OBSTRUCTIONS TO LIFTING *-MORPHISMS INTO THE CALKIN ALGEBRA

BY

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1. Introduction

Let H be a separable infinite dimensional Hilbert space, $\mathscr{B}(H)$ the algebra of bounded operators on H, $\mathscr{K}(H)$ the set of compact operators, $\mathscr{A}(H) = \mathscr{B}(H)/\mathscr{K}(H)$, $\pi\colon \mathscr{B}(H) \to \mathscr{A}(H)$ the quotient map. In their paper [1], Brown, Douglas, and Fillmore investigate for a compact metric space X the group Ext (X) consisting of unitary equivalence classes of unital injective *-morphisms $\tau\colon C(X) \to \mathscr{A}(H)$. This group completely solves (in principle at least) the lifting problem for injective unital *-morphisms τ from $C(X)$ to $\mathscr{A}(H)$: namely, there is a *-morphism $\tilde{\tau}$ which makes the diagram

$$\begin{tikzcd} C(X) \arrow[r, "\tilde{\tau}"] \arrow[dr, "\tau"'] & \mathscr{B}(H) \arrow[d, "\pi"] \\ & \mathscr{A}(H) \end{tikzcd}$$

commutative iff the equivalence class $[\tau]$ of τ in Ext (X) is 0. The lifting problem is meaningful for any injective *-morphism from a C^*-algebra, although in the general case there is no functor around with the pleasant group properties of Ext. In the case A is UHF, we give in this paper an essentially complete answer. We follow throughout the terminology and conventions of Dixmier [3].

2. The semigroup $E(A)$

To solve the lifting problem we use a semigroup of *-morphisms from a C^*-algebra A to $\mathscr{A}(H)$ which is a fairly straightforward generalization of the semigroup Ext of [1]. We let $M(A; H)$ be the set of injective *-morphisms $\tau\colon A \to \mathscr{A}(H)$ and $\tilde{M}(A; H)$ the set of maps $\tau'\colon A \to \mathscr{B}(H)$ such that the composition $\pi\tau'$ is an injective *-morphism. Introduce on $\tilde{M}(A; H)$ the relation $\equiv$ of unitary equivalence modulo the compacts (i.e., $\tau \stackrel{\sim}{\equiv} \rho$ iff there is a unitary U such that $U\rho(x)U^* - \tau(x) \in \mathscr{K}(H)$ for all $x \in A$). On $M(A; H)$ we consider the corresponding relation $\equiv$; (i.e., $\rho \equiv \tau$ iff there is a unitary $U \in \mathscr{B}(H)$ such that $\pi(U)\rho(x)\pi(U^*) = \tau(x)$ for all $x \varepsilon A$.) The quotient sets $\tilde{M}(A; H)/\stackrel{\sim}{\equiv}$ and $M(A; H)/\equiv$ are naturally equivalent; we denote them by $E(A)$; denote the class of $\tilde{\tau} \in \tilde{M}(A; H)$ in $E(A)$ (resp. the class of $\tau \in M(A; H)$) by $[\tilde{\tau}]^\sim$ (resp. $[\tau]$). Observe $E(A)$ is a contravariant functor in the category of C^*-algebras

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and injective *-morphisms. Also, it is clear that direct summation induces a commutative semigroup operation on $E(A)$ as in [1].

