ is a mod-p homology isomorphism and consequently $(\Sigma X/G)_{(p)}$ has the homotopy type of $\mathrm{tel}_N\, (\Sigma X/G)_{(p)}$. We next observe that the natural map

$$\mathrm{tel}_N\, \Sigma r\colon \mathrm{tel}_N\, (\Sigma X/\pi)_{(p)} \to \mathrm{tel}_N\, (\Sigma X/G)_{(p)}$$

is a mod-p homology isomorphism. This fact follows immediately from the previous observation that $T \circ r_* = \theta_*$.

Hence $(\Sigma X/\pi)_{(p)}$ has the homotopy type of $(\Sigma X/G)_{(p)} \vee \mathrm{tel}_{k-N}\, (\Sigma X/\pi)_{(p)}$. To finish 6.1, we define T' to be the composite

$$\Sigma X/G \xrightarrow{\beta} (\Sigma X/G)_{(p)} \xrightarrow{I} (\Sigma X/G)_{(p)} \vee \mathrm{tel}_{k-N}\, (\Sigma X/\pi)_{(p)} \xrightarrow{\gamma} (\Sigma X/\pi)_{(p)} \xrightarrow{\delta} \Sigma X/\pi$$

where β is the natural map, I the inclusion, γ the homotopy "inverse" of the equivalence derived above, and δ is the homotopy "inverse" at p to the natural map $\Sigma X/\pi \to (\Sigma X/\pi)_{(p)}$ (which exists by [13]).

Proof of 5.2. Let π be the p-Sylow subgroup of G and assume that π is abelian. Consider the normalizer of π in G, denoted $N\pi$. It is an exercise in homological algebra to show that the restriction map $i_*\colon H_*(BN\pi;\, Z_p) \to H_*(BG;\, Z_p)$ is an isomorphism (with trivial π and G action on Z_p). (See [14] for example.) The result follows directly from Theorem 5.1.

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