THE EXISTENCE OF UNAVOIDABLE SETS OF GEOGRAPHICALLY GOOD CONFIGURATIONS[1]

BY

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Abstract

A set of configurations is unavoidable if every planar map contains at least one element of the set. A configuration $\mathscr{C}$ is called geographically good if whenever a member country M of $\mathscr{C}$ has any three neighbors N_1, N_2, N_3 which are not members of $\mathscr{C}$ then N_1, N_2, N_3 are consecutive (in some order) about M.

The main result is a constructive proof that there exist finite unavoidable sets of geographically good configurations. This result is the first step in an investigation of an approach towards the Four Color Conjecture.

1. Introduction

This work has been inspired by the work of Heesch [9], [10] on the Four Color Problem, especially by his conjecture [9, p. 11, paragraph 1, and p. 216] that there exists a finite set $\mathscr{S}$ of four-color reducible configurations such that every planar map contains at least one element of $\mathscr{S}$. (This conjecture implies the Four Color Conjecture but is not implied by it.) Furthermore, in 1970 Heesch communicated an unpublished result (described later in this section and as $\mathscr{S}_5$ in Table 1) which he calls a finitization of the Four Color Problem. Our main objective is to develop a tool which we hope may eventually be used to attack the Four Color Conjecture via the Heesch Conjecture. We do not claim, however, that our work should be regarded as a finitization.

We call a set $\mathscr{S}$ of configurations *unavoidable* if every planar map contains at least one element of $\mathscr{S}$. We are interested in developing a theory of unavoidable sets independent of reduction theory and the coloration concept. We shall be specially interested in unavoidable sets of such configurations that have certain properties which appear to be necessary conditions for four color reducibility. At a later stage we plan to consider unavoidable sets of configurations with properties which seem to make them likely candidates for four

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color reducibility. Our objectives will be explained in more detail in Sections 2–7.

2. Discussion of previous results

We will phrase the results in the dual terminology of planar triangulations and vertex colorations (see Ore [15, Chapter 12]).

Both the theory of unavoidable sets and the theory of four color reducibility were initiated in 1879 by Kempe in his attempted proof of the Four Color Conjecture [11]. The unavoidable set $\mathscr{S}_0$ exhibited by Kempe consists of four elements, the single vertices of degrees 2, 3, 4, and 5, $\{V_2, V_3, V_4, V_5\}$. Kempe proved four color reducibility for V_2, V_3, and V_4 but failed in the case of V_5. That $\mathscr{S}_0$ is unavoidable is implied by the fact that the Euler Characteristic of any planar triangulation is positive. The four color reducibility of V_4 requires the Jordan Curve Theorem. Thus the unavoidability of $\mathscr{S}_0$ also holds in triangulations of the projective plane P^2 while the four color reducibility of V_4 does not.

One may regard the Heesch Conjecture as a straightforward attempt to repair Kempe's argument (with the required unavoidable set larger than $\mathscr{S}_0$).

The reduction theory for configurations larger than single vertices has been developed by many investigators to a high degree of perfection. (See Heesch [9], Tutte and Whitney [19], Ore [15], F. Bernhart [5], Allaire and Swart [1] and the literature quoted therein.) On the other hand we found only a few published results on finite unavoidable sets which hold for all planar triangulations (without further restrictions as to the total number of vertices or the degrees of vertices occurring in the triangulations). We list those of principal interest to us in Table 1.

In 1904, Wernicke [20] found that V_5 in Kempe's $\mathscr{S}_0$ can be replaced by two configurations; a pair of adjacent V_5's and an adjacent V_5-V_6 pair ($\mathscr{S}_1$ in Table 1). In 1922, Franklin [7] showed that every triangulation without vertices of degree less than five contains some V_5 adjacent to two vertices of degree at most six. This may be formulated as a replacement of the V_5-V_6 pair in Wernicke's $\mathscr{S}_1$ by a V_5-V_6-V_6 triangle and a V_6-V_5-V_6 path ($\mathscr{S}_2$ in Table 1). Franklin remarks that the reducibility of these configurations would imply the Four Color Theorem but that it "appears to be not possible" to prove the reducibility.

One way to explain the difficulty is this. The single V_5 could not be proved reducible because the five neighbors of the V_5 exterior to the configuration seemed (in some sense) too many to handle simultaneously. But the V_6-V_5-V_6 path of $\mathscr{S}_2$ has two such "five-legger" vertices, i.e., the V_6 vertices each have five external neighbors.

If one takes this point of view one might seek an unavoidable set of configurations each vertex of which has at most four outside legs. However, H. Heesch has pointed out to us that with the exception of the solitary V_4 no known method of proving reducibility has ever succeeded in the presence of a four-legger vertex, nor in the presence of a three-legger articulation vertex (i.e., a

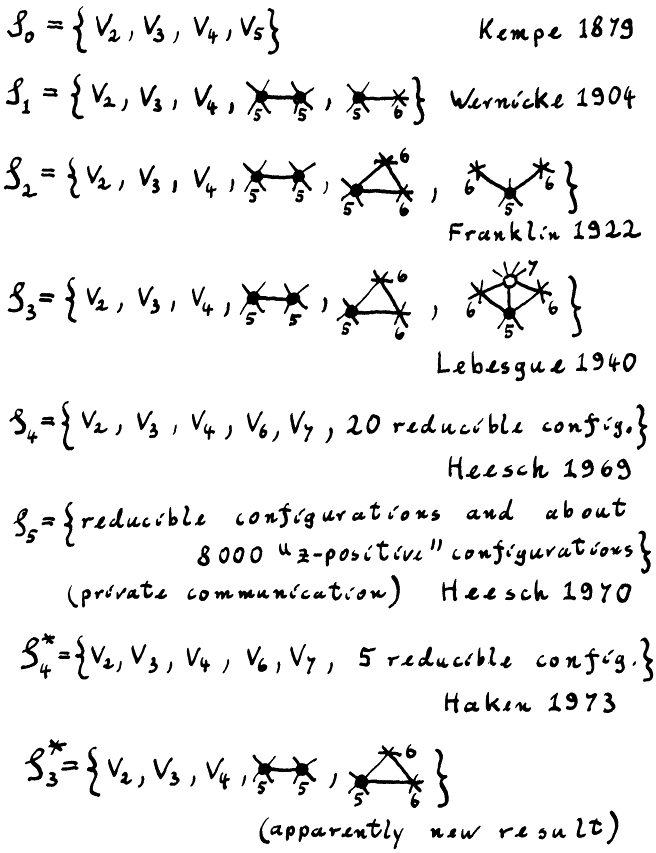


TABLE 1. Some finite, unavoidable sets of configurations

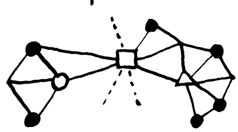
vertex the removal of which disconnects the configuration), nor in the presence of a hanging V_5-V_5 pair (i.e., a pair of adjacent 5-vertices V_5^1 and V_5^2 which are adjacent to some vertex V but such that neither V_5^1 nor V_5^2 is adjacent to any vertex but the other of the pair and the vertex V. For examples see Figure 1a; of course in all cases except the 5-5-5 triangle, the vertex V is an articulation vertex.)

Recently, Stromquist [18] has used the methods of Tutte and Whitney [19] to prove the irreducibility (in a somewhat restricted sense) of configurations which contain certain "reduction obstacles." In Stromquist's work, the obstacles of greatest practical importance are precisely those mentioned above.

In 1940, Lebesgue [12] in generalizing the work of Wernicke and Franklin exhibited several unavoidable sets of configurations. From his table on p. 36 one can derive the corollary that in Franklin's $\mathscr{S}_2$ the V_6-V_5-V_6 path may be replaced by a V_5-V_6-V_7-V_6 configuration as in $\mathscr{S}_3$ of Table 1. $\mathscr{S}_3$ considerably improves $\mathscr{S}_2$ by avoiding all five-legger vertices.

● $deg(V)=5$	$deg_{min}(V)=5$, $deg_{max}(V)=\infty$
$deg(V)=6$	$deg_{min}(V)=6$, $deg_{max}(V)=\infty$
○ $deg(V)=7$	$deg_{min}(V)=7$, $deg_{max}(V)=\infty$
□ $deg(V)=8$	$deg_{min}(V)=8$, $deg_{max}(V)=\infty$
$deg(V)=9$	$deg_{min}(V)=9$, $deg_{max}(V)=\infty$
$deg(V)=10$	$deg_{min}(V)=10$, $deg_{max}(V)=\infty$
$deg(V)=11$	$deg_{min}(V)=11$, $deg_{max}(V)=\infty$
....	....

Example



(The only leg-specifications which need to be indicated by dashed lines are at the articulated 3-legger V_8)

TABLE 2

Heesch, to demonstrate the plausibility of his conjecture, exhibits in [9] and [10] several sets of reducible configurations which are unavoidable in those planar triangulations which satisfy certain restrictive conditions. We call Heesch's approach the *principle of discharging* (see [8]). One of Heesch's sets [9, Kapitel II] consists of V_2, V_3, V_4, and twenty reducible configurations and is unavoidable in triangulations containing no V_6 or V_7.[2] Reformulated for general planar triangulations this gives set $\mathscr{S}_4$ in Table 1. Later Heesch applied the same methods which he used for deriving $\mathscr{S}_4$ to the general case; this led to an unavoidable set ($\mathscr{S}_5$ in Table 1) which consists of reducible configurations and about 8,000 z-positive configurations which are analogous to the 16 z-positive configurations in [9, p. 99]. A further discussion of the z-positive configurations is hoped to lead to an unavoidable set which consists entirely of reducible configurations and thus proves the Heesch Conjecture. The communication of this result stimulated Haken to ask the question whether the required discussion could be simplified by technical improvements to the discharging procedure. Haken [8] improved $\mathscr{S}_4$ to include five instead of twenty reducible configurations (see $\mathscr{S}_4^*$ in Table 1). The approach he used was a modification of Heesch's method. The improvement due to this reexamination by a

[2] *Added in proof.* It was brought to our attention by J. Mayer that $\mathscr{S}_4$ and $\mathscr{S}_4^*$ were preceded by the result of Chojnacki that $\{V_2, V_3, V_4, V_6, V_7$ plus a small set of reducible configurations$\}$ is unavoidable (see H. Chojnacki, *A contribution to the four color problem*, Amer. J. Math., vol. 64 (1942), pp. 36–54).

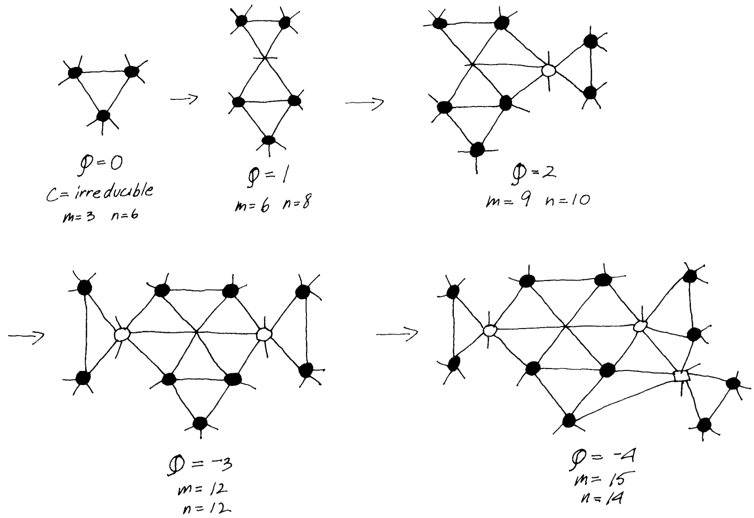


FIGURE 1a

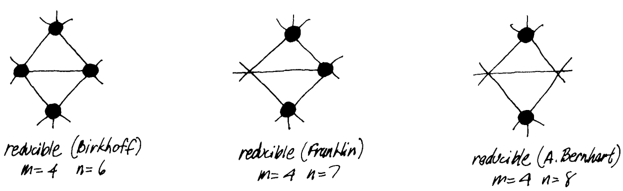


FIGURE 1b

slightly different approach raised the question whether similar reexamination could simplify the entire Heesch program. We conjecture that it can but nevertheless expect the unavoidable set of reducible configurations (if it exists) to include between one thousand and ten thousand configurations after all possible simplifications.

Further investigations with these methods have been conducted by R. Stanik [17] who applied Heesch's method to triangulations not containing any V_6 and by T. Osgood [14] who applied his own version of Haken's method to triangulations containing only V_5, V_6, and V_8.

The Heesch program consists of two parts: finding unavoidable sets and testing their members for reducibility. Heesch emphasizes finding reducible configurations and building unavoidable sets from them. Since checking reducibility for a configuration can be extremely time consuming, we feel that it would be more economical to investigate unavoidable sets in the Kempe-Wernicke-Franklin-Lebesgue line using our modifications of Heesch's method.

The Heesch program should also be compared with the work of Franklin [7], Winn [21], Ore and Stemple [16], and Mayer [13] (where sets of four color reducible configurations are exhibited which are unavoidable in planar triangulations with a restricted number of vertices.)

Our first major goal is to find unavoidable sets of configurations which are *geographically good* in the sense that no vertex of such a configuration has three or more nonconsecutive neighbors outside the configuration. Note that V_4 is the only geographically good configuration which contains a four-legger vertex; moreover, a geographically good articulation vertex must be a two-legger. Thus two of the three major reduction obstacles (mentioned above) are not present. (It is possible to look upon this concept as geopolitical rather than color theoretic. Nations might strive to form alliances which are geographically good for the sake of providing defensible frontiers.)

Although the 5-5-5 triangle is not reducible [3], [9], the next three geographically good configurations, those containing four vertices (see Figure 1b) were proved reducible by Birkhoff [6], Franklin [7], and Arthur Bernhart [3].

Two numbers of significance are m, the number of members (vertices) of the configuration, and n, the number of neighbors (which form an n-circuit about the configuration). The quotient m/n may serve as an estimator of the likelihood of reducibility. The ratio takes on its smallest value when every vertex is a three-legger boundary vertex and this value is $\frac{1}{2}$. Of all the geographically good configurations with $m/n = \frac{1}{2}$ only that of A. Bernhart (Figure 1) is known to be reducible. On the other hand one may expect that almost every geographically good configuration with $m/n > 0.8$ is reducible (provided that it does not contain a hanging pair). This expectation is supported by the recent work of F. Bernhart [5] and Allaire and Swart [1] and is based on the following reasoning. As m increases so does the number of possible (vertex) colorations of the configuration. As this number increases so does the number of those four-colorations of the surrounding n-circuit which can be extended over the configuration. Thus, for any fixed n, the larger m is the more likely it is that the iterated Kempe-chain argument used in C- and D-reductions (see [9] and [19]) will succeed. Furthermore, if we associate a third number $\phi = n - m - 3$ with a configuration, and if the number ϕ associated with an *arbitrary* configuration is not positive then that configuration contains a geographically good subconfiguration, again with ϕ nonpositive. (This can easily be proved by induction on m; see Lemma 1 in Section 8.) This implies that every configuration for which the ratio m/n of "area" to "circumference" (both measured in numbers of vertices) is at least 1 contains a geographically good subconfiguration. Now, it is plausible to expect that every planar triangulation will contain many relatively small configurations with $m/n \geq 1$ since "the area grows faster than the circumference."[3] But Figure 1a indicates how one can construct a configuration

[3] This consideration has been made precise in the recent work of Stromquist. [18, Chapter 3]. There he proves the existence of unavoidable sets of geographically good configurations by showing that every planar triangulation contains configurations of bounded size with $\phi < 0$. Stromquist's proof is much shorter than ours but does not yield a practicable procedure for explicitly constructing such a set.

with $m/n > r$ for each number $r < 1.5$ so that every geographically good subconfiguration contains a hanging pair. One could still speculate that every configuration with $m/n \geq 1.5$ contains a reducible subconfiguration. We do not believe that known techniques would yield proofs of such conjectures. Hence we prefer to look for unavoidable sets of geographically good configurations which are so small that it is practicable to check all members for reducibility individually.

Our second major goal will be to find unavoidable sets of geographically good configurations without hanging pairs. When we have displayed such unavoidable sets we shall be able to judge whether an attack on the Four Color Conjecture is practically possible with these tools. Our reasoning will be based on the number of configurations and their m and n numbers (in the best such set we can find).

Of course it will be of vital importance that we not only find *one* unavoidable set $\mathscr{S}$ of configurations likely to be reducible but that we develop an algorithm which allows us to find many such sets. When certain configurations in $\mathscr{S}$ turn out to be C-irreducible (or despite great effort cannot be proved reducible) then we shall want to find a set $\mathscr{S}'$ which contains none of these configurations but otherwise is similar to $\mathscr{S}$. This procedure must be iterated producing sets $\mathscr{S}'', \ldots$ until hopefully an unavoidable set of reducible configurations can be exhibited. These goals make it necessary to develop as broad a theory of unavoidable sets as possible.

3. Some preliminary new results

Application of Heesch's method [9, Kap. II] to the problem of finding an unavoidable set of configurations containing no vertex with more than four legs yields a stronger result than previously known ($\mathscr{S}_3^*$ of Table 1 as compared with $\mathscr{S}_3$). The V_5-V_6-V_7-V_6 in Lebesgue's $\mathscr{S}_3$ may be omitted. The proof is very simple: Let T be a planar triangulation containing no members of $\mathscr{S}_3^*$. Assign to each vertex V_i of degree i in T a "charge" $q_0(V_i) = 6 - i$. By Euler's formula

$$\sum_{V \in T} q_0(V_i) = 6\chi(T) = 12 > 0$$

where $\chi(T)$ is the Euler characteristic. Now we discharge all V_5's in T, i.e., we obtain a new charge distribution q_1 which assigns every vertex V of T a charge $q_1(V)$ such that $\sum_{V \in T} q_1(V) = \sum_{V \in T} q_0(V)$, as follows. The positive charge $(+1)$ of each V_5 is distributed in equal portions to its negative neighbors (of degree at least seven). By hypothesis, each V_5 has at least three negative neighbors. Thus a $V_k \in T$ ($k \geq 7$) receives contribution at most $\frac{1}{3}$ from any one of its V_5-neighbors. Again by hypothesis, a vertex $V_7 \in T$ has at most three V_5-neighbors (for otherwise a consecutive 5-5 pair would occur) and a V_8 has at most four V_5-neighbors. Trivially, a V_j with $j \geq 9$ cannot have more than

$3(j-6)$ neighbors. Thus $q_1(V_5) = 0$ for all $V_5 \in T$, $q_1(V_6) = 0$ for all $V_6 \in T$, and

$$q_1(V_k) \leq q_0(V_k) + 3(k-6)(\tfrac{1}{3}) \leq 0 \quad \text{for all } V_k \in T,\ k \geq 7,$$

contradicting Euler's formula. Thus T does not exist.

The two configurations in $\mathscr{S}_3^*$ which are not geographically good each contain two four-legger vertices whereas the V_5-V_6-V_7-V_6 in $\mathscr{S}_3$ contains three four-leggers. This suggested a further preliminary study—to find an unavoidable set of configurations which fail the criterion for geographical goodness at no more than one vertex each. Haken found an unavoidable set (unpublished) of 68 configurations (besides V_2, V_3, V_4) with this property. This result required a considerable refinement of Heesch's method and indicated that still further refinements would be required to find an unavoidable set of geographically good configurations. The magnitude of the task indicated the usefulness of a computer for the excessively large number of combinatorial case distinctions. This led to the joint work of the two authors in which Appel wrote a series of rather complicated computer programs. During the first period of the work, using results supplied by initial runs of the basic program, we developed an improved discharging procedure which now allows us to prove the existence of unavoidable sets of geographically good configurations. The proof actually exhibits an algorithm, or rather a family of algorithms, for constructing such sets.

Although the results in this paper indicate bounds on the size and number of the configurations in the set, the actual construction of a reasonably small such set will require some additional work. We hope to achieve this in the near future.

In what follows we describe the discharging procedure and the corresponding algorithm for constructing unavoidable sets of geographically good configurations and prove the existence theorem for such sets.

4. The theory of short-range discharging

Since the principle of discharging seems to yield the most fruitful procedures known for finding unavoidable sets we consider it appropriate to develop a particular theory of these methods, emphasizing possible applications to the Four Color Problem. The discharging arguments which have been applied are essentially *local* arguments based on the hypothesis that the given triangulation T does not contain certain relatively small configurations. Although in principle charges could be shifted from any vertex of T to any other one the discharging procedures which have been applied thus far all involve moving charges from a vertex to another vertex quite close by. These "short range" discharging arguments are entirely independent of the *global* Euler characteristic $\chi(T)$. In fact, knowledge of $\chi(T)$ seems to be of no use in determining a discharging procedure, although the finiteness of T occasionally yields advantages. For this reason, *henceforth we shall consider triangulations T of arbitrary closed 2-manifolds M^2.*

The Euler characteristic is considered for just two purposes in the entire approach. First, it motivates the *original charge distribution*

$$q_0(V) = 60(6 - \deg(V))$$

where $\deg(V)$ means the degree of V in T and the factor 60 is introduced (following Heesch) for convenience. Second, it permits us to conclude, after discharging has been completed, that $\chi(T) \leq 0$ and hence that the set $\mathcal{S}$ of configurations which has been excluded by hypothesis is unavoidable in all triangulations of S^2 and P^2.