A few words about matrix units; first, the definition. A family $\{u_i: 0 \leq i < n\}$ of partial isometries of a C^*-algebra which satisfy $u_k^* u_j = \delta_{kj} u_0$ is called a system of matrix units. For example if A is a full algebra of matrices $M_n(\mathbf{C})$, then the matrices v_i which have entries all zeroes except in the ith row and 0th column, where a 1 appears, form a system of matrix units which generate the algebra. Secondly, any system $\{u_i: i < n\}$ of matrix units in a C^*-algebra A determines a unique *-morphism $\tau: M_n(\mathbf{C}) \to A$ such that $\tau(v_i) = u_i$. Finally, by Calkin's lifting theorem [2], for any system of matrix units $\{u_i: i < n\}$ in $\mathscr{A}(H)$ there is a system of matrix units $\{\tilde{u}_i: i < n\}$ in $\mathscr{B}(H)$ such that $\pi(\tilde{u}_i) = u_i$. This implies any *-morphism from a finite dimensional C^*-algebra to $\mathscr{A}(H)$ lifts to a *-morphism to $\mathscr{B}(H)$. (Actually this is true for any dual C^*-algebra, though we will not need this).

We now determine $E(A)$ for finite dimensional C^*-algebras A.

PROPOSITION 1. *Suppose A is a finite dimensional C^*-algebra and $\phi, \psi: A \to \mathscr{B}(H)$ are *-morphisms such that $\phi(x) - \psi(x) \in \mathscr{K}(H)$ for all $x \in A$. Then there is a partial isometry u in $\mathscr{B}(H)$ such that:*

(a) *The initial projection e of u commutes with $\phi(A)$ and the final projection f commutes with $\psi(A)$.*

(b) $u^*\psi(x)u = e\phi(x)$.

(c) $u - \phi(1)$, $\phi(1) - e$, $\psi(1) - f \in \mathscr{K}(H)$.

(d) $e \leq \phi(1); f \leq \psi(1)$.

Proof. One need only consider the case A is a full matrix algebra. Let $\{u_i: i < n\}$ be a system of matrix units which generate A. As $\pi\phi(u_0) = \pi\psi(u_0)$, by Calkin's lifting theorem [2] there is a partial isometry v with initial projection $e_0 \leq \phi(u_0)$, final projection $f_0 \leq \psi(u_0)$ and so that $\pi(v) = \pi\phi(u_0)$. Let

$$u = \sum_{i \in n} \psi(u_i) v \phi(u_i)^*.$$

Now $\phi(u_i)^*$ is a partial isometry with final projection $\phi(u_0)$ and $\psi(u_i)$ is a partial isometry with initial projection $\psi(u_0)$. Thus $\psi(u_i)v\phi(u_i)^*$ is a partial isometry. As the sum of partial isometries with orthogonal initial projections and orthogonal final projections is also a partial isometry, u is a partial isometry. The initial projection e of u satisfies

$$\begin{aligned} e = u^*u &= \sum_{i<n} \phi(u_i) v^* \psi(u_i)^* \psi(u_i) v \phi(u_i)^* \\ &= \sum_{i<n} \phi(u_i) v^* \psi(u_0) v \phi(u_i)^* \\ &= \sum_{i<n} \phi(u_i) v^* v \phi(u_i) \\ &= \sum_{i<n} \phi(u_i) e_0 \phi(u_i)^* \leq \sum_{i<n} \phi(u_i u_i^*) = \phi(1). \end{aligned}$$

Thus $e \leq \phi(1)$. A similar calculation shows $f = uu^* \leq \psi(1)$.

To prove (a) it suffices to show e commutes with $\phi(u_k u_l^*)$ for all k, l, and f commutes with $\psi(u_k u_l^*)$ for all k, l. Now

$$\begin{aligned}\left\{\sum_{i<n} \phi(u_i)e_0\phi(u_i)^*\right\} \phi(u_k u_l^*) &= \phi(u_k)e_0\phi(u_k)^*\phi(u_k)\phi(u_l)^* \\ &= \phi(u_k)e_0\phi(u_0)\phi(u_l)^* \\ &= \phi(u_k)e_0\phi(u_l)^* \\ &= \phi(u_k u_l^*)\phi(u_l)e_0\phi(u_l)^* \\ &= \phi(u_k u_l^*)\left\{\sum_{i<n} \phi(u_i)e_0\phi(u_i)^*\right\}.\end{aligned}$$