For further convenience we observe the following convention.

Convention. By a triangulation T we mean a triangulation of a closed 2-*manifold M^2 with the following properties.*

(a) T contains no vertices of degree less than five.

(b) T contains no 5-5-5-5 diamonds (the reducible configuration of Birkhoff displayed in Figure 1).

(c) T contains no circuit of length two.

(d) T contains no circuit of length three other than the boundary of a triangle.

(e) T contains no circuit of length four other than the boundary of the union of two triangles with a common edge.

(f) If C is a circuit of length five in T then C bounds a disk in M^2 which contains at most one vertex of T in its interior.

(g) If C is a circuit of length six in T such that C contains a vertex of degree five then C bounds a disk in M^2 which contains at most three vertices of T in its interior.

The excluded circuits are known to be four-color reducible in planar maps (for (c), (d), (e), (f) see [6] and for (g) see [3]) so we consider them to be geographically good and include them, together with the 5-5-5-5 diamond and V_2, V_3, and V_4 in each unavoidable set we describe (without special mention).

DEFINITION A. By a *charged triangulation* (T, q) we mean a triangulation T together with a function q which assigns to each vertex V of T a rational number $q(V)$ so that

$$\sum_{V \in T} q(V) = \sum_{V \in T} q_0(V) = 360\chi(T).$$

(T, q) is *completely discharged* if $q(V) \leq 0$ for every V in T.

By a "discharging" we mean the replacement of a charge function q' by another charge function q''. The theory of unavoidable sets lends itself rather naturally to "short-range" dischargings. The main question of the theory will be which dischargings lead to the best results.

In [9, Kapitel III and IV], Heesch described a very interesting example of a discharging procedure. For each vertex V of a triangulation T and each nonnegative integer n, he defines a *curvature of order n at V*, $k_n(V)$. Here k_0 is (except for a multiplicative factor 60) our initial charge function q_0. The function k_n is a charge function such that for any vertex V of T, $k_n(V)$ is obtained by a certain averaging procedure from the k_0 charges (and thus from the degrees) of all vertices in the nth neighborhood of V in T. Heesch conjectures [9, p. 216] that whenever $k_5(V)$ is positive then the fifth neighborhood of V contains some (four-color-) reducible configuration belonging to a finite set $\mathscr{S}$. (In our terminology this means that the discharging procedure which replaces k_0 by k_5 yields a finite unavoidable set of reducible configurations.)

Heesch proves a stronger conjecture (with k_4 replacing k_5) for a special case in which every vertex has degree five or seven [9, IV.2]. The only obstacle to proving the conjecture in general seems to be the combinatorial complexity of the task. It would be necessary to treat all possible fifth neighborhoods of a vertex V such that $k_5(V)$ is positive. We believe this would require a number of case distinctions which appears too large to be practicable. The discharging procedure $k_0 \rightarrow k_5$ appears particularly theoretically simple and elegant (probably best possible in these respects), but from a practical point of view seems far from optimal. We believe that more sophisticated procedures can be presented which are much more practical.

5. The family of discharging procedures treated in this paper

The remainder of this paper deals with a special family of discharging procedures which depend on several parameters and which we believe to be relatively close to the practical optimum. These discharging procedures are based on the following reasoning. In order to minimize combinatorial complexity we try to avoid exhaustive consideration of large neighborhoods. (We regard the third neighborhood of a vertex as a practical limit.) We are willing to sacrifice some theoretical simplicity for practicality. One essential step in our discharging procedure is to move the positive charges of the 5-vertices in fractions to their major (i.e., degree at least seven) neighbors. We call this step *fractional discharging*. The simplest possible fractional discharging has been used by Heesch in [9, Kapitel II] (and also in his unpublished work which led to the set $\mathscr{S}_5$), where the positive charges are moved in equal fractions to the negative neighbors. However there are many situations (16 in [9] and about 8,000 in the general case) in which this simple fractional discharging makes formerly negative vertices positive (we call this phenomenon *overcharging*) but no reducible configuration is present. Heesch treats these newly positive ("overcharged" or "z-positive") vertices by a second discharging step. It turns out that the number of overcharging situations can be greatly decreased by use of a more sophisticated fractional discharging in which the distribution of the positive charge is done according to suitably chosen weight factors. In order to determine what part of the positive charge of a 5-vertex V_5 is to be moved to some particular negative neighbor V of V_5 we take into account the amount of negative charge of V, the number of positive 5-neighbors of V and, of course, the corresponding data for all other negative neighbors of V_5. This means that we determine the fractional discharging of V_5 by considering its first neighborhood and the 5-vertices in its second neighborhood. We call a configuration consisting of these vertices a *plugged* V_5*-neighborhood.*

In Section 19 we shall describe a *fractional discharging algorithm* which yields the fraction of charge going from V_5 to V as a function of the above data. There is some freedom in the choice of such an algorithm and we think that our algorithm is close to optimal.

A crucial task is to examine those cases in which the fractional discharging algorithm overcharges a major vertex but no geographically good configuration is present. For this purpose we must exhaustively examine all configurations which consist of a negatively charged major vertex V, all 5-neighbors V_5^i of V, and plugged neighborhoods of the V_5^i. We call these configurations *plugged clusters of* V_5*-neighborhoods.* They contain a large part (in many cases all) of the second neighborhood of V and additional "plug V_5" vertices at distance three from V. Thus they are, roughly speaking, larger than second neighborhoods but smaller than third neighborhoods of V. This exhaustive examination can be done with acceptable effort by using a rather sophisticated computer program which directs the computer to print all relevant cases and their discussion (details may be found in Section 27).

It is rather obvious, unfortunately, that there exist some plugged clusters which overcharge but contain no geographically good subconfigurations. Thus our fractional discharging algorithm alone does not constitute a satisfactory discharging procedure. A more sophisticated fractional discharging procedure would require the examination of more and larger configurations and we think this inadvisable. Therefore we prefer to treat individually those cases in which the fractional discharging procedure is unsatisfactory. Using the conventions listed in Table 2, in Tables 3 and 4 we display 46 particular configurations. We prove (by exhaustive examination, see details in Sections 25–27) that every overcharging plugged cluster which does not contain a geographically good subconfiguration contains at least one of these 46 configurations. For these configurations we define particular dischargings which are carried out *before* the general fractional discharging is applied. Our strategy is thus to "improve" the original charge function q_0 by *preliminary dischargings* so that the fractional discharging can be applied without the occurrence of unsatisfactory cases. Thus we call the 46 configurations mentioned the *primary preliminary discharging situations*. There is considerable freedom in the choice of primary preliminary discharging situations and of the dischargings defined on them. We have tried many different choices and the list presented in this paper is the best we have found so far.

6. Problems which must be handled

At this point it should be obvious that a great deal of machinery must be developed in order to handle the theoretical and practical complexity of the project described above. This helps explain the length of the paper and the complexity of some of the definitions.

To briefly describe some problems which arise we consider the example of three primary preliminary discharging situations, $A1$, $B1$, and $C1$ (see Figure 2). In these situations, the central V_5 has only one negative neighbor which we call the *pivot* of the configuration. We choose to move the entire positive charge of 60 from the central V_5 to the pivot as indicated by the arrow. We must immediately ask the Overcharging Question: If these dischargings were carried out simultaneously whenever an image of $A1$, $B1$, or $C1$ is contained in T, can overcharging result without T containing a geographically good subconfiguration (out of a fixed finite set)? In our example it is not difficult to check that the answer is "no." We must examine all cases in which $k - 5$ such dischargings go from central V_5's to a (pivot) vertex V_k, of degree $k \geq 7$. No such cases exist if $k \geq 16$ since there is not enough room for $k - 5$ copies of $A1$, $B1$, or $C1$ (with pairwise distinct central V_5's) around the V_k. If $12 \leq k \leq 15$ then each case involves at least one of the two geographically good configurations of Figure 3. If k is 10 or 11 then each case involves either one of the configurations of Figure 3 or a V_{10} or V_{11} which is entirely surrounded by minor vertices, at least five of which are of degree five. The cases $k = 7$, 8, and 9 can be easily enumerated and examined individually. (It seems to be a general rule in this work that the higher the degree of the major vertices the easier the analysis.)

Now we must ask whether the discharging which acted acceptably on the central V_5's in situations $A1$, $B1$, and $C1$ may have created new situations in which a positive V_5 has only one negative neighbor without a geographically good configuration being involved. One may check that there are precisely eight such "secondary situations." It appears best to carry out dischargings (as indicated by the arrows in Figure 2) in these secondary situations also. This leads us to again ask the overcharging question and determine all possible "tertiary situations." There are only four tertiary situations (see Figure 2); we carry out dischargings in these situations also. Fortunately, no "quaternary situations" are created and hence the three primary situations lead to a collection of fifteen discharging situations.

In order to handle the question of secondary, tertiary, etc., situations properly, we shall define the concept of a *modification* (details in Section 10). The essential idea is this: if C' is a configuration then we obtain a modification, say C, of C' by (i) raising the degrees of some non-5-vertices of C', (ii) lowering the degrees of some 6-vertices of C' to five, and (iii) attaching discharging situations (by "merging") in such a way that (after the discharging) every vertex of C' has the same charge as it originally had in C'. For example, all of the secondary situa-

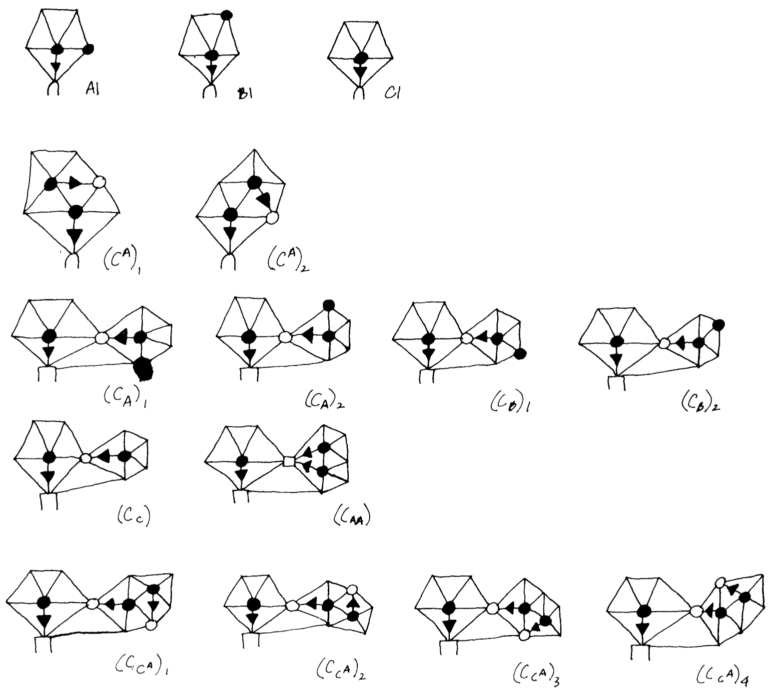


FIGURE 2

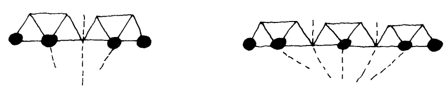


FIGURE 3

tions in Figure 2 are modifications of primary situation $C1$. $(C^A)_1$ and $(C^A)_2$ are obtained from $C1$ by raising a V_6 to a V_7, lowering another V_6 to a V_5 and attaching a copy of $A1$ so that the discharging in $A1$ produces charges of zero at both the V_7 and the V_5. C_{AA} is obtained by raising a V_6 to a V_8 and attaching two copies of $A1$ so as to produce charge zero at the V_8. It is essential to define the modification procedure in such a way that if C' contains a geographically good configuration, say C_0', then every modification C of C' contains a geographically good subconfiguration (which is essentially a modification of C_0'). On the other hand, in many cases a configuration (such as $A1$ or $B1$) which does not contain any geographically good configuration will have geographically good subconfiguration in *all* of its (nontrivial) modifications.

We must not only define the modification procedure precisely but we must also develop a practical method of exhaustively examining all possible modifications of a given set of configurations (such as, for instance, the 46 primary preliminary discharging situations in Tables 3 and 4). This leads to the somewhat more complicated concepts of premodification and extension of a premodification which shall allow us to build up modifications conveniently, one step at a time. Using the modification concept recursively, we define the general (primary, secondary, etc.) preliminary discharging situations to be modifications of primary situations which do not contain geographically good subconfigurations. The first application of the machinery for exhaustively examining all modifications will be the proof that our 46 primary discharging situations lead to precisely 253 preliminary discharging situations (as indicated in Tables 5 and 6).

For formal reasons we distinguish two sorts of preliminary discharging situations, namely (see Tables 3 and 4) *integral discharging situations* in which a central V_5 is *completely* discharged to a major vertex, the degree of which has no upper bound, and *partial discharging situations* in which several V_5's are *partially* discharged to V_7's or V_8's so that these *receiving vertices* are *completely* discharged. We carry out all integral dischargings simultaneously, which yields a new charge function q_1. Next we carry out all the partial dischargings simultaneously and obtain a charge function q_2. Then, we treat the overcharging question for q_2 (Section 18).

After these preliminary improvements of the charge function we carry out the fractional discharging as described before based on the q_2 (rather than the q_0) charges of the vertices. In particular, a V_7 or V_8 with q_2-value of zero will be treated as if it were a V_6 and will not receive any charge. Finally, we must ask whether it is possible that $q_3(V)$ (the charge at V after the fractional discharging) is positive for any vertex V of T without the occurrence of a geographically good subconfiguration in T. The essential part of this question will be the overcharging question for q_3.

Our definition of the fractional discharging (Section 19) immediately implies that overcharging can occur only at vertices of degree seven or eight and it is feasible to exhaustively examine all plugged clusters P of V_5-neighborhoods which have the following property: Suppose that each preliminary discharging situation *entirely* contained in P is discharged but no discharging of situations partially inside and partially outside of P is considered. This gives each vertex V of P a charge which is either $q_0(V)$ or the result of these "interior" dischargings applied to $q_0(V)$. Under this charge function, P overcharges (i.e., the q_3-value of the "pivot" V_7 or V_8 of P is positive).

We shall prove that *all possible* overcharging cases can be derived by modifications from the special cases just mentioned. Thus it appears that the modification machinery is not only the theoretically most complicated but also the practically most important part of this paper.

We shall prove several theorems and lemmas. Each lemma states that under certain hypotheses a triangulation or configuration contains a geographically good configuration in a certain "size class" (where a size class $\langle p, q \rangle$ is defined by the maximal diameter p of its members and the maximal degree q of the vertices of its members). The final lemma will be the Discharging Lemma which states that if a triangulation T contains a vertex V with $q_3(V) > 0$ then T contains a geographically good configuration in size class $\langle 14, 23 \rangle$. Some lemmas will be proved by exhaustive examination of all cases, others will be proved by more general considerations.

7. Future applications

In order to obtain unavoidable sets of geographically good configurations which are not only finite but also reasonably small we shall have to reformulate the lemmas to explicitly list all relevant geographically good configurations instead of only stating the size class of the set. Correspondingly all proofs must be given by explicit enumeration of all cases. The machinery developed in this paper will enable us to handle this problem and we hope to finish this task in the near future.

Next we must replace the concept "geographically good" by "permissible" in all definitions, lemmas, and proofs. At this stage "permissible" will mean "geographically good and not containing a hanging pair." Then we shall need more preliminary discharging situations, longer case enumerations, and larger sets of configurations. It appears to us, however, that nothing need be changed in principle.[4]

If the unavoidable sets required at this stage encourage us to continue, we will have to make the concept of "permissible" somewhat stronger by forbidding certain individually listed (irreducible) configurations.

In order to illustrate our methods we have treated, in a subsequent paper [2], the case of isolated V_5's, i.e., triangulations which do not contain any pair of adjacent 5-vertices, where we regard as permissible all geographically good configurations without hanging pairs and with $\phi = n - m - 3 \leq 1$. The unavoidable set so obtained contains 47 members.

Remark. It is important to note that if we say that a triangulation T *contains* a configuration G then we mean that in T there is an image of G under some *immersion.* That means, if the diameter of G is greater than two, it may be that the first neighborhood N of G is *not "properly embedded"* in T (as is usually assumed in reduction theory, see [9, p. 31], but for instance, N may "overlap itself" in T. If N is four color reducible under the hypothesis of proper embedding it remains to check whether it also allows a reduction in the case of immersion. We do not expect any difficulties if N is D-reducible [9], [19] but some additional work may be required if N is C-reducible but D-irreducible.

[4] In more recent work we have found it possible to replace "permissible" by "likely to be reducible" which means satisfying the conditions required in [2]. Moreover it appears possible to add the condition $n \leq 17$, making it possible to check the reducibility of all required configurations with available computational techniques.

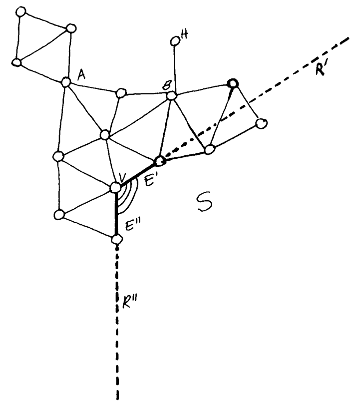


FIGURE 4

8. Configurations with specified degrees and charges

When considering configurations we find it convenient, following Heesch, to think of some vertex degrees as precisely specified and others as specified within certain limits. Thus, we effectively discuss classes of configurations whose members all satisfy certain desired conditions without necessarily enumerating all of their members. Those vertices with specified degrees may be assigned "charges," given as rational numbers. We define a configuration to be a connected and simply connected planar complex along with functions which specify degrees and charges. (In practice, intuition is usually safe as the formal definitions correspond to the intuitive notions of the properties involved.)

DEFINITION B. Let K be a *finite simplicial complex* in the *plane* $\mathbf{E}^2$ (see Figure 4) such that the edges of K are straight, or at least piecewise straight, and such that K is *connected* and *simply connected.* An edge of K is called a *boundary edge* of K if it is incident to at most one triangle of K, otherwise it is called an *interior edge* of K. A vertex V of K is called a *boundary vertex* of K if it is incident to a boundary edge of K or if $V = K$; otherwise V is called an *interior vertex.* A boundary vertex is called an *articulation vertex* of K (see A, B in Figure 4) if its removal disconnects K. A boundary vertex which is incident to precisely one edge is called a *hanging vertex* (see H in Figure 4). We denote by deg $(V \mid K)$ the *degree of vertex V of K in K* (i.e., the number of edges of K which are incident to V).

We restrict our attention to complexes for which deg $(V \mid K) \geq 5$ for every interior vertex of K. The (*absolute*) *cyclic order* in which those edges, say $E_1, \ldots, E_v$ of K incident upon a vertex V of K lie about V in $\mathbf{E}^2$ means the set of $2v$ orderings of $E_1, \ldots, E_v$ one can obtain by starting at any E_i and reading clockwise or counter-clockwise. If K is not a single vertex then by an *outer sector of K* we mean (see Figure 4) a sector S in $\mathbf{E}^2$ of angle greater than 0 (but possibly 360°), consisting of a vertex V of K, two rays R' and R'' originating from V (if the angle is 360° then $R' = R''$) together with the interior of S such that the following conditions are satisfied.

(i) V is a boundary vertex of K.

(ii) Some initial segments of R' and R'' are identical with initial segments of boundary edges, say E', E'' of K.

(iii) Some initial part of S (i.e., some small neighborhood of V in S) has its interior disjoint from K.

We say then that S is an *outer sector of K at V between E' and E''*. If K is a single vertex V then the outer sector at V consists of V and $\mathbf{E}^2 - V$.

DEFINITION C. By a *configuration* $C = (K, l_{\min}, l_{\max})$ we mean a simplicial complex K as in Definition B together with functions $l_{\min}$ and $l_{\max}$ called *leg specifications*. The leg specifications associate with each outer sector S of K a positive integer $l_{\min}(S)$ and a value $l_{\max}(S)$ which is either a positive integer or ∞. These values are called the *specified minimal or maximal number of legs* (*of* V) *lying in sector* S. If $l_{\min}(S) = l_{\max}(S)$ we call the common value $l(S)$. A vertex V is called a *fully specified vertex* if it is either interior or satisfies $l_{\min}(S) = l_{\max}(S)$ for each outer sector S at V. The configuration C is called *fully specified* if each of its vertices is fully specified. A fully specified vertex V of K is called an *l-legger vertex of C* if l is the sum of $l(S)$ over all outer sectors at V. (For an interior vertex $l = 0$, for an articulation vertex $l \geq 2$.)

From $l_{\min}$ and $l_{\max}$ we derive functions $\deg_{\min}$ and $\deg_{\max}$ called *degree specifications* which associate with each vertex of K a value (a positive integer or ∞).

$$\deg_{\min}(V) = \deg(V \mid K) + \sum l_{\min}(S),$$

$$\deg_{\max}(V) = \deg(V \mid K) + \sum l_{\max}(S)$$

where the sums are taken over all outer sectors S of K at V.

If $\deg_{\max}(V) = \deg_{\min}(V)$ then the common value is denoted by $\deg(V)$.