The other statements can be proved in a similar vein and we omit the details. ∎

Observe that if A is finite dimensional, then a representation τ of A is determined up to unitary equivalence by giving the multiplicity $m_\alpha(\tau)$ of each $\alpha \in \hat{A}$ in τ together with the dimension $N(\tau)$ of the null space of τ. Conversely, given any family $\{m_\alpha\}_{\alpha \in \hat{A}}$ in $\mathbf{Z}^+ \cup \{\omega\}$ and $N \in \mathbf{Z}^+ \cup \{\omega\}$ there is up to unitary equivalence a unique representation of A which we denote $\sum_\alpha^\oplus m_\alpha \cdot \alpha \oplus N(0)$ with the multiplicities $\{m_\alpha\}$ and nullity N.

In the following, for a finite dimensional C^*-algebra A, we will let $d(A)$ be the greatest common divisor of the integers $\{\dim \alpha : \alpha \in \hat{A}\}$ (where $\dim \alpha$ is the dimension of the representation space of α).

PROPOSITION 2. *Let A be a finite dimensional C^*-algebra, ϕ, $\psi \in \tilde{M}(A)$ *-morphisms; a necessary and sufficient condition for $\phi \cong \psi$ is that*

$$\operatorname{Codim} \phi(1) \equiv \operatorname{Codim} \psi(1) \quad (\operatorname{Mod} d(A))$$

understood as an equality if one side is infinite.

Proof. Let $s = d(A)$ and $r_\alpha = \dim \alpha$ for $\alpha \in \hat{A}$. To show necessity one may assume $\pi\phi$, $\pi\psi$ are both unital; for otherwise both Codim $\phi(1)$ and Codim $\psi(1)$ are infinite and the stated congruence holds trivially. Consider first the case when $\pi\phi = \pi\psi$. This implies there are e, f, u which satisfy (a) through (d) of Proposition 1. Now by (c) e, f are of finite codimension and u is a compact perturbation of the identity. As the index is unchanged by compact perturbation, it follows that index $u = 0$, and so codim e = codim f. Since $\phi(A)$ commutes with e, it commutes with $\phi(1) - e$. Thus we have

$$\operatorname{Codim} \phi(1) - \operatorname{Codim} e = -\dim(\phi(1) - e) \equiv 0 \pmod{s}.$$

This last congruence holds because any nondegenerate representation of A must be on a hilbert space whose dimension is divisible by s. Similarly,

$$\operatorname{Codim} \psi(1) - \operatorname{Codim} f \equiv 0 \pmod{s}$$

so that Codim $\phi(1) \equiv$ Codim $\psi(1)$ (mod s). In the general case, if $\phi \cong \psi$ there is a unitary $U \in \mathscr{B}(H)$ such that $\phi(x) - U\psi(x)U^* \in \mathscr{K}(H)$ so by the preceding remarks

$$\text{Codim } \psi(1) = \text{Codim } U\psi(1)U^* \equiv \text{Codim } \phi(1) \pmod{s}.$$

To show sufficiency, suppose Codim $\phi(1) \equiv$ Codim $\psi(1)$ (mod s). Then ϕ, ψ are unitarily equivalent to

$$\sum_\alpha^\oplus m_\alpha(\phi)\alpha \oplus N(\phi)(0); \quad \sum_\alpha^\oplus m_\alpha(\psi)\alpha \oplus N(\psi)(0)$$

resp. where $m_\alpha(\phi) = m_\alpha(\psi) = \omega$ for all α. Now either $N(\phi) = N(\psi) = \omega$ or

$$N(\phi) - N(\psi) = \sum_{\alpha \in \hat{A}} (a_\alpha - b_\alpha)r_\alpha$$

with a_α, $b_\alpha \geq 0$. In the first case ϕ, ψ are unitarily equivalent. In the second case