We restrict our attention to configurations C which fulfill Conditions (a)–(g) as stated in the convention on triangulations; in particular $\deg_{\min}(V) \geq 5$ for every vertex V of C. By a *major vertex* of C we mean a vertex V of C with $\deg_{\min}(V) \geq 7$. If a vertex V belongs to several configurations, say C', C'', simultaneously we write $\deg^{C'}_{\min}(V)$, $\deg^{C''}_{\min}(V)$, etc., to denote the degree specifications for V in C' and C''.

As a superficial measure for the size of a configuration we may consider its *diameter* D and the highest degree, $\deg^{C}(V)$ which occurs at any (fully specified) vertex V of C. By $\langle D, d\rangle$ we denote the (finite) set of all fully specified configurations of diameter not greater than D and highest degree not greater than d. For a fully specified configuration C we define $m(C)$ to be the number of vertices of C; $n(C) = \sum_S [l(S) - 1]$, where the sum is taken over all outer sectors S of C; and $\phi(C) = n(C) - m(C) - 3$.

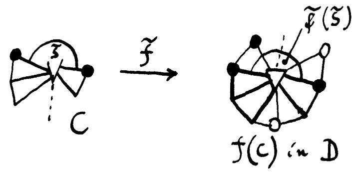


FIGURE 5

A configuration $C' = (K, l'_{\min}, l'_{\max})$ is called a *specialization* of a configuration $C = (K, l_{\min}, l_{\max})$ if it is obtained from C by narrowing the leg specifications, i.e., if $l'_{\min}(S) \geq l_{\min}(S)$ and $l'_{\max}(S) \leq l_{\max}(S)$ for each outer sector S of K. The complex K is called the *carrier* of C and is written $K = |C|$.

Usually we define configurations by drawings, using the conventions of Heesch [9] for indicating the degree specifications, see Table 2. In most cases this will make the leg specifications clear; but occasionally we may indicate legs by outgoing dashed lines.

DEFINITION D. By a charged configuration $\mathscr{C} = (C, q)$ we mean a configuration along with a function q which assigns to every fully specified vertex V a number $q(V)$ called the *charge of* V. As we did for triangulations we define the *original charge function* on a configuration C by

$$q_0^C(V) = 60[6 - \deg^C(V)]$$

for all fully specified vertices V of C.

DEFINITION E. Let $C = (K, l_{\min}, l_{\max})$ be a configuration. Let D be either a configuration, say $(K^*, l^*_{\min}, l^*_{\max})$, or a triangulation of a 2-manifold M^2. By a (*combinatorial*) *immersion* of $|C|$ *into* $|D|$ we mean a simplicial map $f\colon K \to K^*$ (or $f\colon K \to M^2$) with the following properties.

(i) f is *locally* 1-1 to the extent that it maps each triangle (each edge) of $|C|$ in a 1-1 fashion onto a triangle (an edge) of $|D|$ and that $f(V') \neq f(V'')$ for any two different vertices V', V'' of $|C|$ if the *distance between* V' *and* V'' *in* $|C|$ (the number of edges in the shortest edge path joining V' and V'' in $|C|$) is smaller than six.

(ii) f can be extended to a simplicial map $f^\sim\colon K^\sim \to K^{\sim *}$ (or $f^\sim\colon K^\sim \to M^2$) where $K^\sim$ and $K^{\sim *}$ are small neighborhoods of K and K^* respectively in $\mathbf{E}^2$ in such a way that $f^\sim$ is still locally 1-1. (This implies that f preserves the cyclic order of edges about vertices.) The immersion f is called an *embedding* if it is 1-1; f is called an *isomorphism* if it is 1-1 and onto. We say that an immersion f *respects the leg specifications of C and D* if the specifications for D are equivalent to the specifications for C as carried over by f. By this we mean that for every outer sector S of C the following holds (see Figure 5).

(iii) Let V be the vertex of S and let $S^\sim$ be some initial part of S in $K^\sim$. Let $S^{*(1)}, \ldots, S^{*(p)}$ be those outer sectors of D at $f(V)$ which have initial parts in

$f(S^\sim)$ (p may be zero). Let e^* be the number of edges of D which have initial segments in the interior of $f(S^\sim)$. Then

$$e^* + \sum_{i=1}^{p} l^*_{\min}(S^{*(i)}) = l_{\min}(S),$$

$$e^* + \sum_{i=1}^{p} l^*_{\max}(S^{*(i)}) = l_{\max}(S).$$

We say that D *contains* (*an image of*) C if there is an immersion $f\colon |C| \to |D|$ which respects the leg specifications of C and D.

DEFINITION F. Let $C = (K, l_{\min}, l_{\max})$, $C^* = (K^*, l^*_{\min}, l^*_{\max})$ be two configurations. An isomorphism $f\colon |C| \to |C^*|$ is called a *full equivalence* between C and C^* if for each outer sector S of C, $l_{\min}(S) = l^*_{\min}(S^*)$ and $l_{\max}(S) = l^*_{\max}(S^*)$, where S^* means the outer sector of D which contains the $f^\sim$-image of an initial part of S in K. f is called a *degree equivalence between* C^* *and* C if

$$\deg^{C^*}_{\min}(f(V)) = \deg^{C}_{\min}(V) \quad \text{and} \quad \deg^{C^*}_{\max}(f(V)) = \deg^{C}_{\max}(V)$$

for every vertex V of C. For brevity we shall usually say configuration C when we mean (*degree*)-*equivalence class* represented by configuration C. It will be clear from the context what is meant.

We say that C is a *subconfiguration* of D (where D is a configuration or a triangulation) if:

(a) $|C|$ is a subcomplex of $|D|$;

(b) the $\deg_{\min}$ and $\deg_{\max}$ functions of C and D agree on C; and

(c) the inclusion map $i\colon |C| \hookrightarrow |D|$ respects the leg specifications of C and D.

A subconfiguration of C is called *the specified part of* C if it contains precisely those vertices of C which are fully specified. If A and B are subconfigurations of a configuration C we use the expression $A \cup B$ (expression $A \cap B$) for the subconfiguration of C which contains precisely those vertices which belong to A or B (to A and B), provided that such a subconfiguration exists.

Remark. Let C be a fully specified configuration, T a triangulation, and $f\colon |C| \to T$ an embedding which respects the leg specifications of C (and T). Suppose that there exists a subconfiguration, say $C^\sim$, of T which contains precisely the vertices of $f(C)$ and those vertices of T which are adjacent to vertices of $f(|C|)$. Then the number of boundary vertices of $C^\sim$ is $n(C)$. (This follows immediately from the definitions.)

DEFINITION G. If a vertex V of a configuration C is fully specified then we may treat the legs in each outer sector at V as additional "edges" originating from V into S and thus assign a cyclic order to legs in those outer sectors on V which have more than one leg. This, along with the cyclic order of edges at V in $|C|$ induces a complete cyclic order of edges and legs about V. With this convention we say that a fully specified configuration C is *geographically good*

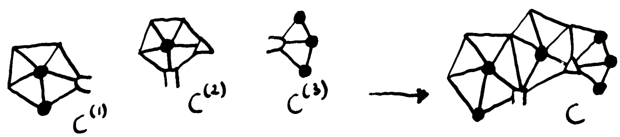


FIGURE 6

if it has no vertex which lies on three (or more) legs not all consecutive (in the cyclic order around the vertex). A vertex which is either an l-legger, $l \geq 4$, or a 3-legger articulation vertex cannot lie in a geographically good configuration. We call such a vertex a *bad* vertex. We shall often need to know of a configuration whether or not it possesses a geographically good subconfiguration. Clearly, this can be determined by "wiping off" a bad vertex V, if one exists, deleting all edges and triangles on V, and iterating. If this procedure stops before all vertices of C have been wiped off then C has a geographically good subconfiguration, namely the residue after the procedure has been carried out. Thus we will use the term *wipeout* configuration for one which contains no geographically good subconfiguration.

Tools we shall use in the proof of later lemmas include the following two lemmas.

LEMMA 1. *If C is a fully specified configuration with $\phi(C) \leq 0$ then C contains a geographically good subconfiguration, say C_0, with $\phi(C_0) \leq 0$.*

An observation which makes Lemma 1 more immediately applicable is the following.

LEMMA 2. *If $C = A \cup B$ is a fully specified configuration such that $A \cap B$ is also a configuration then $\phi(C) + \phi(A \cap B) \leq \phi(A) + \phi(B)$. Thus if C is the union of two configurations such that $\phi(A \cap B) \geq \phi(A) + \phi(B)$ then C has a geographically good subconfiguration.*

DEFINITION H. We say that a *configuration C is obtained from configurations* $C^{(1)}, \ldots, C^{(n)}$ by merging (see Figure 6) if there are combinatorial immersions $f^{(1)}: |C^{(1)}| \to |C|, \ldots, f^{(n)}: |C^{(n)}| \to |C|$ so that the following conditions hold.

(i) Every vertex of C is the image of at least one vertex of one or more of $C^{(1)}, \ldots, C^{(n)}$

(ii) The leg specifications of C are such that

$$\deg_{\min}(V) = \max\{\deg_{\min}(V')\} \quad \text{and} \quad \deg_{\max}(V) = \min\{\deg_{\max}(V')\}$$

where the minimum and maximum are taken over all preimages V' of V.

(iii) There exist specializations say $C^{\vee(1)}, \ldots, C^{\vee(n)}$ of $C^{(1)}, \ldots, C^{(n)}$ so that $f^{(i)}$ respects the leg specifications of $C^{\vee(i)}$ and C (for all i, $1 \leq i \leq n$).

We say that the merging *identifies* vertex $V^{(i)}$ of $C^{(i)}$ and vertex $V^{(j)}$ of $C^{(j)}$

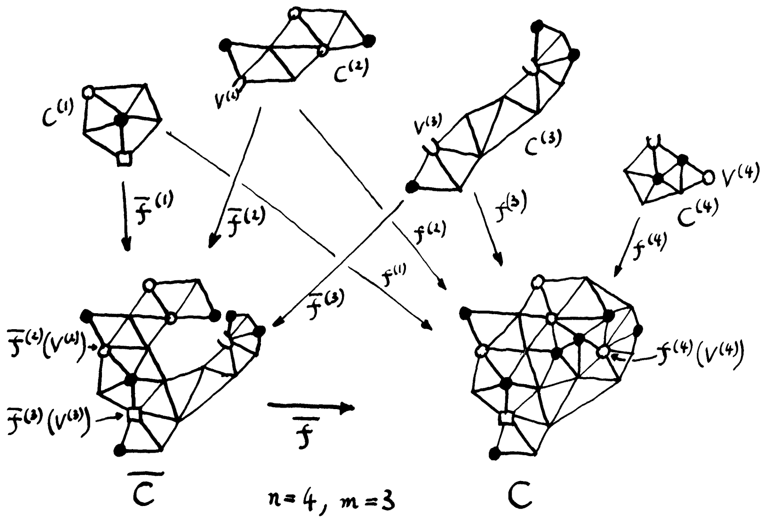


FIGURE 7

if $f^{(i)}(V^{(i)}) = f^{(j)}(V^{(j)})$ etc. We also say that an edge F of $C^{(i)}$ *is identified to a leg of* outer sector S of $C^{(j)}$ if (with notation analogous to Definition E(iii)) the interior of $f^{\sim(j)}(S^{\sim})$ contains an initial segment of $f^{(i)}(E)$.

If C is as above and if $1 \leq m \leq n$ then we say that a configuration $\bar C$ is obtained from $C^{(1)}, \ldots, C^{(m)}$ by merging *compatible with the merging of* $C^{(1)}, \ldots, C^{(n)}$ *to* C if the following holds (see diagram and Figure 7).

$$\begin{tikzcd}
C^{(1)} \arrow[d, "\bar f^{(1)}"'] \arrow[ddrr, "f^{(1)}"] & \cdots & C^{(m)} \arrow[dll, "\bar f^{(m)}"'] \arrow[dd, "f^{(m)}"] & C^{(m+1)} \arrow[ddl, "f^{(m+1)}"] & \cdots & C^{(n)} \arrow[ddlll, "f^{(n)}"] \\
\bar C \arrow[drr, "\bar f"'] & & & & & \\
& & C & & &
\end{tikzcd}$$

Denote the immersions of the merging of $C^{(1)}, \ldots, C^{(m)}$ to $\bar C$ by

$$\bar f^{(1)}: |C^{(1)}| \to |\bar C|, \ldots, \bar f^{(m)}: |C^{(m)}| \to |\bar C|.$$

Then C can be obtained by merging from $\bar C$ and $C^{(m+1)}, \ldots, C^{(n)}$ by some immersion $\bar f: |\bar C| \to |C|$ and $f^{(m+1)}, \ldots, f^{(n)}$ so that for each i, $1 \leq i \leq n$, $f^{(i)} = \bar f \circ \bar f^{(i)}$.

Later there will be several occasions at which we shall have to deal with the case that several configurations, say $C^{(2)}, \ldots, C^{(n)}$ are attached (by merging) to one configuration, say $C^{(1)}$. It will be helpful to know that we can carry out this attaching procedure in several steps, e.g., attaching $C^{(2)}, \ldots, C^{(m)}$ first and $C^{(m+1)}, \ldots, C^{(n)}$ thereafter. This is expressed by the following theorem (see Figure 7).

THEOREM 1. *Assume that a configuration C is obtained by merging from configurations $C^{(1)}, \ldots, C^{(n)}$ according to immersions*

$$f^{(1)}: |C^{(1)}| \to |C|, \ldots, f^{(n)}: |C^{(n)}| \to |C|.$$

Further assume that there are vertices $V^{(2)}, \ldots, V^{(n)}$ of $C^{(2)}, \ldots, C^{(n)}$, respectively so that $f^{(i)}(V^{(i)}) \in f^{(1)}(|C^{(1)}|)$ for each i, $2 \leq i \leq n$.

Let $1 \leq m \leq n$. Then there exists a configuration, say $\bar{C}$, which is obtained from $C^{(1)}, \ldots, C^{(m)}$ by merging according to immersions, say

$$\bar{f}^{(1)}: |C^{(1)}| \to |\bar{C}|, \ldots, \bar{f}^{(m)}: |C^{(m)}| \to |\bar{C}|$$

so that the following hold.

(i) *The merging is compatible with the merging of $C^{(1)}, \ldots, C^{(n)}$ to C.*

(ii) $\bar{f}^{(j)}(V^{(j)}) \in \bar{f}^{(1)}(|C^{(1)}|)$ *for each j, $2 \leq j \leq m$.*

9. Plugged configurations and primary discharging situations

DEFINITION I. A *V_5-neighborhood* is a fully specified configuration consisting of a "*central*" V_5 and its five neighbor vertices (along with the corresponding edges and triangles). A *plugged V_5-neighborhood* is a configuration, say C, which consists of a V_5-neighborhood, say C^*, and (possibly zero) additional 5-vertices each of which is adjacent to some vertex of C^* (and of the corresponding edges and triangles). The V_5-neighborhood C^* is called the *core of* C and the additional 5-vertices are called the *plug* V_5's *of* C.

A *k-wheel* is a configuration which consists of a "pivot" vertex, say Q, of degree $k \geq 7$ and its neighbor vertices (and the corresponding edges and triangles) so that for each vertex V adjacent to Q the degree specification is either $\deg(V) = 5$ or $\deg_{\min}(V) = 6$ and $\deg_{\max}(V) = \infty$.

A *cluster* is a configuration, say C^*, (see Figure 8a, 8b) which is obtained by merging from a k-wheel, say C_0, and V_5-neighborhoods $C_1^*, \ldots, C_\nu^*$, (ν may be zero) so that each 5-vertex of C_0 is identified to the central V_5 of precisely one of $C_1^*, \ldots, C_\nu^*$. The (images of the) V_5's of C_0 are called the *central* V_5's of C^*. A *plugged cluster* is a configuration, say C (see Figure 8c), which consists of a cluster, say C^*, and (possibly zero) additional 5-vertices each of which is adjacent to some *fully-specified* vertex of C^* (and of the corresponding edges and triangles). The cluster C^* is called the *core* of C; the additional 5-vertices are called the *plug-V_5*'s *of* C; the pivot of C_0 is called the *pivot of* C and of C^*; and the central V_5's of C^* are called *central V_5's of* C.

Note. If a plugged V_5-neighborhood, say C_1, is a subconfiguration of a plugged cluster, say C, then each plug-V_5 of C which belongs to C_1 is also a plug-V_5 of C_1; but some plug-V_5 of C_1 may belong to the core of C and thus not be a plug-V_5 of C.

We shall need the complicated but important concept of modification for plugged clusters and for certain configurations (specializations of) which can

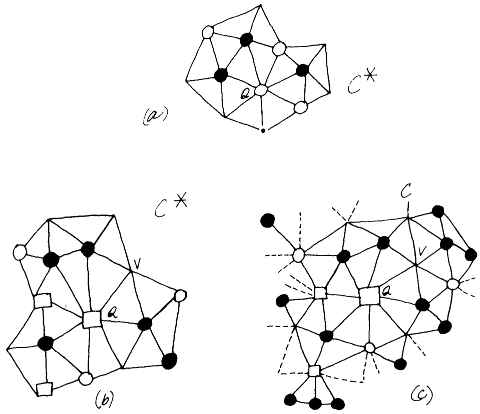


FIGURE 8

be embedded into plugged clusters. We restrict the class of configurations for which we shall define modifications as follows:

DEFINITION J. Let C be a configuration. By a *plug-specification on C* we mean a two-valued function p (with domain the 5-vertices of C) which specifies each V_5 of C to be *plug* or *nonplug*. A configuration C together with a plug specification p on C is called a *plugged configuration* if the following conditions are fulfilled:

(P.1) If V is a vertex of C then either $\deg(V)$ is fully specified or $\deg_{\min}(V) \geq 6$ and $\deg_{\max}(V) = \infty$.

(P.2) Removing all plug-V_5's (and the corresponding edges and triangles) from C yields a subconfiguration, say C^*, of C (which may be empty or may consist of a single vertex). C^* is called the *core* of (C, p) or simply the core of C.

(P.3) Every edge of C^* is incident to at least one triangle of C^*.

(P.4) There exists a plugged cluster P and an embedding $f\colon |C| \to |P|$ which respects the leg specifications of some specialization $C^\vee$ of C and of P such that f maps the core of C into the core of P.

(P.5) If V is a plug-V_5 of C then there exist P, f, and C as in condition (P.4) such that $f(V)$ is a plug-V_5 of P. (For different plug-V_5's of C the corresponding $P, f, C^\vee$ may be different.)

(P.6) If V is a vertex of C with $\deg_{\min}(V) = 6$ and $\deg_{\max}(V) = \infty$ then *every* embedding $f\colon |C| \to |P|$ as described in (P.3) maps V to a vertex adjacent to the pivot of P.

If C contains a vertex, say S, such that *every* embedding $f\colon |C| \to |P|$ as described in (P.3) maps S to the pivot of P then S is called the *special vertex* of C.

THEOREM 2. *If C^* is a cluster with pivot Q then the following hold.*

(A) *If V is a vertex of C^* which is different from Q and from the central V_5's of C^* then one of the following cases applies (see Figure 9).*

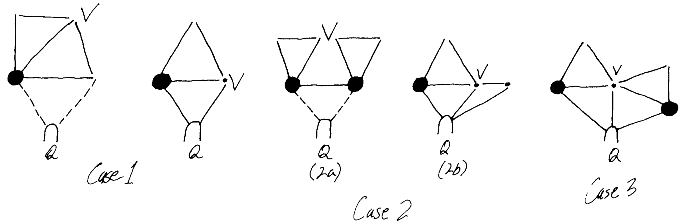


FIGURE 9

Case 1. *V has precisely three neighbors in C^*; these are consecutive and the middle one is a central V_5.*

Case 2. *V has precisely four neighbors in C^*; these are consecutive and the two middle ones are either* (*Case* 2a) *central V_5's or* (*Case* 2b) *a central V_5 and Q.*

Case 3. *V has precisely five neighbors in C^*; these are consecutive and the three middle ones are a central V_5, Q, and another central V_5 in that order.*

(B) *Let C be a plugged cluster with core C^*. Let V_5 be a plug-vertex of C. Then V_5 has at most three neighbors in C^* and these are consecutive around V_5.*

COROLLARY. *Let $f\colon |C| \to |C^\vee|$ be an immersion of a configuration C into a plugged cluster $C^\vee$ which respects the leg specifications of C and $C^\vee$. Let V_5 be a 5-vertex of C which has three nonconsecutive non-5-neighbors in C. Then $f(V_5)$ is a central V_5 of $C^\vee$. Further if V is a non-5-vertex of C then $f(V)$ is adjacent to the pivot $C^\vee$ or to a central V_5 of $C^\vee$.*

DEFINITION K. The *primary preliminary discharging situations of Classes* (A), (B), (C), (D), (E), (F), and (G) are the configurations drawn in Table 3 and are called *primary integral discharging situations.* The primary preliminary discharging situations of *Classes* (H), (I), (J), (K), and (L) are the configurations drawn in Table 4 and are called the *primary partial discharging situations.* In these configurations we define certain pairs (V_5, V) of vertices (V a major vertex) to be (*endpoints of*) *discharging tracks* as marked by arrows in the drawings in Tables 3 and 4. The 5-vertex of the track is called the *discharging vertex*; the major vertex is called the *receiving vertex.* In case the vertices of a track are joined by an edge we call that edge a *discharging edge.* We distinguish between *integral discharging tracks,* marked ↓ and partial discharging tracks. The latter include 15-*discharging tracks,* marked , 30-*discharging tracks,* marked , 45-*discharging tracks,* marked .