$$\begin{aligned}\phi &\cong \sum_\alpha^\oplus m_\alpha(\phi)\alpha \oplus N(\phi)(0) \oplus \sum_\alpha^\oplus b_\alpha \cdot \alpha \\ &\cong \sum_\alpha^\oplus m_\alpha(\phi)\alpha \oplus \{N(\phi) + \sum b_\alpha r_\alpha\}(0)\end{aligned}$$

and

$$\begin{aligned}\psi &\cong \sum_\alpha^\oplus m_\alpha(\psi)\alpha \oplus N(\psi)(0) \oplus \sum_\alpha^\oplus a_\alpha \alpha \\ &\cong \sum_\alpha^\oplus m_\alpha(\psi)\alpha \oplus \{N(\psi) + \sum a_\alpha r_\alpha\}(0)\end{aligned}$$

so that $\phi \cong \psi$. ■

We can now compute $E(A)$ for any finite dimensional C^*-algebra A. Given $N \in \mathbf{Z}^+ \cup \{\omega\}$ define $\tilde{\delta}_A(N) \in \tilde{M}(A)$ as $\sum_\alpha^\oplus n_\alpha \alpha \oplus N(0)$ where $n_\alpha = \omega$ for all $\alpha \in \hat{A}$. $\tilde{\delta}_A$ induces a map δ_A: $\mathbf{Z}^+ \cup \{\omega\} \to E(A)$. It follows immediately that δ_A is a morphism of semigroups, $\mathbf{Z}^+ \cup \{\omega\}$ considered additively. Furthermore δ_A is surjective. By Proposition 2 we have that $\delta_A(m) = \delta_A(n)$ iff $m \equiv n$ (mod $d(A)$). Thus:

PROPOSITION 3. *If A is finite dimensional, then*

$$E(A) \cong \mathbf{Z}/d(A)\mathbf{Z} \cup \{\omega\}$$

where ω acts as a zero in the semigroup.

3. Application to the UHF case

A C^*-algebra A with unit is uniformly hyperfinite (UHF) iff there is an increasing sequence $\{A_n\}$ of full matrix subalgebras containing the unit of A and such that $A = (\bigcup_n A_n)^-$. These algebras have been studied in detail and classified by Glimm [4]; it is easy to show they are all simple.

Now suppose $\{A_n\}$ is a directed sequence of sub-C^*-algebras of A such that $A = \overline{\bigcup_n A_n}$. Denoting the inclusions $A_n \to A_{n+1}$ by k_{n+1}, we have a projective sequence of semigroups $\{E(A_n), E(k_n)\}$; denoting the inclusions $A_n \to A$ by h_n, we have also the morphisms of semigroups $E(h_n)\colon E(A) \to E(A_n)$ which induce the morphisms $\underleftarrow{E} = \varprojlim E(h_n)\colon E(A) \to \varprojlim E(A_n)$.

PROPOSITION 4. *The morphism* $\underleftarrow{E}$ *is surjective.*

Proof. Let $\alpha \in \varprojlim EA_n$. Then $\alpha = \{[\phi_n]\}_{n\geq 0}$ where $\phi_n \in M(A_n)$ and $\phi_{n+1} \mid A_n \equiv \phi_n$. This means there is a unitary $U_n \in \mathscr{B}(H)$ such that if $x \in A_n$

$$\pi(U_n)\phi_{n+1}(x)\pi(U_n^*) = \phi_n(x).$$

Now let $\psi_{n+1} \in M(A_{n+1})$ be given by

$$\psi_{n+1}(x) = \pi(U_0)\pi(U_1)\cdots\pi(U_n)\phi_{n+1}(x)\pi(U_n^*)\cdots\pi(U_0^*).$$

Clearly $\psi_{n+1} \equiv \phi_{n+1}$. Furthermore $\psi_{n+1} \mid A_n = \psi_n$. Thus there is a unique injective *-morphism $\psi \in M(A)$ such that $\psi \mid A_n = \psi_n$; thus $E(h_n)[\psi] = [\phi_n]$ and $\underleftarrow{E}[\psi] = \alpha$. ∎