By a *partial discharging track system* for V we mean the set of those partial discharging tracks whose receiving vertex is V together with the *discharging designations* 15, 30, 45, respectively, for each member. In particular, (30, 30)-, (30, 15, 15)-, (30, 30, 30, 30)-, (45, 45, 30)-*track systems* occur in Table 4.

In each of the preliminary discharging situations a vertex is distinguished to be the *pivot* (the bottommost major vertex in the drawing in Tables 3 and 4) as follows. In an integral discharging situation it is the not-fully-specified receiving

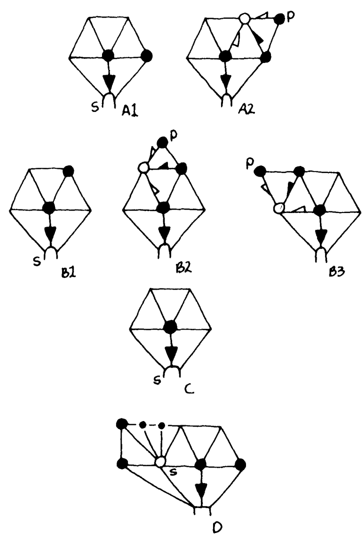


TABLE 3. Part 1

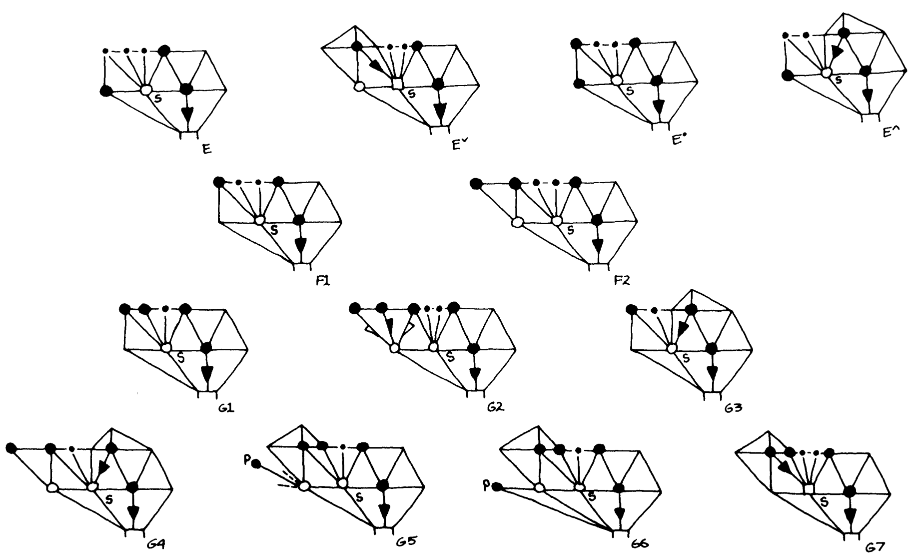


TABLE 3. Part 2

vertex. In a partial discharging situation of class other than (H) or (I), the pivot is the (unique) receiving vertex which is a bad vertex (see Definition G). In a situation of Class (H) or (I) it is the receiving vertex which is adjacent to one or two bad vertices. The discharging V_5's adjacent to the pivot are called *main* V_5's and the edges from the main V_5's to the pivot are called the *main discharging edges* (of the situation). The *essential part of a discharging situation C* is the subconfiguration of C obtained by removing all hanging V_5's and all those vertices

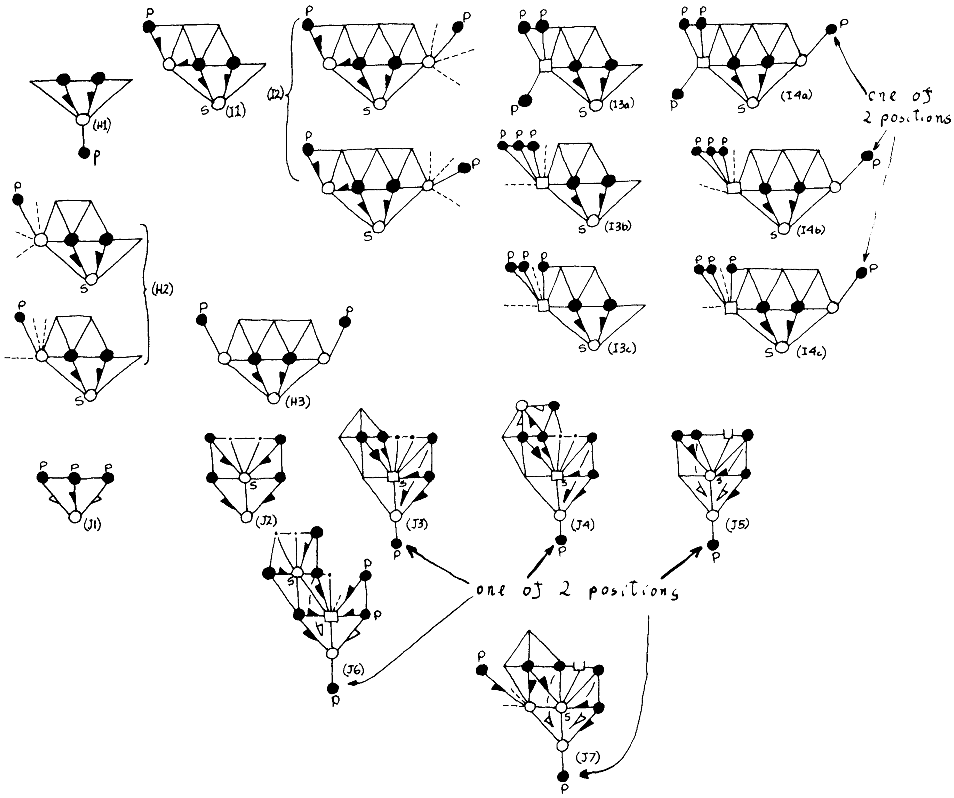


TABLE 4. Part 1

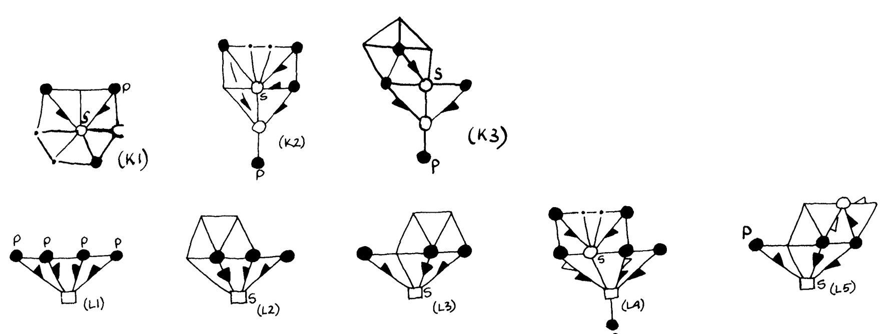


TABLE 4. Part 2

except the pivot which are not fully specified. The essential part of a specialization C' of a discharging situation C is defined to be the subconfiguration of C' which is a specialization of the essential part of C. *Plug-specifications* are indicated in Tables 3 and 4, by letters p written at the plug-V_5's. *Special vertices* are marked S.

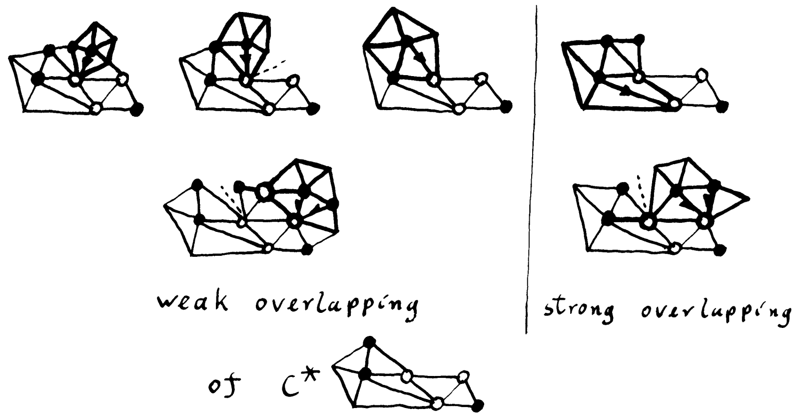


FIGURE 10

THEOREM 3. *The primary preliminary discharging situations together with the plug specifications indicated in Tables* 3 *and* 4 *are plugged configurations. Special vertices of these plugged configurations are precisely those vertices marked S in Tables* 3 *and* 4. *Moreover, the plug-specifications are maximal* (*i.e., any further p-specifications would violate* (P.5) *in Definition* J).

10. The modification procedure

In this section we shall define the *procedure for modifying a plugged configuration by preliminary discharging situations.* This procedure is used for recursively defining preliminary discharging situations in Section 11, Definition P, as modifications of primary discharging situations (defined above) by primary secondary, etc., situations.

DEFINITION L. Let C be a configuration which is obtained by merging from a plugged configuration C' with core C^* and specializations of preliminary discharging situations $C^{(1)}, \ldots, C^{(n)}$ so that the pivots $P^{(i)}$ of the $C^{(i)}$ are identified to vertices of C^*. Then for any i, $1 \leq i \leq n$, we say $C^{(i)}$ *weakly overlaps* C^* if one or two cases applies (see Figure 10).

(1) $C^{(i)}$ is (a specialization of) an integral discharging situation and no integral discharging edge of $C^{(i)}$ is identified to an interior edge of C^*.

(2) $C^{(i)}$ is (a specialization of) a partial discharging situation and no vertex of the essential part of $C^{(i)}$ which is adjacent to $P^{(i)}$ is identified to any vertex of C^*.

Otherwise $C^{(i)}$ is said to *strongly overlap* C^*.

In order to formally define the procedure of modification we shall first introduce some notation to be used in the definitions which follow. Let C' be a plugged configuration (see Figure 11) with core C^*. If C' has a special vertex denote this by Q. Let $V^{(1)}, \ldots, V^{(a)}$ (a may be zero) be those fully specified boundary vertices of C^* which are not 5-vertices and are different from Q. Let

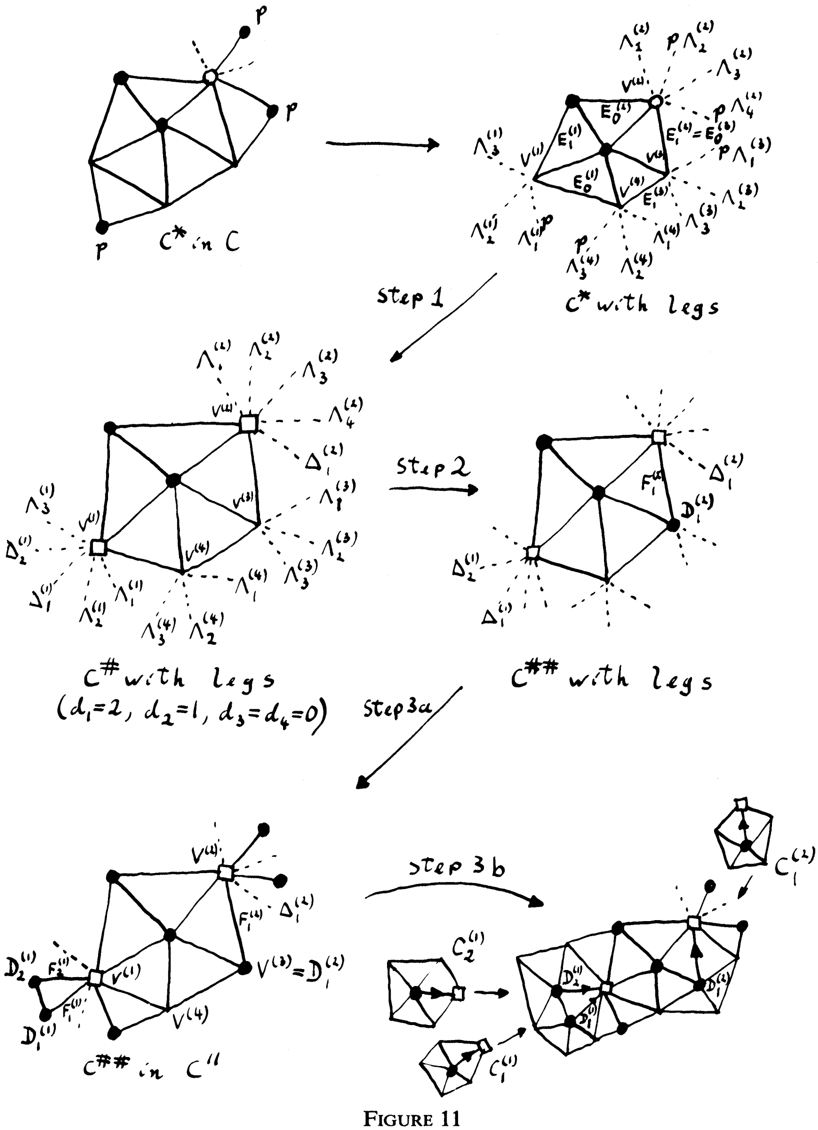


FIGURE 11

$S^{(1)}, \ldots, S^{(a)}$ be the outer sectors of C^* at $V^{(1)}, \ldots, V^{(a)}$ respectively, chosen in such a manner that all plug-neighbors of $V^{(\alpha)}$, $1 \leq \alpha \leq a$, in C' lie at the ends of legs in $S^{(\alpha)}$. (If $V^{(\alpha)}$ is an articulation point of C^* and has no plug-neighbors in C' then either of the outer sectors at $V^{(\alpha)}$ can arbitrarily be chosen to be $S^{(\alpha)}$.) If C^* is not a single vertex, let $E_0^{(\alpha)}$, $E_1^{(\alpha)}$ be boundary edges of $S^{(\alpha)}$ chosen so that in the enumeration of the $V^{(\alpha)}$, $E_0^{(1)}$, $E_1^{(1)}$, $E_0^{(2)}, \ldots, E_1^{(a)}$ lie in that cyclic order in the boundary of C^* where $E_1^{(\gamma)}$ may be the same as $F_0^{(\gamma+1)}$ (upper indices modulo a).

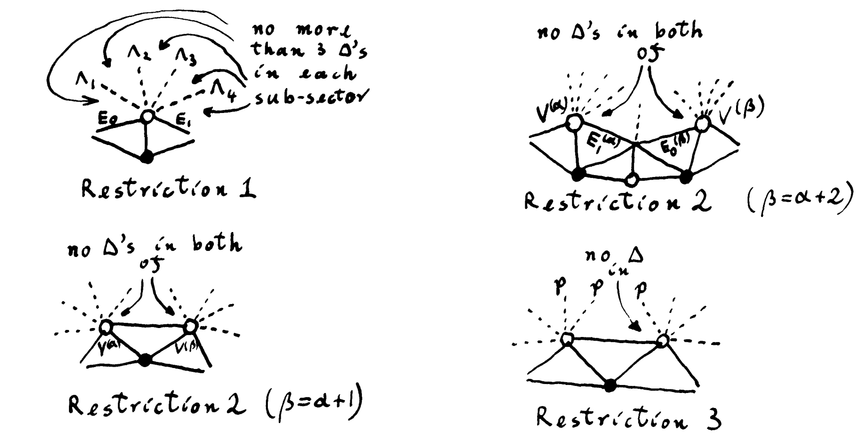


FIGURE 12

We denote the legs of C^* in $S^{(\alpha)}$ by $\Lambda_1^{(\alpha)}, \ldots, \Lambda_{l_\alpha}^{(\alpha)}$ in the order in which they occur between $E_0^{(\alpha)}$ and $E_1^{(\alpha)}$. Those $\Lambda_i^{(\alpha)}$ incident to plug-neighbors of $V^{(\alpha)}$ in C' are called *plug-legs*; the others are called *nonplug legs.*

DEFINITION M. In the above notation, a configuration C *is obtained from* C' *by a premodification, or simply is a premodification of* C' if C can be obtained from C' by the three-step *premodification procedure* consisting of *degree raising*, *degree lowering from* 6 *to* 5, and *attaching preliminary discharging situations* described below. (See Figure 11 and, for more complicated examples, Figure 14 and Tables 5 and 6. For the full notation for figures past 13 one must look past this definition.)

Step 1 (degree raising). The degree raising is performed by insertion of additional legs into sectors $S^{(\alpha)}$ at vertices $V^{(\alpha)}$ whose degrees we wish to raise. Formally, into each sector $S^{(\alpha)}$, $1 \leq \alpha \leq a$, we insert additional legs $\Delta_1^{(\alpha)}, \ldots, \Delta_{d_\alpha}^{(\alpha)}$ (where d_α may be zero) changing the number of legs in $S^{(\alpha)}$ from l_α to $l_\alpha + d_\alpha$ (and increasing the degree of $V^{(\alpha)}$ by d_α). C^* is thus changed into a new configuration $C^\#$ with $|C^\#| = |C^*|$ which has larger leg numbers specified for some of the $S^{(\alpha)}$ (and correspondingly larger degrees for some of the $V^{(\alpha)}$). The new legs Δ must satisfy the following restrictions (see Figure 12).

Restriction 1. No more than three $\Delta^{(\alpha)}$ lie between any $\Lambda_i^{(\alpha)}, \Lambda_{i+1}^{(\alpha)}$ or between $E_0^{(\alpha)}$ and $\Lambda_1^{(\alpha)}$ or between $\Lambda_{l_\alpha}^{(\alpha)}$ and $E_1^{(\alpha)}$.

Restriction 2. If in the boundary of C^* there are no vertices other than 1-leggers between $V^{(\alpha)}$ and $V^{(\beta)}$ (in the cyclic order of increasing upper indices) then no $\Delta^{(\alpha)}$ lies between $\Lambda_{l_\alpha}^{(\alpha)}$ and $E_1^{(\alpha)}$ *or* no $\Delta^{(\beta)}$ lies between $E_0^{(\beta)}$ and $\Lambda_1^{(\beta)}$.

Restriction 3. If $E_1^{(\alpha)} = E_0^{(\alpha+1)}$ and if both $\Lambda_{l_\alpha-1}^{(\alpha)}$ and $\Lambda_{l_\alpha}^{(\alpha)}$ are plug (both $\Lambda_1^{(\alpha+1)}$ and $\Lambda_2^{(\alpha+1)}$ are plug) then no $\Delta^{(\alpha+1)}$ lies between $E_0^{(\alpha+1)}$ and $\Lambda_1^{(\alpha+1)}$ (no $\Delta^{(\alpha)}$ lies between $\Lambda_{l_\alpha}^{(\alpha)}$ and $E_1^{(\alpha)}$).

Step 2 (degree lowering from 6 to 5). From $C^{\#}$ we derive a new configuration $C^{\#\#}$ as follows. Suppose that for some α, i, we have:

(a) $\Delta_i^{(\alpha)}$ is adjacent to $E_1^{(\alpha)}$ (to $E_0^{(\alpha)}$) (see $\Delta_1^{(2)}$ in Figure 11);

(b) $E_1^{(\alpha)} = E_0^{(\alpha+1)}$ ($E_1^{(\alpha+1)} = E_0^{(\alpha)}$);

(c) $V^{(\alpha+1)}$ is a V_6 ($V^{(\alpha-1)}$ is a V_6) in $C^{\#}$;

(d) $V^{(\alpha+1)}(V^{(\alpha-1)})$ is not adjacent to Q (and thus, by Theorem 2, is at least a 2-legger of C^*);

(e) $\Lambda_2^{(\alpha+1)}, \ldots, \Lambda_{l_{\alpha+1}}^{(\alpha+1)}$ are ($\Lambda_{l_{\alpha-1}-1}^{(\alpha-1)}, \ldots, \Lambda_1^{(\alpha-1)}$ are) nonplug-legs of C^*. Then we *may* (but need not) remove the leg $\Lambda_1^{(\alpha+1)}$ (the leg $\Lambda_{l_{\alpha-1}}^{(\alpha-1)}$) so as to change the degree specification of $V^{(\alpha+1)}$ (of $V^{(\alpha-1)}$) from 6 to 5. If this operation is performed we denote the 5-vertex $V^{(\alpha+1)}$ (5-vertex $V^{(\alpha-1)}$) by $D_i^{(\alpha)}$ and the edge $E_1^{(\alpha)}$ (the edge $E_0^{(\alpha)}$) by $F_i^{(\alpha)}$.

Step 3 (attaching preliminary discharging situations). First (Step 3a below) we attach V_5's to $C^{\#\#}$ which are (in the obvious way) in 1-1 correspondence with the plug-V_5's of C'; further we attach V_5's, denoted by $D_i^{(\alpha)}$ (in addition to those introduced in Step 2) which are eventually to be identified to main V_5's of integral discharging situations. Then (Step 3b) we attach the preliminary discharging situations; thus "compensating" most (and if the premodification is a *modification, all*) of the degree raisings.

Step 3a. We construct a configuration C'' (which contains $C^{\#\#}$ as a subconfiguration) by identifying certain legs of $C^{\#\#}$ to edges incident to 5-vertices outside of $C^{\#\#}$ as follows.

(i) If $\Lambda_j^{(\alpha)}$ is a plug-leg of C^* and is also a leg of $C^{\#\#}$ (i.e., has not been removed in the degree lowering) then $\Lambda_j^{(\alpha)}$ is identified to an edge incident to a V_5.

(ii) If $\Delta_i^{(\alpha)}$ lies between $E_0^{(\alpha)}$ and $\Lambda_1^{(\alpha)}$ or between $\Lambda_{l_\alpha}^{(\alpha)}$ and $E_1^{(\alpha)}$ then $\Delta_i^{(\alpha)}$ is identified to an edge, denoted by $F_i^{(\alpha)}$, which leads to a 5-vertex, denoted by $D_i^{(\alpha)}$, provided that $F_i^{(\alpha)}$ has not been defined to be $E_0^{(\alpha)}$ or $E_1^{(\alpha)}$ in Step 2. If $F_i^{(\alpha)}$ has been so defined then $\Delta_i^{(\alpha)}$ is not identified to any edge.

(iii) If $\Delta_i^{(\alpha)}$ lies between legs $\Lambda_j^{(\alpha)}$ and $\Lambda_{j+1}^{(\alpha)}$ then $\Delta_i^{(\alpha)}$ *may* (but need not) be identified to an edge, denoted $F_i^{(\alpha)}$ leading to a 5-vertex $D_i^{(\alpha)}$.

(iv) Additional edges and triangles are added where required to obtain a configuration (without introducing new vertices). Note that by Restriction 2, $D_i^{(\alpha)} \neq D_j^{(\beta)}$ if $\Delta_i^{(\alpha)} \neq \Delta_j^{(\beta)}$.

Step 3b. We obtain C from C'' and preliminary discharging situations $C_i^{(\alpha_k)}$ ($1 \leq k \leq b \leq a$, where $\{\alpha_1, \ldots, \alpha_b\}$ is a subset of $\{1, \ldots, a\}$; $i = 1, \ldots, d_{\alpha_k}$) by merging in such a way that the following conditions are fulfilled for each k, $1 \leq k \leq b$. Denote the immersion $|C_i^{(\alpha_k)}| \to |C|$ by $f_i^{(\alpha_k)}$ and $|C''| \to |C|$ by f''.

(0) $f'' \mid |C^{\#\#}|$ is the identity map on $|C^{\#\#}|$.

(1) The pivot of $C_i^{(\alpha_k)}$ is identified to a vertex $V^{(\alpha_k)}$ of C''.

(2a) If $C_i^{(\alpha_k)}$ is an integral discharging situation then the main discharging edge of $C_i^{(\alpha_k)}$ is identified with the edge $F_i^{(\alpha_k)}$ of C''.

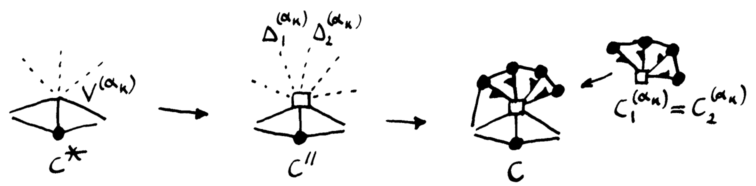


FIGURE 13a

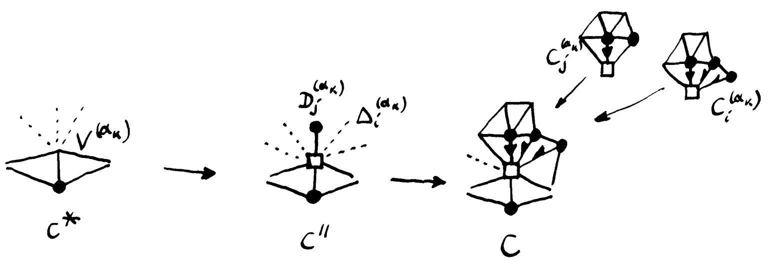


FIGURE 13b

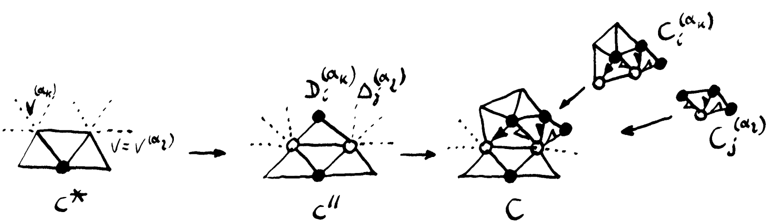


FIGURE 13c

(2b) If $C_i^{(\alpha_k)}$ is a partial discharging situation then some interior edge of $C_i^{(\alpha_k)}$ is identified with the leg $\Delta_i^{(\alpha_k)}$ at C'' (which, in this case, was *not* replaced by an edge $F_i^{(\alpha_k)}$ in Step 3a).

(2c) If $C_i^{(\alpha_k)}$ is one of $L1$, $L4$, (and $V^{(\alpha_k)}$ is an 8-vertex in C'') then $d_{\alpha_k} = 2$ and $C_1^{(\alpha_k)}$ and $C_2^{(\alpha_k)}$ are degree equivalent and are identified with one another (see Figure 13a).

(3) $C_i^{(\alpha_k)}$ weakly[5] overlaps $C^{\#\#}$.

(4a) If $C_i^{(\alpha_k)}$ is a partial discharging situation (e.g., $L2$, see Figure 13b) which contains as a subconfiguration some specialization $C^{\vee}$ of an integral discharging situation with the same pivot, then $d_{\alpha_k} = 2$ and there is some integral discharging situation $C_j^{(\alpha_k)}$ ($j = 1$ or 2, $j \neq 1$) so that $f_j^{(\alpha_k)}(|C_j^{(\alpha_k)}|) = f_j^{(\alpha_k)}(|C^{\vee}|)$.

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[5] The main reason for not allowing strong overlappings is to ensure that a modification of a geographically good configuration contains a geographically good subconfiguration. Note that for partial discharging situations no vertex of C^i which is adjacent to the pivot is identified to any vertex of C''. Here we could allow stronger overlapping but in the case of $H1$ and $J1$ this would yield many additional discharging situations.

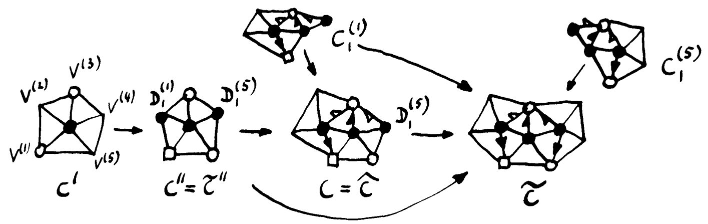


FIGURE 14a

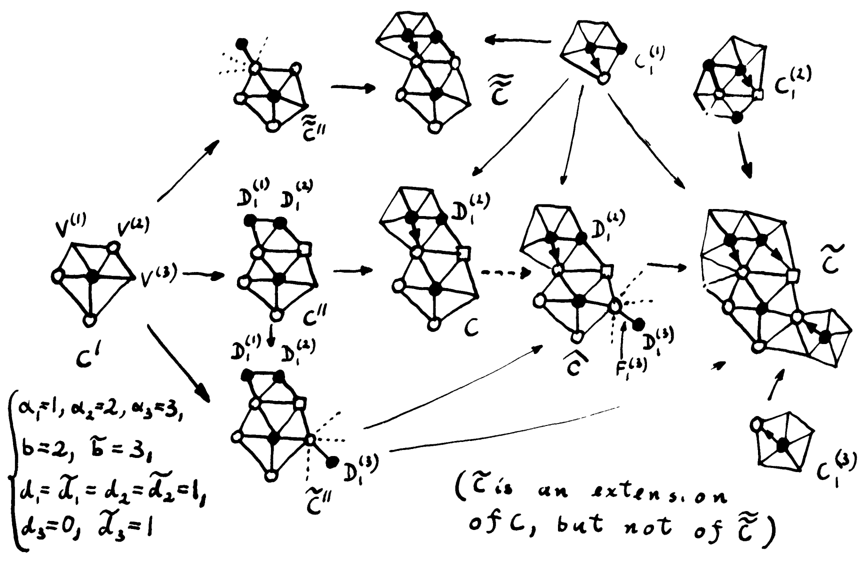


FIGURE 14b

(4b) If $C_i^{(\alpha_k)}$ is a preliminary discharging situation (e.g., B3, see Figure 13c) which contains a specialization $C^\vee$ of some other preliminary discharging situation so that $f_i^{(\alpha_k)}(|C^\vee|)$ weakly overlaps $C^{\#\#}$ with pivot identified to a vertex $V \neq V^{(\alpha_k)}$ of $C^{\#\#}$ then $V = V^{(\alpha_l)}$ (for some l, $1 \leq l \leq b$; $l \neq k$) and there is some $C_j^{(\alpha_l)}$ $(1 \leq j \leq d_{\alpha_l})$ so that

$$f_j^{(\alpha_l)}(|C_i^{(\alpha_l)}|) = f_i^{(\alpha_k)}(|C^\vee|).$$

(5) ("noncompensated degree raising" is only permitted under certain "crowded conditions"). If $\alpha \neq \alpha_1, \ldots, \alpha_b$, $(1 \leq \alpha \leq a)$ and $d_\alpha > 0$ then we demand that the following conditions are satisfied.

(5.1) No fully specified vertex of any $C^{(\alpha_k)}$ is identified with $V^{(\alpha)}$.

(5.2) For each $j = 1, \ldots, d_\alpha$, at least one of the following cases applies.

(i) A 5-vertex $D_j^{(\alpha)}$ has been introduced in Step 2 and is identified to some vertex of some $C_i^{(\alpha_k)}$ (see $D_1^{(5)}$ in Figure 14b).

(ii) $\Delta_j^{(\alpha)}$ is identified to some edge of C which joins $V^{(\alpha)}$ to some vertex of some $f_i^{(\alpha_k)}(|C_i^{(\alpha_k)}|)$; (if $\Delta_j^{(\alpha)}$ has been identified with an edge $F_j^{(\alpha)}$ in Step 2 then this is again (i) above; for examples of other cases see $\Delta_1^{(2)}$ in Figure 14c and $\Delta_1^{(1)}$ in Figure 14d).

(iii) $\Delta_j^{(\alpha)}$ has been identified with an edge $F_j^{(\alpha)}$ in Step 2 so that $F_j^{(\alpha)}$ is an interior edge of C, and $D_j^{(\alpha)}$ is a 0- or 1-legger vertex of C (see $D_1^{(2)}$ in Figure 14e).

(iv) $\Delta_j^{(\alpha)}$ is a leg of C which is (the only leg) in a 1-legger outer sector, say (G_1, G_2) of C at $V^{(\alpha)}$ so that at least one of the edges G_1, G_2 leads from $V^{(\alpha)}$ to a 1-legger vertex of C (see $\Delta_1^{(2)}$ in Figure 14f).

DEFINITION N. The number $\sum_{k=1}^{b} d_{\alpha_k}$ of attached discharging situations is called the *order* of the premodification C. We also call C a premodification of C' *by the* $C_i^{(\alpha_k)}$ $(1 \leq k \leq b,\ 1 \leq i \leq d_{\alpha_k})$. If $b = 1$, (i.e., if all pivots of the $C_i^{(\alpha_k)}$ are identified to the same vertex $V^{(\alpha_1)}$ of C') then we call C a premodification at $V^{(\alpha_1)}$.

A premodification C of C' is called a *modification of* C' if (in the above notation) $b = a$, i.e., if the degree-raisings in Step 1 are in 1-1 correspondence with the "compensating" attachments in Step 3. (The premodifications C of Figures 14a, ..., f are not modifications of C').

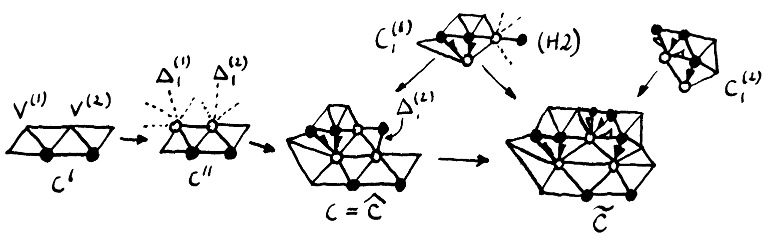


FIGURE 14c

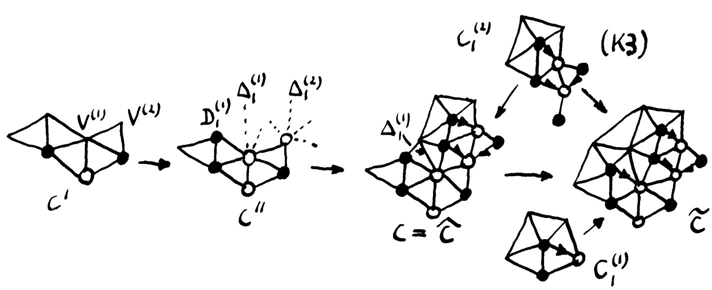
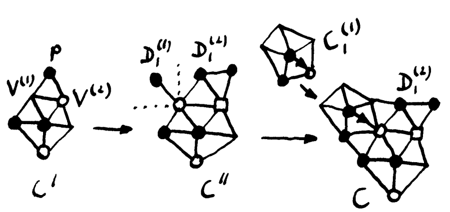


FIGURE 14d



(the pre-modific. C of C′ cannot be extended to a modification of C′ since no integral discharging situat. is suitable for $C_1^{(2)}$)

FIGURE 14e

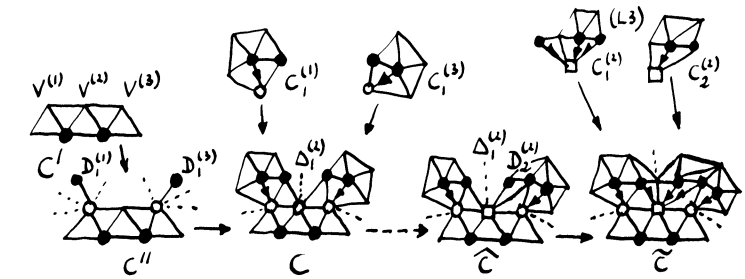


FIGURE 14f

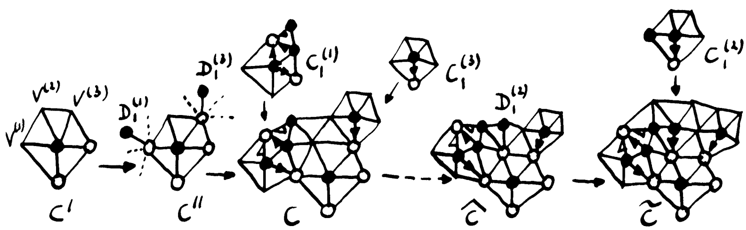


FIGURE 14g

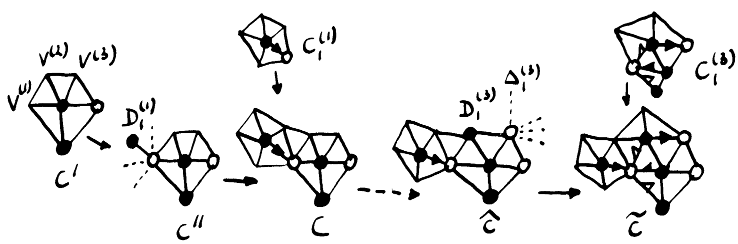


FIGURE 14h

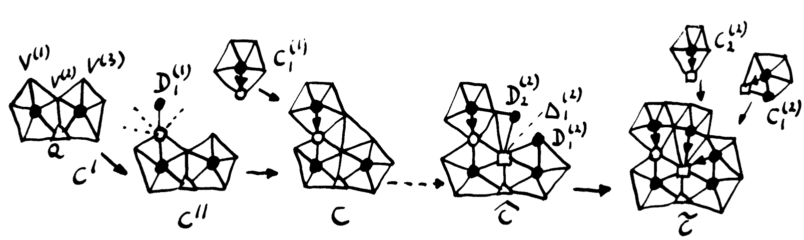


FIGURE 14i

DEFINITION O. A premodification $C^{\sim}$ of C' (see Figure 14a, ..., i) is called an *extension of a premodification* C of C' if $C^{\sim}$ can be obtained from C' by additional degree-raisings, degree-lowerings, and/or attachments beyond those leading to C. We use the obvious notation $b^{\sim}$ ($\geq b$), $d_{\alpha}^{\sim}$ ($\geq d_{\alpha}$), $C^{\sim\prime\prime}$, etc., for the description of the procedure for obtaining $C^{\sim}$ from C'. We denote by $C^{\wedge}$ the configuration which is obtained from $C^{\sim\prime\prime}$ and the old $C_i^{(\alpha_k)}$ ($k = 1, \ldots, b$)

by merging compatible with the merging of $C^{\sim\prime\prime}$ and the $C_i^{(\alpha_{k^\sim})}$ ($k^\sim = 1, \ldots, b^\sim$) to $C^\sim$. We impose the following restriction (which informally says that no additional degree raisings are permitted under crowded conditions, i.e., under conditions that $V^{(\alpha)}$ is overlapped by some "old" $C^{(\alpha_k)}$ or in which "noncompensated degree raisings" had been permissible in constructing C according to Condition (5) in Step 3 of Definition M).

(i) No fully specified vertex of any old $C_i^{(\alpha_k)}$ is identified with $V^{(\alpha)}$.

(ii) If a 5-vertex $D_r^{(\alpha)}$ is introduced (in Step 2 or 3a of the premodification procedure leading to $C^\sim$) the $D_r^{(\alpha)}$ is not identified with any vertex of any "old" $C_i^{(\alpha_k)}$.

(iii) $C^\wedge$ can be derived from C by a sequence of $e = \sum_{\alpha=1}^{a} (d_\alpha^\sim - d_\alpha)$ operations as follows. Denote the sequence of configurations by $C^0 = C, \ldots, C^e = C^\wedge$. Then C^{f+1} is derived from C^f by one of two operations.

Operation 1. A new leg $\Delta_r^{(\alpha)}$ ($\alpha \in \{1, \ldots, a\} - \{\alpha_1, \ldots, \alpha_b\}$; $d_\alpha \leq r \leq d_\alpha^\sim$) is inserted into some outer sector of C^f at $V^{(\alpha)}$. Subsequently, $\Delta_r^{(\alpha)}$ may or may not be identified with an edge $F_r^{(\alpha)}$ which leads from $V^{(\alpha)}$ to a new 5-vertex $D_r^{(\alpha)}$. Also, if $\Delta_r^{(\alpha)}$ is not identified with an edge and is adjacent to an edge $E_0^{(\alpha)}$ or $E_1^{(\alpha)}$ which leads to a 6-vertex $V^{(\alpha+1)}$ or $V^{(\alpha-1)}$, not adjacent to Q and not identified with any vertex in any old $C^{(\alpha_k)}$, then the degree specification of this vertex may be lowered to 5 (by removing a leg). (In Figures 14b, 14f, and 14h, $C^\wedge$ is derived from C by Operation 1 alone.)

Operation 2. A boundary edge E of C^f which neither belongs to $C^{\#\#}$ nor to the image of any $C_i^{(\alpha_k)}$ and which lies opposite to $V^{(\alpha)}$ ($\alpha \in \{1, \ldots, a\} - \{\alpha_1, \ldots, \alpha_b\}$) in a triangle H of C^f is removed and the triangle H is replaced by a 1-legger outer sector the leg of which is $\Delta_r^{(\alpha)}$ ($d_\alpha \leq r \leq d_\alpha^\sim$), thus raising the degree specification of $V^{(\alpha)}$ by 1. Subsequently $\Delta_r^{(\alpha)}$ may or may not be identified to an edge $F_r^{(\alpha)}$. Also, if $\Delta_r^{(\alpha)}$ is adjacent to an edge $E_0^{(\alpha)}$ or $E_1^{(\alpha)}$ which leads to a 6-vertex not adjacent to Q then the degree specification of this vertex may be lowered to 5. (In Figure 14g, $C^\wedge$ is derived from C by one application of Operation 2. In Figure 14i, $C^\wedge$ is derived from C by application of two operations, either both Operation 2, introducing $D_2^{(2)}$ first and $\Delta_1^{(2)}$ second or one application of Operation 2 introducing $\Delta_1^{(2)}$ and one of Operation 1 introducing $D_2^{(2)}$.)

Theorems 4 and 5 are important consequences of the definitions above.

THEOREM 4 (see diagram). *Suppose that C is a modification of either a plugged cluster or a plugged V_5-neighborhood, C', with core C^* (notation as in Definition* M). *Suppose that C_0' is a plugged configuration with core C_0^*, that $C_0^{\vee\prime}$ is a specialization of C_0' and that $g_0'\colon |C_0'| \to |C'|$ is a combinatorial embedding which respects the leg-specifications of $C_0^{\vee\prime}$ and C' and which maps $|C_0^*|$ into $|C^*|$.*

Then there exist configurations C_{00}', C_{00}, C_0, $C_0^\vee$, and an immersion $g_0\colon |C_0| \to |C|$ so that the following hold:

(i) *C_{00}' is a plugged configuration and is obtained from C_0' by lowering the degree-specifications "six" to "five" at boundary vertices, say $V^1, \ldots, V^s$,*

(s *may be zero*) *of* C_0' *which are not adjacent to the special vertex of* C_0' (*by removing one leg from each such vertex*).