In the case the algebras A_n are full matrix algebras and the inclusions h_n are unital we can use the available information to give necessary and sufficient conditions for a unital map $\phi \in M(A)$ to lift to a *-morphism $\tilde{\phi}\colon A \to \mathscr{B}(H)$. Observe first that the morphisms $\delta_{A_i}\colon \mathbf{Z}^+ \cup \{\omega\} \to E(A_i)$ define a morphism of semigroups $\delta\colon \mathbf{Z}^+ \cup \{\omega\} \to \varprojlim E(A_i)$. If $\phi \in M(A)$ is unital and has a lifting to a *-morphism $\tilde{\phi}\colon A \to \mathscr{B}(H)$, then letting $n = \operatorname{Codim} \tilde{\phi}(1) \in \mathbf{Z}^+ \cup \{\omega\}$ gives

$$\delta_{A_i}(n) = [\phi \mid A_i] = E(h_i)[\phi]$$

so $\delta(n) = \underleftarrow{E}[\phi]$.

In order to prove a converse, we state two lemmas:

LEMMA 1. *Let B be a full matrix algebra, η a *-morphism $B \to \mathscr{B}(H)$. If f' is a projection in $\mathscr{B}(H)$ of finite codimension, then there is a projection of finite codimension $f \leq f'$ which reduces $\eta(B)$.*

Proof. Standard. ∎

LEMMA 2. *Suppose B_i, $i = 1, 2$ are full matrix algebras and k is a unital *-morphism $B_1 \hookrightarrow B_2$, (viewed as an inclusion). If $\eta \in M(B_2)$ is a unital *-morphism and $\eta_1\colon B_1 \to \mathscr{B}(H)$ is a *-morphism which lifts $\eta \mid B_1$ and*

$$[\eta] = \delta_{B_2}\ (\operatorname{Codim} \eta_1(1))$$

*then there is a *-morphism $\eta_2\colon B_2 \to \mathscr{B}(H)$ which extends η_1 and which lifts η.*

Proof. Let ψ be a *-morphism $B_2 \to \mathscr{B}(H)$ which lifts η. By Proposition 1, there is a partial isometry u' with initial projection e' and final projection f' such that:

(a) e' reduces $\eta_1(B_1)$, f' reduces $\psi(B_1)$.
(b) $u'^*\psi(x)u' = e'\eta_1(x)$ for $x \in B_1$.
(c) $u' - \eta_1(1), \eta_1(1) - e', \psi(1) - f' \in \mathcal{K}(H)$.
(d) $e' \leq \eta_1(1), f' \leq \psi(1)$.

Now as $f' \leq \psi(1)$ is of finite codimension by Lemma 1 there is an $f \leq f'$ of finite codimension which reduces $\psi(B_2)$. Let $e = u'^*fu'$ and $u = fu'$. Clearly $uu^* = f$, $u^*u = e$, and (a)–(d) are still valid if we replace u, f, e for u', f', e'. Now if rank $B_i = r(i)$, then:

(A) Codim $\psi(1) \equiv$ Codim $\eta_1(1)$ (mod $r(2)$).
(B) Codim $e \equiv$ Codim $\psi(1)$ (mod $r(2)$).

(A) follows from Proposition 2 and

$$\delta_{B_2}(\text{Codim } \psi(1)) = [\psi]^{\sim} = [\eta] = \delta_{B_2}(\text{Codim } \eta_1(1)).$$

To show (B) observe $u^*\psi u$ is a *-morphism $B_2 \to \mathcal{B}(H)$ which lifts η. Thus by Proposition 2, (b) and (d),

$$\text{Codim } e = \text{Codim } e\eta_1(1) = \text{Codim } u^*\psi(1)u \equiv \text{Codim } \psi(1) \pmod{r(2)}.$$