(ii) C_{00} *is a modification of* C_{00}'.

(iii) C_0 *is obtained from* C_{00} *by attaching integral discharging situations, say* $C^1, \ldots, C^s$, *so that for each* $r = 1, \ldots, s$, *the main* V_5 *of* C^r *with identified to* V^r *and the pivot of* C^r *is not identified with any fully specified vertex of* C_{00}.

(iv) $C_0^\vee$ *is a specialization of* C_0 *and* g_0 *respects the leg-specifications of* $C_0^\vee$ *and* C.

(v) $g_0 \mid |C_0^*| = g_0' \mid |C_0^*|$.

$$\begin{tikzcd}
C_0' \arrow[r, "\text{6 to 5}"] \arrow[ddd, "g_0'"'] & C_{00}' \arrow[rrrr, "\text{modification}"] \arrow[dr] & & & & C_{00} \arrow[rr, "\text{attach } C^r\text{s}"] \arrow[dddr, "g_{00}"] & & C_0 \arrow[ddd, "g_0"] \\
& & C_{00}^{\#} \arrow[r] & C_{00}^{\#\#} \arrow[r] & C_{00}'' \arrow[ur] \arrow[d, phantom, "g_{00}''"] & & & \\
& & C^{\#} \arrow[r, "\text{st 2}"] & C^{\#\#} \arrow[r, "\text{st 3a}"] & C'' \arrow[drrr, "\text{st 3b}"] & & & \\
C' \arrow[urr, "\text{st 1}"] \arrow[rrrrrrr, "\text{modification}"'] & & & & & & & C
\end{tikzcd}$$

THEOREM 6. *Let* C'' *be derived from* C' *by Steps* 1, 2, *and* 3a *of the pre-modification procedure* (*notation as in Definition* M). *Then there is a one-to-one correspondence between the* 5-*vertices of* C' *and those* 5-*vertices of* C'' *which are not* $D_i^{(\alpha)}$'*s so that the correspondence has the following properties.*

If a 5-*vertex* V_5'' *of* C'' *is joined by an edge* E'' *of* C'' *to some vertex* $V^{(\alpha)}$ *of* $C^{\#\#}$ *then the corresponding* 5-*vertex* V_5' *of* C' *is joined by an edge, say* E', *of* C' *to the same vertex* $V^{(\alpha)}$ *of* C^* (*recall that* $|C^*| = |C^{\#\#}|$); *moreover, if* V_5'' *belongs to* $C^{\#\#}$ *then* $V_5' = V_5''$ *and* $E' = E''$, *and if* V_5'' *does not belong to* $C^{\#\#}$ *then* E' *and* E'' *correspond to the same leg* $\Lambda_j^{(\alpha)}$.

11. The preliminary discharging

DEFINITION P. The *k-ary preliminary discharging situations* and the *corresponding discharging tracks*, etc., are defined by recursion on k as follows. By a *preliminary discharging situation* we shall mean a k-ary preliminary discharging situation for any $k = 1, 2, \ldots$.

($k = 1$) The primary preliminary discharging situations are defined in Definition K (Tables 3 and 4).

($k = n + 1$) The ($n + 1$)-ary preliminary discharging situations are defined to be those configurations C which have the following properties.

(i) C is a modification by primary, secondary, ..., n-ary preliminary discharging situations (where at least one n-ary situation is actually attached) of some plugged primary preliminary discharging situation C' (with plug-specifications as given in Tables 3 and 4).

(ii) C does not contain any geographically good configuration. The discharging tracks, etc., in C are defined to be induced by the modification $C' \to C$ (using the notation of Definition M) as follows.

(iii) If (V_5, V) is an integral discharging edge (if $(V_5^1, V), \ldots, (V_5^u, V)$, $u = 2, 3$, or 4, are the members of a partial discharging track system) in $C_i^{(\alpha)}$ then $(f_i^{(\alpha)}(V_5), f_i^{(\alpha)}(V))$ is an integral discharging edge (then $(f_i^{(\alpha)}(V_5^1), f(V_i^{(\alpha)}))$, $\ldots, (f(_i^{(\alpha)}V_5^u), f_i^{(\alpha)}(V))$ with the discharging designations of $(V_5^1, V), \ldots,$ (V_5^u, V) respectively, form a track system) in C. Also if (V_5', V) is an integral discharging edge (if $(V_5'^1, V), \ldots, (V_5'^u, V)$ are the members of a track system) in C' then $(f''(V_5''), f''(V))$ is an integral discharging edge (then $(f''(V_5''^1),$ $f''(V)), \ldots, (f''(V_5''^u), f''(V))$ form a track system) in C, where V_5'' is either identical to V_5' (if V_5' is not a plug vertex of C') or corresponds to V_5' in the sense of Theorem 6, etc.

(iv) We designate the *pivot* of C to be the f''-image of the pivot of C'.

(v) C does *not* contain any subconfiguration which is an m-ary preliminary discharging situation with $m \leq n$ and with same discharging tracks and pivot as defined for C in (iii) and (iv).

If C' is an integral discharging situation (a partial discharging situation) then C is also called an *integral discharging situation* (a *partial discharging situation*). A V_5 that discharges to the pivot of C is called a *main* V_5 of C. The *essential part of* C is defined as in Definition K.

DEFINITION Q. Let D be any triangulation or configuration. We define *integral discharging edges* and *partial discharging track systems* in D to be induced by immersions of preliminary discharging situations: Suppose $f\colon |C| \to |D|$ is an immersion which respects the leg specifications of C and D, where C is a specialization of a preliminary discharging situation. Let (V_5, V) be an integral discharging edge (let $(V_5', V), \ldots, (V_5^u, V)$ be the members of a track system in C). Then $(f(V_5), f(V))$ is an integral discharging edge $((f(V_5^1),$ $f(V)), \ldots, (f(V_5^u), f(V))$ with the discharging designations of $(V_5^1, V), \ldots,$ (V_5^u, V), respectively forming a track system) in C.

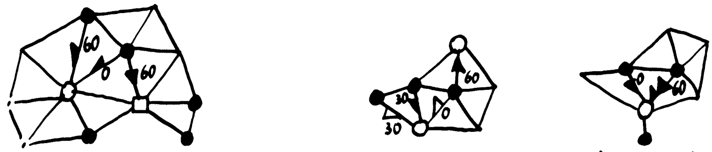
We define the *charge function* q_1^D to be derived from q_0^D by *integral discharging* as follows. If (V_5, V) is an integral discharging edge in D and if the degree of V is fully specified in D then charge 60 is transferred from V_5 to V. (This is done simultaneously for all integral discharging edges in D.)

We define the *charge function* q_2^D derived from q_1^D by *partial discharging* as follows. Suppose that $(V_5^1, V), \ldots, (V_5^u, V)$, $u = 2, 3$, or 4 are the members of a track system $\mathscr{T}$ in D. For each track (V_5^i, V), $i = 1, \ldots, u$, we define its (*partial*) *discharging value* $P^{\mathscr{T}}(V_5^i, V)$ in $\mathscr{T}$ by the following rules.

(PD1) If $u = 2$ and if *some* track in $\mathscr{T}$ is also an integral discharging edge in D then $P^{\mathscr{T}}(V_5^i, V) = 0$ for both $i = 1, 2$.

(PD2) If (PD1) does not apply and if either $q_1^D(V_5^i) > 0$ for all $i = 1, \ldots, u$ or $q_1^D(V_5^i) \leq 0$ for all $i = 1, \ldots, u$, then

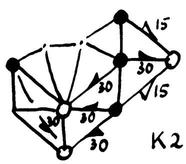
$$\begin{aligned} P^{\mathscr{T}}(V_5^i, V) &= 45 \text{ if } (V_5^i, V) \text{ is a 45-discharging track in } \mathscr{T} \\ &= 30 \text{ if } (V_5^i, V) \text{ is a 30-discharging track in } \mathscr{T} \\ &= 15 \text{ if } (V_5^i, V) \text{ is a 15-discharging track in } \mathscr{T}. \end{aligned}$$



(a) (merging of I4a and E) (b) J1 and A2 (c) H1 and A1 (PD1 applies)

The charges transferred are indicated by numbers

FIGURE 15



K2 and J1

FIGURE 16

(PD3) If (PD1) and (PD2) do not apply and if $q_1^D(V_5^i) > 0$ precisely for $i = i_1, \ldots, i_v$ where $1 \leq v \leq u$, then

$$\begin{aligned} P^{\mathscr{T}}(V_5^i, V) &= 60/v \text{ if } i \in \{i_1, \ldots, i_v\} \text{ and } \mathscr{T} \text{ is a } (30, 30)\text{- or } (30, 15, 15)\text{-system,} \\ &= 120/v \text{ if } i \in \{i_1, \ldots, i_v\} \text{ and } \mathscr{T} \text{ is a } (30, 30, 30, 30)\text{- or } (45, 45, 30)\text{-system,} \\ &= 0 \quad \text{if } i \in \{i_1, \ldots, i_v\}. \end{aligned}$$

We transfer the charge $P^D(V_5^i, V) = \mathrm{Max}_{\mathscr{T}} \{P^{\mathscr{T}} = (V_5^i, V)\}$ from V_5^i to V (where the maximum is taken over all track systems $\mathscr{T}$ in D to which the track (V_5^i, V) belongs. $P^D(V_5^i, V)$ is called the *partial discharging value* of (V_5^i, V) in D. This is done simultaneously for all tracks in D (for examples see Figures 15 and 16).

It should be noted that integral and partial dischargings each always increase the charge of a major vertex by a multiple of 60 units of charge. This may be restated as follows:

THEOREM 7. *If V_k is a major vertex of a triangulation or configuration D then $q_2^D(V_k)$ is an integral multiple of* 60.

DEFINITION R. Let D be any configuration or triangulation and let V_k be any fully specified vertex with degree $k \geq 6$. For $t = 0, 1, 2$, we define v_t^D to be the number of positive V_5-neighbors of V_k in (D, q_t^D); and $L_t^D = v_t^D(V_k) + (1/60)q_t^D(V_k)$ the *load of V_k in (D, q_t^D)*.

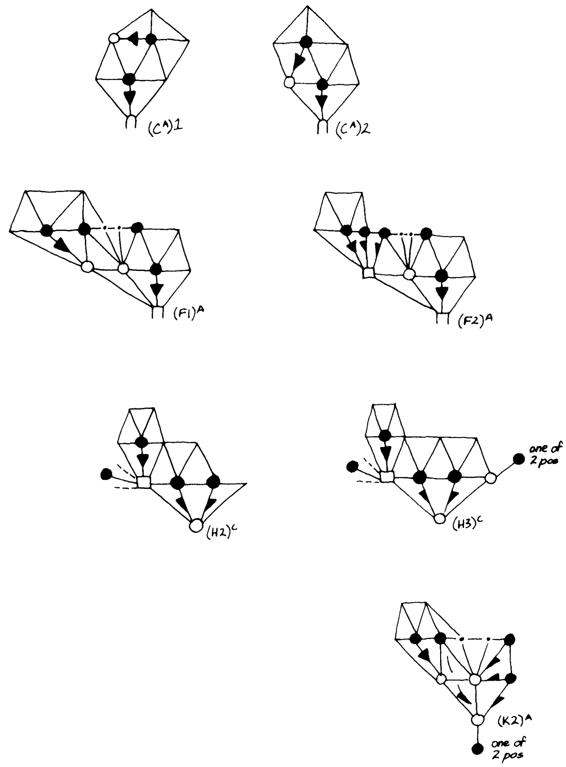


TABLE 5

12. Finiteness and classification of the preliminary discharging situations

DEFINITION S. We distinguish the following *classes of secondary and tertiary preliminary discharging situations* (see Tables 5 and 6). We have:

Class (C^A) (two members up to degree equivalence), *Class* (F^A) (two members), *Class* (H^c), (two members), *Class* (K^A) (one member). These classes arise from modification of members of Classes (C), (F), (H), (K), respectively, by $A1$ or C so that no new neighbors of the pivot are introduced.

We have *Class* (C_A) (four members), *Class* (C_B) (six members), *Class* (C_C) (one member), *Class* (C_{C^A}) (four members), *Class* (C_{AA}) (one member), *Class* (C_J) (twelve members), *Class* (C_K) (five members), *Class* (C_{K^A}) (two members), *Class* (C_L) (nine members). These classes arise when C is modified by a member of (A), (B), (C), (C^A), (J), (K), (K^A), (L), or by two situations $A1$, so that at least one new neighbor to the pivot is introduced (44 situations in Classes (C_X)). We have *Class* (F_A) (eight members), *Class* (F_B) (twelve members), *Class* (F_C) (two members), *Class* (F_{C^A}) (eight members), *Class* (F_{AA}) (two members), *Class* (F_J) (twelve members), *Class* (F_K) (five members), *Class* (F_{K^A}) (two

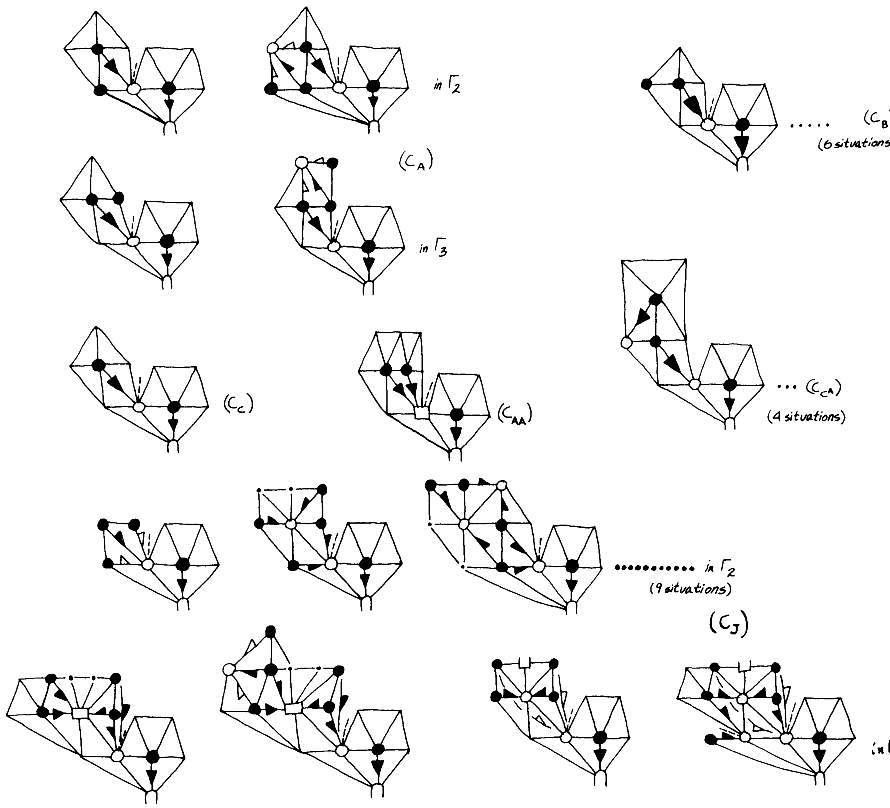


TABLE 6. The 207 secondary and tertiary preliminary discharging situations of Classes (X_Y). Part 1.

members), *Class* (F_L) (nine members). These classes arise when $F1$ or $F2$ is modified so that at least one new neighbor to the pivot is introduced (60 situations in Classes (F_X)). We have *Class* (H_A) (sixteen members), *Class* (H_B) (twelve members), *Class* (H_C) (two members), *Class* (H_{CA}) (eight members), *Class* (H_{AA}) (two members), *Class* (H_J) (twenty-four members), *Class* (H_K) (ten members), *Class* (H_{KA}) (four members), *Class* (H_L) (eighteen members). These classes arise when a member of Class (H) is modified so that after removal of not fully specified vertices and hanging V_5's an articulation vertex remains adjacent to the pivot (96 situations in Classes (H_X)).

The classes defined above and in Definition K contain 128 integral discharging situations, up to degree equivalence. For these we introduce another (coarser) classification as follows (see Tables 3, 5, and 6). *Class* Γ_k is defined to be the set of all those integral discharging situations which contain precisely k non-5-neighbors of the pivot. In detail, we have the following (by inspection).

$\Gamma_1 = (A) \cup (D)$ (three members).

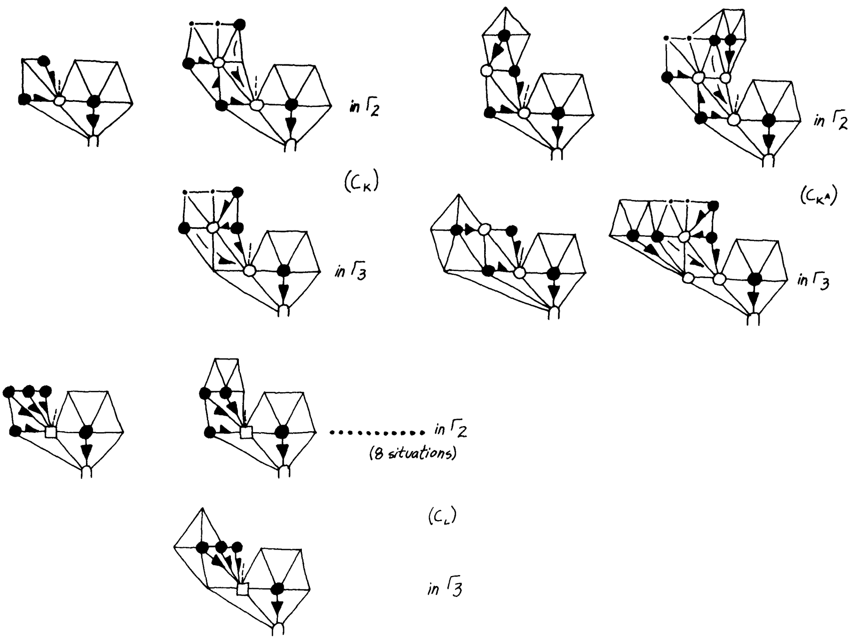


TABLE 6. Part 2

If we include in $(\bar{E})$ situations E, $E^{\cdot}$, $E^{\vee}$, and $E^{\wedge}$,

$\Gamma_2 = (B) \cup (C) \cup (\bar{E}) \cup (C^A) \cup$ {those 22 members of (C_A), (C_J), (C_K), (C_{K^A}), (C_L) in which a V_5 of the modifying situation is adjacent to the pivot of the modified situation} (32 members);

$\Gamma_3 = (F) \cup (G) \cup (F^A) \cup (C_B) \cup (C_C) \cup (C_{AA}) \cup (C_{C^A}) \cup$ {those ten members of (C_A), (C_J), (C_K), (C_{K^A}), (C_L) which are not in Γ_2} $\cup$ {24 members of Classes (F_X), $X = A, B, C, C^A, AA, J, K, K^A, L$} (57 members);

$\Gamma_4 =$ {those 36 members of Classes (F_X) which are not in Γ_3}.

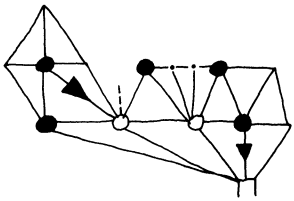
Moreover we define $\Gamma_{2.1}$ and $\Gamma_{2.2}$ to be the subclasses of Γ_2 consisting of those configurations C with $\phi(C) = 1$ or $\phi(C) = 2$, respectively. In detail we have

$$\Gamma_{2.2} = \{\text{Situations } C \text{ and } F\}.$$

$$\Gamma_{2.1} = \Gamma_2 - \Gamma_{2.2}.$$

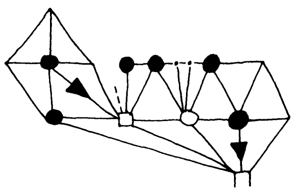
In the following sections we shall prove that the 253 preliminary discharging situations of Definitions K and S are all the preliminary discharging situations which exist (by Definition P).

Remark. One may use the ϕ-values of primary discharging situations to partially determine their behavior under modification. The situations of

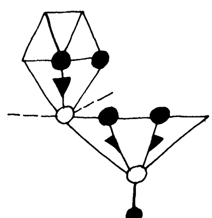


44 modifications of F1
analogous to those of C
(22 in Γ_3, 22 in Γ_4)

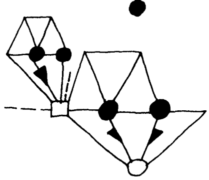
(F_x) $x = A, B, C, C^A, AA, J, K, K^A, L$



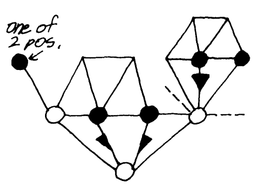
16 modifications of F2
by integral discharging situations
(2 in Γ_3, 14 in Γ_4)



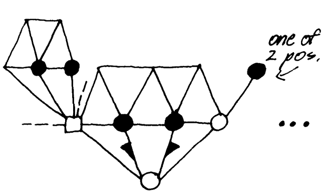
44 modifications of H1



4 modifications of H2 by A1, A2



44 modifications of H2



4 modifications of H3 by A1, A2

(H_x)

TABLE 6. Part 3

Classes (A), (B), (D), (G), (I), (J), $(K3)$, (L), $(E^\vee)$, $(E^\wedge)$, and $(E^\cdot)$ have ϕ-values of 1. Every modification of one of these situations will be seen to have a geographically good subconfiguration. (It appears that every nontrivial modification reduces the ϕ-value of the configuration which is modified; but we have not attempted to prove this.) The situations of Classes (C), (E), (F), (H), $(K1)$, $(K2)$ have ϕ-values of 2 and admit modifications without geographically good subconfigurations.

MAIN LEMMA 1. *Let C' be a primary preliminary discharging situation with plug-specifications as defined in Tables* 3 *and* 4. *Let $C^\sim$ be a modification of C'. Then either $C^\sim$ contains a geographically good subconfiguration in Size Class* $\langle 12, 18\rangle$ *or C is* (*degree equivalent to*) *one of the* 253 *situations described above* (*Definitions* K *and* S).

COROLLARY (FINITENESS THEOREM). *There are only finitely many* (253) *preliminary discharging situations. In particular, each of these lies in one of the* 44 *classes* $(A), \ldots, (H_L)$ *and thus in Size Class* $\langle 7, 9\rangle$ *while the maximum distance of a vertex from the pivot is* 4.

COROLLARY (DIAMETER THEOREM). *If C is a premodification of a plugged configuration of diameter D then the diameter of C is at most* $D + 8$.

Proof. We prove Main Lemma 1 via a sequence of lemmas and theorems some of which will be used again elsewhere. Using the recursive definition of preliminary discharging situations (Definition P) we proceed formally as follows. By a *classified preliminary discharging situation* we mean a preliminary discharging situation which belongs to one of the classes described above. A *restricted* (*pre*)*modification* is a (pre)modification in which all preliminary discharging situations used are classified preliminary discharging situations. In each of the theorems and lemmas in the following sections, the *restricted version of the theorem or lemma* will be the statement with "preliminary discharging situation" replaced by "classified preliminary discharging situation" and "(pre)modification" replaced by "restricted (pre)modification". We shall first prove the restricted versions of these lemmas and theorems. Finally, in Section 17, the restricted lemmas and theorems will be used to prove the restricted version of Main Lemma 1. But this will immediately imply (by Definition P) the unrestricted Main Lemma 1 (which says that there are no preliminary discharging situations other than the classified ones). This will immediately imply the unrestricted lemmas and theorems. Thus, in what follows we restrict ourselves to the consideration of classified preliminary discharging situations without special mention.

THEOREM 8. *If C is a preliminary discharging situation not in Classes* (H), (H^c), (H_X), *or* (I), *then no vertex of the essential part of C except the pivot is bad. If C is a partial discharging situation in* (H) (*in* (H^c), (H_X), *or* (I)) *then precisely two* (*precisely one*) *vertices of the essential part of C are bad and these are* 4-*leggers adjacent to the pivot.*

THEOREM 9. *If C is an integral discharging situation the following conditions are satisfied.*

(i) *The main V_5 of C has at most one neighbor of degree five in C.*

(ii) *If the main V_5 of C has a neighbor of degree five which is also a neighbor of the pivot then C lies in Γ_1.*

COROLLARY (SIZE-CLASS THEOREM). *Let C' be a plugged configuration of diameter D such that* $\deg^{C'}(V)$ *is specified,* $L_0^{C'}(V) > 0$ *and* $\deg^{C'}(V) \leq k$ *for each vertex V of C'. If C is a modification of C' then C is in Size Class $\langle D + 8, k + 10\rangle$.*

13. The extension lemma

In this section we show that the operation of extending a premodification cannot change a nonwipeout configuration into a wipeout configuration.

LEMMA 4 (EXTENSION LEMMA). *Let C be a premodification of a plugged configuration C'. Let $C^{\sim}$ be a modification of C' which is an extension of C. If C contains a geographically good subconfiguration so does $C^{\sim}$.*

COROLLARY. *Let C' be a configuration which contains a geographically good subconfiguration. If $C^{\sim}$ is a modification of C' then $C^{\sim}$ contains a geographically good subconfiguration.*

14. The load lemma

The following lemmas deal with the case in which (specializations of) several preliminary discharging situations discharge to the same pivot.

By detailed analysis, we may conclude that no more than two preliminary discharging situations can discharge to a vertex V_k of positive load L_0 without forcing the existence of a geographically good configuration. If the load of V_k is greater than one we obtain even stronger results. We express this in the following Load Lemma.

LEMMA 8. *Let C_0 consist of a vertex V_k of specified degree $k \geq 7$, some 5-neighbors of V_k and the corresponding edges and triangles. Let C be obtained from C_0 and δ (possibly zero) preliminary discharging situations $C_1, \ldots, C_\delta$ by merging so that the following conditions are satisfied.*

(i) *The pivots of the C_i are identified with V_k, the "pivot" of C.*

(ii) *If C_i is an integral discharging situation then its main discharging edge is not identified with any integral discharging edge, nor with any track of a 30-30-track system of any other C_j.*

(iii) *If C_i and C_j are partial discharging situations then neither of $|C_i|$, $|C_j|$ is identified with any subcomplex of the other.*

(iv) *The number of 5-neighbors of V_k in C is at least $k - 5$.*

Then C contains a geographically good subconfiguration in Size Class $\langle 8, k + 10\rangle$ or one of the following applies (see Figure 19).

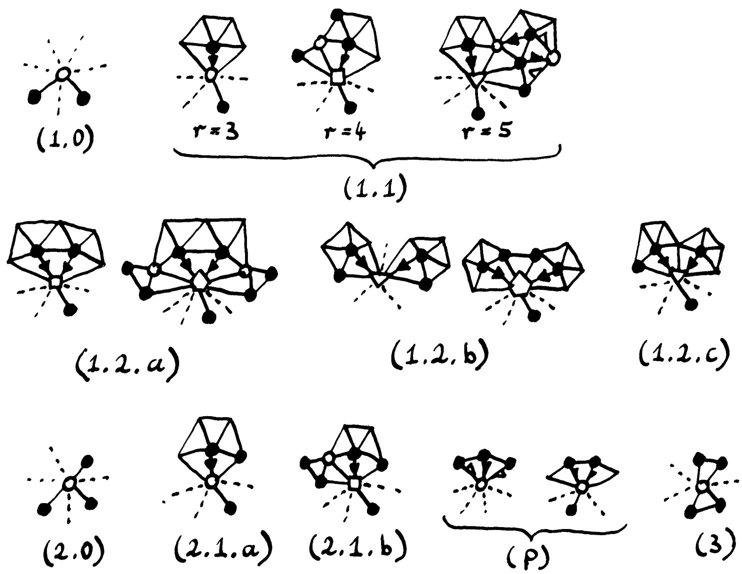


FIGURE 19

Case (1) $v = k - 5$, *i.e.*, $L_0^C(V_k) = 1$, *and one of three subcases applies.*

Subcase (1.0) $\delta = 0$.

Subcase (1.1) $\delta = 1$ *and* C_1 *is in* Γ_1 *or* Γ_2 *with* $r = 3$, 4, *or* 5 *neighbors to the pivot and* $k \geq r + 4$.

Subcase (1.2) $\delta = 2$, $k \geq 8$ *and one of the following specifications is fulfilled* (*see* a, b, c, *in Lemma* 5).
(1.2.a) C_1 *and* C_2 *are in* Γ_1 *with main* V_5*'s adjacent in* C.
(1.2.b) C_1 *and* C_2 *are in* Γ_1 *with no vertices but* V_k *in common in* C *and* $k \geq 9$.
(1.2.c) *One of* C_1, C_2 *belongs to* Γ_1, *the other to* $\Gamma_{2.2}$ *and their images in* C *have precisely one vertex besides* V_k *in common and* $k \geq 9$.

Case (2) $v = k - 4$, *i.e.*, $L_0^C(V_k) = 2$ *and one of three subcases applies.*

Subcase (2.0) $\delta = 0$

Subcase (2.1) $\delta = 1$ *and one of the following specifications is fulfilled.*
(2.1.a) C_1 *is in Class* (A).
(2.1.b) C_1 *is in Class* (D) *and* $k \geq 8$.

Case (3) $v = k - 3$, *i.e.*, $L_0^C(V_k) = 3$ *and* $\delta = 0$.

Case (P) *C is obtained from a partial discharging situation by attaching at most one hanging* V_5 *to its pivot and* $\delta = 1$.

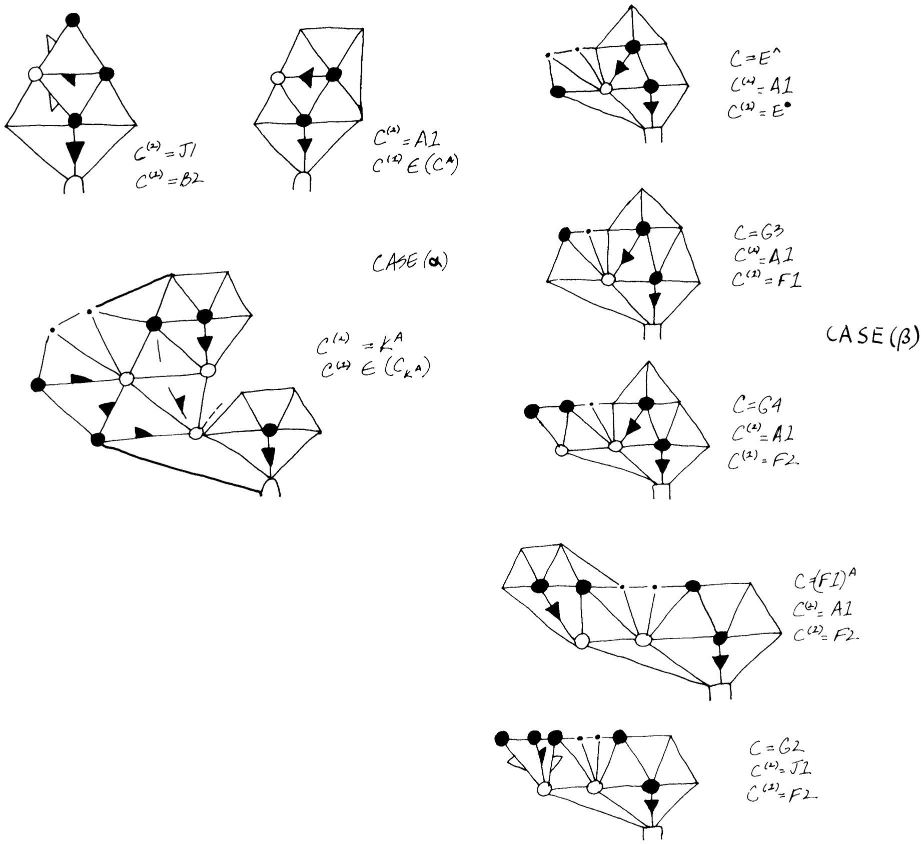


FIGURE 20

15. Pivot to non-pivot merging

LEMMA 9. *Let C be a configuration which is obtained by merging from preliminary discharging situations $C^{(1)}$ and $C^{(2)}$ so that the pivot $V^{(2)}$ of $C^{(2)}$ is identified to a major vertex V_k of the essential part of $C^{(1)}$ which is different from the pivot $V^{(1)}$ of $C^{(1)}$.*

If so, either C contains a geographically good subconfiguration in Size Class $\langle 12, 9\rangle$ or one of the following cases applies (see Figure 20).

Case (α) A specialization of $C^{(2)}$ is degree-equivalent to a subconfiguration of $C^{(1)}$ and C is degree-equivalent to $C^{(1)}$.

Case (β) $C^{(1)}$ is E, $F1$, or $F2$, $C^{(2)}$ is $A1$ or $J1$, and C is $E^\wedge$, $G3$, $G4$, $(F1)^A$, or $G2$, respectively.

16. Building critical modifications in simple steps

To prove Main Lemma 1 we must consider all possible modifications (Definition N) of primary preliminary discharging situations. The following observations show that the task is manageable.

(1) We may ignore cases in which a simple application of one of Lemmas 8 or 9 reveals a geographically good subconfiguration.

(2) As we shall show in this section, every "critical" modification can be built in simple steps by premodifications, each of which extends the preceding one.

(3) By Lemma 4, we may ignore all extensions of any premodification which contains a geographically good subconfiguration.

To make these observations precise we employ the following definitions:

DEFINITION T. By a *critical combination* (*of preliminary discharging situations*) we mean a pivoted configuration C as described in the hypothesis of the Load Lemma (Lemma 8) such that one of Cases (1), (2), (3), (P) of Lemma 8 applies.

A *premodification* $C^{\sim}$ of a plugged configuration C' is called *critical* if the following hold.

(i) If C is a configuration as described in the hypothesis of Lemma 8 and if some specialization of C is contained in $C^{\sim}$ then C is a critical combination.

(ii) If C is a configuration as described in the hypothesis of Lemma 9 and if some specialization of C is contained in $C^{\sim}$ then one of Cases (α), (β) described in Lemma 9 applies.

A premodification C of C' is called a *simple premodification* of C' (*at* $V^{(\alpha)}$) if C (see Figure 21; notation as in Definition M) can be obtained from C'' by merging with some critical combination, say $K^{(\alpha)}$, with pivot identified to $V^{(\alpha)}$ so that every 5-neighbor of $V^{(\alpha)}$ in C'' is identified to some 5-neighbor of the pivot in $K^{(\alpha)}$. (Note that $K^{(\alpha)}$ may contain the image of some $C_i^{(\beta)}$ with $\beta \neq \alpha$; see Figure 21).

$C^{\sim}$ is called a *simple extension of a premodification C of C'* (at $V^{(\beta)}$) if $C^{\sim}$ (see Figure 22; notation as in Definition O) is an extension of C which can be obtained from $C^{\wedge}$ by merging with some critical combination, say $K^{(\beta)}$, with pivot identified to $V^{(\beta)}$ so that every 5-neighbor of $V^{(\beta)}$ in $C^{\wedge}$ is identified with some 5-neighbor of the pivot in $K^{(\beta)}$. ($C^{\sim}$ may be the trivial extension which is degree-equivalent to C).

THEOREM 10. *Let C' be a plugged configuration such that every fully specified major vertex of C' has positive load $L_0^{C'}$. Let C be a critical modification* (*notation as in Definitions* M *and* O) *of C'. Let $V^{(\beta_1)}, \ldots, V^{(\beta_p)}$ be those boundary vertices of C^* whose degree-specifications are raised in constructing C* (*i.e.*, $d_{\beta_1}, \ldots, d_{\beta_p} > 0$; *the β_i enumerated in some arbitrary order*). *Then there are premodifications $C_{[0]} = C', C_{[1]}, \ldots, C_{[p]} = C$ of C' so that for each m, $1 \leq m \leq p$, $C_{[m]}$ is a simple extension of $C_{[m-1]}$ at $V^{(\beta_m)}$ and C is an extension of $C_{[m]}$.*

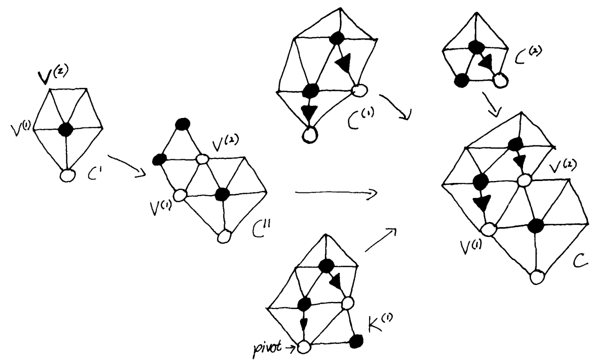


FIGURE 21

Figure 21

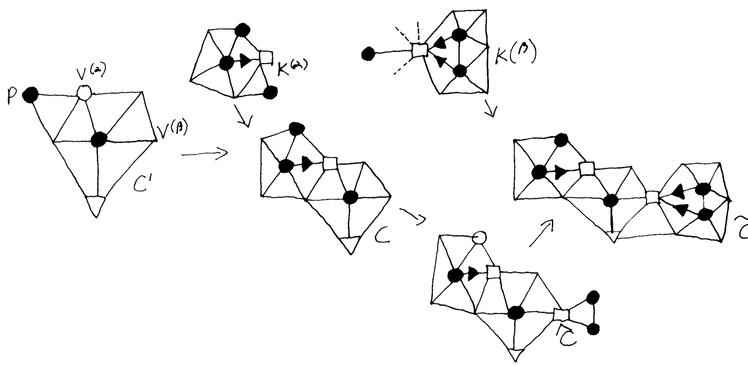


FIGURE 22

17. The proof of Main Lemma 1

With the tools developed in the preceding sections, Main Lemma 1 can be proved with moderate effort by inspection. It is sufficient to consider all modifications of the *primary* preliminary discharging situations (Tables 3 and 4) by preliminary discharging situations of Classes $(A), \ldots, (H_L)$ mentioned in the statement of Main Lemma 1. We have to show that each such modification either contains a geographically good subconfiguration or is itself in one of the classes listed.

By Lemmas 8 and 9, we may restrict ourselves to *critical* modifications. Using Theorem 10, we can construct these, step by step, by *simple* premodifications and simple extensions. If, in this process, some step leads to a premodification which contains a geographically good subconfiguration, then, by Lemma 4, we need not consider any extension steps.

We consider the 46 primary situations (Tables 3 and 4) one by one, as C' (notation as in Definition M). In most cases the first nontrivial premodification

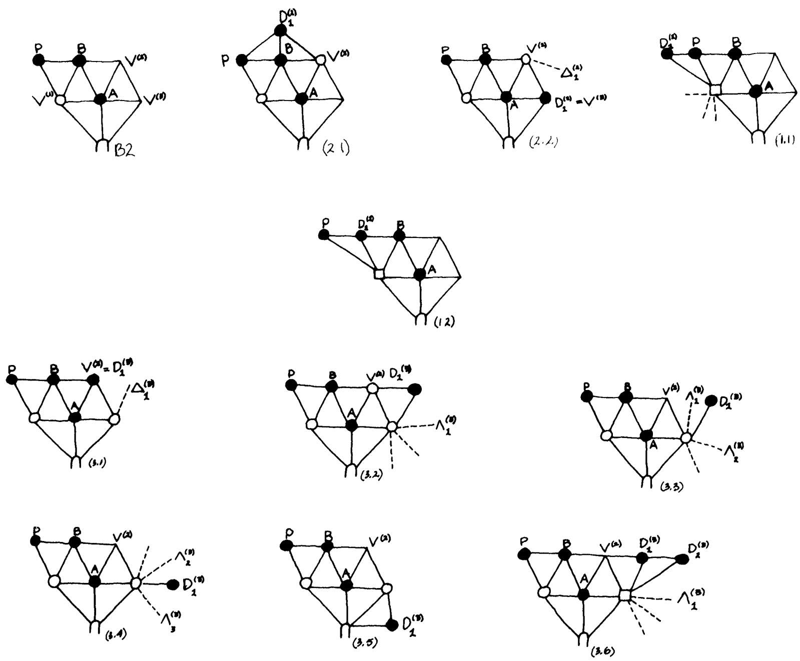


FIGURE 23. Part 1

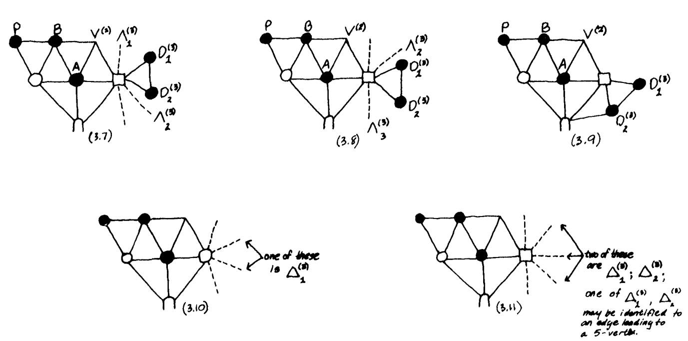


FIGURE 23. Part 2

already contains a geographically good subconfiguration. In order to demonstrate how relatively easy it is to check this we will treat the case $C' = B2$ (Figure 23).

First we consider all simple premodifications C at $V^{(2)}$. Since $L_0^C(V^{(2)}) = 2$ we have $L_0^C(V^{(2)}) \geq 2$. Thus if C is nontrivial ($d_2 \neq 0$) we have (see Lemma 8, Case 2) $d_2 = 1$ and $C_1^{(2)}$ in Class (A). (Since $d_2 = 1$ implies $\deg^C(V^{(2)}) = 7$, the case $C_1^{(2)} = D$ is excluded, and because of the 5-vertices A and B, Case P of Lemma 8 cannot occur).