Combining (A) and (B)

$$\dim(\eta_1(1) - e) = \text{Codim } e - \text{Codim } \eta_1(1) \equiv \text{Codim } e - \text{Codim } \psi(1) \equiv 0 \pmod{r(2)}.$$

Now by (a), $x \to \eta_1(x)\{\eta_1(1) - e\}$ is a *-morphism $B_1 \to \mathcal{B}(H)$; as $r(2)$ divides $\dim(\eta_1(1) - e)$ there is a *-morphism $\rho\colon B_2 \to \mathcal{B}(H)$ such that

$$\rho(x) = \eta_1(x)\{\eta_1(1) - e\} \quad \text{for } x \in B_1.$$

Now we can define

$$\eta_2(x) = \rho(x) + u^*\psi(x)u.$$

η_2 is a *-morphism as $\rho(1)$, $u^*\psi(1)u = e$ are orthogonal projections. Also, if $x \in B_1$,

$$\eta_2(x) = \eta_1(x)\{\eta_1(1) - e\} + e\eta_1(x) = \eta_1(x).$$

Clearly $\pi\eta_2 = \eta$. This proves the lemma. ∎

PROPOSITION 5. *Suppose the C^*-algebras A_i are full matrix algebras, the inclusions h_i are unital, and $\phi \in M(A)$ is unital. Then ϕ lifts to a *-morphism $\tilde{\phi}$ iff there is an $n \in \mathbf{Z}^+$ such that $\delta(n) = \underleftarrow{E}[\phi]$.*

Proof. In one direction it has already been proved. In the converse direction one has to show given ϕ with the stated conditions there is a sequence $\tilde{\phi}_i$ of *-morphisms $A_i \to \mathcal{B}(H)$ such that:

(1) $\tilde{\phi}_i$ lifts $\phi \mid A_i$.
(2) $\tilde{\phi}_{i+1} \mid A_i = \tilde{\phi}_i$.

Assume $\tilde{\phi}_1, \ldots, \tilde{\phi}_N$ are *-morphisms such that (1) holds for $i \leq N$, (2) holds for $i < N$ and in addition Codim $\tilde{\phi}_i(1) = n$ for $i \leq N$. Now

$$[\phi \mid A_{N+1}] = E(h_{N+1})[\phi] = \delta_{A_{N+1}}(n) = \delta_{A_{N+1}}\,(\text{Codim } \tilde{\phi}_N(1)).$$

By Lemma 2 therefore, $\tilde{\phi}_N$ can be extended to a *-morphism $\tilde{\phi}_{N+1}\colon A_{N+1} \to \mathscr{B}(H)$ which is a lifting for $\phi \mid A_{N+1}$. ■

If $d(A_i) = r_i$ then it is possible to explicitly describe the map $\delta \mid \mathbf{Z}^+$; first there is a canonical imbedding

$$G = \varprojlim \mathbf{Z}/r_i \to \varprojlim E(A_i)$$

and it is easy to see $\delta(\mathbf{Z}^+) \subseteq G$. Secondly, the group $\varprojlim \mathbf{Z}/r_i$ is well known to be

$$\prod \{\mathbf{Z}_{p,n_p}\colon p \in \mathbf{Z} \text{ a prime}\}$$

where

$$\mathbf{Z}_{p,m} = \begin{cases} \mathbf{Z}/(p^m) & \text{if } m < \omega \\ \mathbf{Z}_p \text{ (the ring of } p\text{-adic integers)} & \text{if } m = \omega \end{cases}$$

and $n_p = \text{Sup}\,\{n\colon p^n \text{ divides } r_i \text{ for some integer } i\}$.

The map $\delta \mid \mathbf{Z}^+ : \mathbf{Z}^+ \to G$ is the obvious one. From Propositions 4 and 5 it follows in particular there are plenty of unital *-morphisms from a UHF algebra into the Calkin algebra $\mathscr{A}(H)$ which admit no lifting.

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