Further, the nonmain 5-neighbor of the pivot of $C_1^{(2)}$ must be identified with A or B (since otherwise $L_0^C(V^{(2)}$ would be three). Thus, for the configuration C'' we have only the two possibilities (2.1) and (2.2) shown in Figure 23 with $D_1^{(2)}$ adjacent to A or B. In Case (2.1), C'' contains the geographically good 5-5-5 triangle B, $D_1^{(2)}$, P. In Case (2.2), C contains the image of $C_1^{(2)}$ with pivot $V^{(2)}$ of degree 7 and the additional 5-vertex B. This is a geographically good configuration.

Hence we may assume that the degree-specification of $V^{(2)}$ is not raised in constructing C.

Next we consider all simple premodifications C at $V^{(1)}$. By essentially the same argument as in the case of $V^{(2)}$ we conclude that the only possibilities for C'' are (1.1) and (1.2) in Figure 23 and $d_1 = 1$ with $C_1^{(1)}$ in Class (A). In Case (1.1) the image of $C_1^{(1)}$ with pivot $V^{(1)}$ of degree 8, together with the two additional V_5's A and B, is geographically good. In Case (1.2) the merging of any integral discharging situation with central V_5 identified to $D^{(1)}$ is impossible.

Now we may assume that the degree specifications of $V^{(1)}$ and $V^{(2)}$ are not raised in C.

It remains to consider the simple premodifications C at $V^{(3)}$. Since $L_0^C(V^{(3)})$ may be as small as one, we have to consider all cases of Lemma 8. Thus d_3 is either one or two. If $d_3 = 1$ ($\deg^C(V^{(3)}) = 7$) then $C_1^{(3)}$ is either a member of Γ_1 or of Γ_2 with three neighbors to the pivot, or a partial discharging situation with V-pivot. If $d_3 = 2$ ($\deg^C(V^{(3)}) = 8$) then $C_1^{(3)}$ and $C_2^{(3)}$ are either both in Γ_1 with adjacent main V_5's or are both partial discharging situations or one is a partial discharging situation and the other is in Class (A). Thus we have the eleven possible cases (3.1), ..., (3.11) for C'' shown in Figure 23.

In each case it is easily seen (using Theorem 8) that the image of $C_1^{(3)}$ or the images of $C_1^{(3)}$ and $C_2^{(3)}$, together with the vertices A, B, $V^{(2)}$ form a configuration which contains a geographically good subconfiguration. This finishes the treatment of $B2$.

It turns out that the most complicated case of the 46 is presented by the only member of Class (C). This case, $C' \in (C)$, can be treated as follows (see Figure 24). One difficulty is that there are many simple premodifications of C' in which "noncompensated degree raising" (Definition M, Step 3b, Condition 5) occurs and which wipe out (for examples see Figure 24b). But it is not difficult to see that in each such premodification, say C, some 5-vertex $D_j^{(\alpha)}$ has been introduced in one of the places K, L, M (Figure 24a) adjacent to the vertex $V^{(\alpha)}$ at which the "noncompensated degree raising" occurs (according to Case (i) in Condition 5, Definition M). This means that if any critical modification $C^{\sim}$ extends C then $C^{\sim}$ must contain a specialization of some *integral* discharging situation $C_j^{\sim(\alpha)}$ with main V_5 at $D_j^{(\alpha)}$ and the pivot at $V^{(\alpha)}$. But then, the specialization of $C_1^{\sim(\alpha)}$, if $d_\alpha^{\sim} = 1$, (the specializations of $C_1^{\sim(\alpha)}$ and $C_2^{\sim(\alpha)}$ if $d_\alpha^{\sim} = 2$) together with the main V_5 of C' form a geographically good subconfiguration of $C^{\sim}$ (in contradiction to our hypothesis that $C^{\sim}$ was critical). Thus

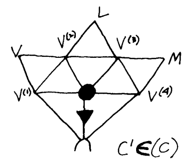


FIGURE 24a

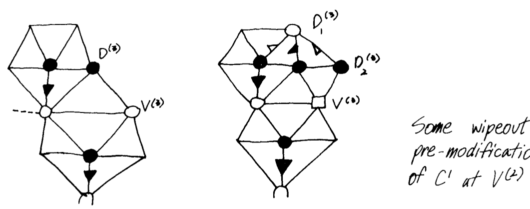


FIGURE 24b

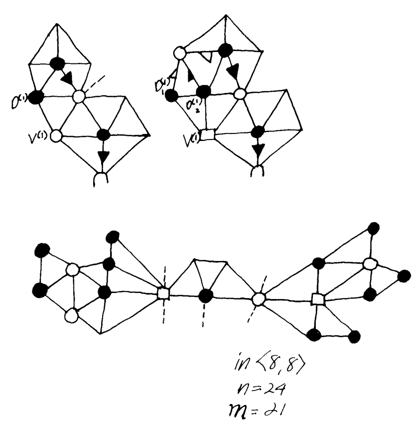


FIGURE 24c

we may ignore all those simple premodifications of C' which are not also modifications.

Next we consider all simple modifications of C' at $V^{(2)}$ and it is easy to see that only one of them wipes out; this is $(C^A)1$ in Table 5. Further, we have to consider all critical modifications of C' which are nontrivial extensions of $(C^A)1$. But these must be simple extensions at $V^{(1)}$ (since $V^{(2)}$, $V^{(3)}$, and $V^{(4)}$ are overlapped by the image of $|C_1^{(2)}|$ in $(C^A)1$). But each of these contains a geographically good subconfiguration. By symmetry, the simple modifications at $V^{(3)}$ do not yield any new cases. Thus we have discussed all critical modifications of C' in which the degrees of $V^{(2)}$ or $V^{(3)}$ are raised, and henceforth we restrict ourselves to critical modifications in which $V^{(2)}$ and $V^{(3)}$ have degrees six or five.

Now we consider all simple modifications of C' at $V^{(1)}$. Precisely 45 of them (up to degree-equivalence) wipe out. These are the 44 configurations in Classes (C_X) with $X = A, B, C, C^A, AA, J, K, K^A$, and L and Configuration $(C^A)2$ in Tables 6 and 5. By symmetry, the simple modifications at $V^{(4)}$ do not yield any new cases. It remains to consider all critical modifications of C' which are simple, nontrivial extensions at $V^{(4)}$ of one of the 45 configurations described above. It is not difficult to see that each of these contains some geographically good subconfiguration.

Remark. This is one of the very few occasions where our discharging procedure leads to rather large geographically good configurations ($n > 20$, diameter > 6) which do not contain smaller geographically good subconfigurations (see the example in Figure 24c). Note that each of the over-size configurations has two articulation vertices.

The remaining configurations of Tables 3 and 4 are discussed similarly. The only other cases which lead to over-size geographically good configurations are $H1$, $H2$, and $H3$ (and again each of the large configurations has two articulation vertices). This finishes the proof of Main Lemma 1. ∎

18. The overcharging lemma for q_2

Now we can prove that q_2-overcharging implies the presence of geographically good configurations.

THEOREM 11. *Let D be a triangulation or configuration. Suppose that V_k is a major vertex of D with fully specified degree k and that $q_2^D(V_k) > 0$. Then at least one of the following cases applies:*

Case 1. *D contains a specialization of a configuration C as considered in the Load Lemma (Lemma* 8) *so that either* (1.a) *$\delta > k - 6$ or* (1.b) *$\delta = k - 6 = 2$ and at least one of C_1, C_2 is a partial discharging situation.*

Case 2. *D contains a specialization of a configuration C (as considered in Lemma* 9) *such that $C^{(1)}$ is a partial discharging situation and such that $C^{(2)}$ is not identified with a subconfiguration of $C^{(1)}$.*

Case 3. *D contains a specialization of a configuration C such that C is obtained from two partial discharging situations $C^{(1)}$ and $C^{(2)}$ by merging in such a way that the following conditions are satisfied.*

(3.1) *Some vertex $V_k^{(2)}$ ($k = 7$ or* 8) *of $C^{(2)}$ which is the receiving vertex of a partial discharging track system $\mathscr{T}^{(2)}$ in $C^{(2)}$ is identified with some vertex $V_k^{(1)}$ of $C^{(1)}$ which is the receiving vertex of a partial discharging track system $\mathscr{T}^{(1)}$ in $C^{(1)}$.*

(3.2) *$V_k^{(1)}$ is different from the pivot of $C^{(1)}$, and $V_k^{(2)}$ is different from the pivot of $C^{(2)}$.*

(3.3) *The q_2^C-value of the image of $V_k^{(1)}$ (which is the q_2^C-value of the image of $V_k^{(2)}$) is positive.*

LEMMA 11. (OVERCHARGING LEMMA FOR Q_2). *Let D be a triangulation or configuration. Suppose that V is a fully specified major vertex of D and $q_2^D(V) > 0$. Then D contains some geographically good configuration in Size Class $\langle 10, 15\rangle$.*

19. The fractional discharging

DEFINITION U. Let D be a triangulation or configuration. We define *the charge function* q_3^D, which is obtained from q_2^D *by fractional discharging*, by the following *fractional discharging algorithm.* Suppose V_5 is a 5-vertex of D which fulfills the following conditions.

(FD1) V_5 is an interior vertex of D and all five neighbors of V_5 are fully specified in D; moreover $q_2^D(V_5) > 0$.

(FD2) V_5 has at least one q_2^D-negative major neighbor in D. Moreover if $q_2^D(V_5) > 30$ then V_5 has at least two q_2^D-negative major neighbors in D.

(FD3) If V is a q_2^D-negative major neighbor of V_5 in D then

$$\begin{array}{ll} \text{if } -180 \le q_2^D(V) \le -60 & \text{then } L_1^D(V) \le 3, \\ \text{if } q_2^D(V) = -240 & \text{then } L_1^D(V) \le 4, \\ \text{if } q_2^D(V) = -300 & \text{then } L_1^D(V) \le 5. \end{array}$$

We define the *capacity of V in D* by[6]

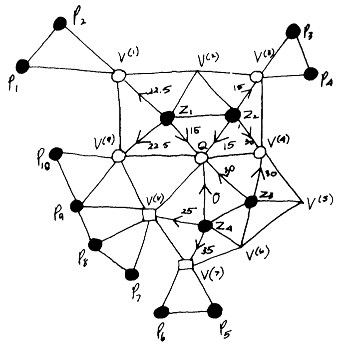
$$\mathrm{Cap}^D(V) = \begin{cases} |q(_2^D V)|/v_1^D(V) & \text{if } \deg^D(V) \ge 7 \text{ and } q_2^D(V) < 0 \\ 0 & \text{if } \deg^D(V) < 7 \text{ or } q_2^D(V) \ge 0. \end{cases}$$

Under these conditions, charge $q_2^D(V_5)$ is transferred from V_5 to its q_2^D-negative neighbors in fractions by the following step-by-step procedure (see Figure 26).

Let $h(V_5)$ be the sum of the capacities of those k neighbors of V_5 which have capacities at least 30 (k may be zero as in the case of Z_1 in Figure 26). If $h(V_5) < q_2^D(V_5)$ (as in the cases of Z_2, Z_3, in Figure 26) then each such neighbor is assigned additional charge from V_5 equal to its capacity and the sequential procedure below is applied. If $h(V_5) \ge q_2^D(V_5)$ (as in the case of Z_4 in Figure 26) then the k neighbors V_i of capacity greater than 30 are assigned charges cap $(V_i) - (h(V_5) - q_2^D(V_5))/k$. In this case the discharging is completed at the V_5.

If the first case above has applied let $d_0 = q_2^D(V_5) - h(V_5)$. By (FD3) the remaining neighbors with negative charge fall into three classes. Class α consists of vertices of capacity 15 ($q_2^D = -60, L = 3$); Class β consists of vertices of capacity 20 ($q_2^D = -60, L = 2$); Class γ consists of vertices of

[6] We regard the integrally discharged V_5's as completely taken care of but the partially discharged V_5's are still considered critical. Thus we use v_1 rather than v_2. Note that *FD*1 implies $v_1 > 0$.



Fractional Discharging
in (D, q_2^D), $q_2^D = q_1^D = q_0^D$;

	q_2^D	V_1	L_1	cap	q_3^D
$V^{(1)}$	-60	3	2	20	-37.5
$V^{(3)}$	-60	3	2	20	-45
$V^{(4)}$	-60	2	1	30	0
$V^{(7)}$	-120	3	1	40	-85
$V^{(8)}$	-120	4	2	30	-95
$V^{(9)}$	-60	3	2	20	-37.5
Q	-60	4	3	15	0

FIGURE 26

capacity 24 ($q_2^D = -120$, $L = 3$). Let $n(c)$ ($c = \alpha, \beta, \gamma$, or $\alpha \cup \beta$) denote the number of vertices in Class c. We define the discharging step $S(c, b)$ with initial charge d and parameter b as follows. If $n(c) = 0$, do nothing. Otherwise let $r = \min(d/n(c), b)$. Assign additional charge r to each neighbor of Class c. Let $d' = d - r \cdot n(c)$. If $d' = 0$ the procedure is completed, otherwise proceed to the next step.

With d initially d_0 the following steps are applied in the indicated order until the procedure halts: $S(\gamma, 16)$, $S(\alpha \cup \beta, 10)$, $S(\gamma, 4)$, $S(\alpha \cup \beta, 5)$, $S(\gamma, 3)$, $S(\beta, 5)$, $S(\gamma, 7)$, $S(\beta, 10)$, $S(\alpha, 15)$.

The total charge which is transferred according to this algorithm from V_5 to a major neighbor, say V, of V_5 is denoted by $F^D(V_5, V)$ and called the *fractional discharging value of* (V_5, V) *in* D. (In Figure 26 the $F^D(V_5, V)$-values are indicated in the drawing.)

This discharging algorithm is applied *simultaneously* to all V_5's in D which fulfill conditions (FD1), (FD2), and (FD3). The charges of those V_5's which do not fulfill the conditions are not changed. We define $F^D(V_5, V)$ to be zero in these cases. The charge function on D so obtained is called q_3^D. Note that the algorithm significantly distinguishes large capacities (≥ 30) from small capacities (< 30, i.e., 15, 20, or 24) in such a way that a vertex of large capacity cannot receive more charge than its capacity from any one of its V_5-neighbors and thus can never be "overcharged."

Our remaining task is to exhibit, for every triangulation T, a geographically good configuration in the neighborhood of every q_3^T-positive (non-6)-vertex in T (so that *one* finite set of such configurations suffices for *all* triangulations T). Trivially $q_3^T(V_6) = 0$ for each vertex of degree 6. Since we have already proved Lemma 11, we need not consider the case of q_2-positive major vertices.

20. q_t neighborhoods

If V_j is a vertex of degree $j \neq 6$ in a triangulation T and if V_j belongs to a subconfiguration C of T, then it may happen that $q_t^C(V_j) \neq q_t^T(V_j)$ for $t = 1$, 2, or 3. But if C contains a sufficiently large neighborhood N of V_j in T then

q_t^C and q_t^T will agree on V_j and if $q_t^T(V_j) > 0$ we shall have to exhibit a geographically good configuration in N. In the following sections this task will be reduced, in several stages, to the consideration of smaller and more restricted configurations than N (such as plugged clusters of V_5-neighborhoods).

DEFINITION V. Let T be a triangulation and let V_j be a vertex of T of degree $j \neq 6$. Let N be a configuration which is obtained by merging from a configuration C^0 and preliminary discharging situations $C^1, \ldots, C^u$. Denote the merging immersions: $|C^k| \to |N|$ by f^k ($k = 1, \ldots, u$). Let $f: |N| \to |T|$ be an immersion which respects the leg-specifications of (some specialization of) N and of T so that f maps some vertex Q of N to V_j. Then the pair (N, f) is called a *q_t-neighborhood of V_j in T* (*with pivot* Q) if the following conditions are fulfilled (according to whether $t = 1$, 2, or 3 and whether $j = 5$ or $j \geq 7$).

Case I. $t = 1, j = 5$. Then C^0 is a single 5-vertex, say Q^0, and $C^1, \ldots, C^u$ are integral discharging situations satisfying the following conditions:
(I.i) Q^0 and the main V_5's of the C's are identified with Q.
(I.ii) If E is any integral discharging edge in T which leaves V_j then E has as preimage (under f) an integral discharging edge in N.

Case II. $t = 1, j \geq 7$. Then C^0 is a single j-vertex, say Q^0, and $C^1, \ldots, C^u$ are integral discharging situations satisfying the following conditions.
(II.i) Q^0 and the pivots of the C's are identified with Q.
(II.ii) If E is any integral discharging edge in T which leads to V_j then E has as a preimage an integral discharging edge in N.

Case III. $t = 2, j = 5$. Then C^0 is a single 5-vertex, say Q^0, which is identified to Q, and $C^1, \ldots, C^u$ satisfy the following conditions.
(III.i) For each $i = 1, \ldots, u$, at least one of the following holds:
(i.a) Some discharging V_5 of C^i is identified with Q.
(i.b) C^i is an integral discharging situation with main V_5 identified with some discharging V_5 of some C^k which satisfies (i.a).
(III.ii) If $\mathscr{T}$ is any track system in T one of whose tracks leaves V_j when $\mathscr{T}$ has as a preimage a track system in N (with the same discharging designation of corresponding tracks).
(III.iii) If E is any integral discharging edge in T which leaves either V_j or any discharging V_5 of some track system $\mathscr{T}$ as considered in (III.ii) then E has a preimage which is an integral discharging edge in N.

Case IV. $t = 2, j \geq 7$. Then C^0 is a single j-vertex, say Q_0, which is identified with Q, and $C^1, \ldots, C^u$ satisfy the following conditions.
(IV.i) For each $i = 1, \ldots, u$, some receiving vertex of C^i is identified with Q.
(IV.ii) If $\mathscr{E}$ is any integral discharging edge (any track system) in T which discharges to V_j then $\mathscr{E}$ has an inverse image which is an integral discharging edge (a track system) in N.

Case V. $t = 3, j = 5$. Then C^0 is a (fully-specified) plugged V_5-neighborhood, the central V_5, say Q^0, of which is identified with Q, and $C^0, \ldots, C^u$ satisfy the following conditions.

(V.i) Every 5-vertex of T which is adjacent to a major neighbor of V_j in T has a preimage (under $f \circ f^0$) in C^0.

(V.ii) For each $i = 1, \ldots, u$, at least one of the following holds.

(i.a) Some discharging V_5 of C^i is identified to Q.

(i.b) C^i is an integral discharging situation with main V_5 identified with some discharging V_5 of some C^k which satisfies Condition (i.a) or with some V_5 of C^0.

(i.c) Some receiving vertex of C^i and some neighbor vertex of Q^0 in C^0 are identified.

(V.iii) (N, f) contains a q_2-neighborhood of V_j in T with pivot at Q. By this we mean that there exists a q_2-neighborhood $(N^\sim, f^\sim)$ of V_j in T with pivot $Q^\sim$ as defined in Case III and an immersion $f^\vee : |N^\sim| \to |N|$ which respects the leg specifications of (some specialization of) $N^\sim$ and N so that $f^\sim = f \circ f^\vee$ and $f^\vee(Q^\sim) = Q$.

(V.iv) For each major neighbor V_k of Q in N, (N, f) contains a q_2-neighborhood of $f(V_k)$ in T with pivot at V_k.

(V.v) For each V_5 in C^0, (N, f) contains a q_1-neighborhood of $f(f^0(V_5))$ in T with pivot at $f^0(V_5)$.

Case VI. $t = 3, j \geq 7$. Then C^0 is a plugged cluster (Definition I), the pivot, Q_0, of which is identified with Q, and $C^0, \ldots, C^u$ satisfy the following conditions.

(VI.i) If a major vertex V^* of T is adjacent to a V_5-neighbor of V_j, and if V^* is adjacent to some other 5-vertex, say V_5^*, of T, then V_5^* has a preimage (under $f \circ f^0$) in C^0.

(VI.ii) For every fully specified major vertex V_k of C^0, N contains a q_2-neighborhood of $f(f^0(V_k))$ in T with pivot at $f^0(V_k)$.

(VI.iii) For every 5-vertex (for every central 5-vertex) V_5^0 of C^0, N contains a q_1-neighborhood (N contains a q_2-neighborhood) of $f(f^0(V_5^0))$ in T with pivot at $f^0(V_0^5)$.

(VI.iv) N can be obtained by merging from C^0 and the q_2- and q_1-neighborhoods mentioned in (VI.ii) and (VI.iii).

The configuration N itself is called a *q_t-neighborhood* (*of* Q). The f^0-image of the core C^0 is called the *core* of N. A q_t-neighborhood of a major vertex Q is called *q_t-overcharging* if $q_t(Q) > 0$ and $q_{t-1}(Q) \leq 0$.

THEOREM 12. *Let T be a triangulation and V_j a vertex of T with degree $j \neq 6$. Then for each $t = 1, 2, 3$, there exists a q_t-neighborhood N of V_j in T, and for the pivot Q of N we have $q_t^N(Q) = q_t^T(V_j)$.*

MAIN LEMMA 2. *If N is a q_3-neighborhood of a vertex Q such that $q_3^N(Q) > 0$ then N contains a geographically good configuration in Size Class $\langle 14, 23\rangle$.*

The proof of this lemma is described in the following sections. The case that Q is a major vertex with $q_2^N(Q) > 0$ is taken care of by Lemma 11. Thus we must consider the cases that Q is a V_5, and that Q is a major vertex and N is q_3-overcharging.

In each case we must exhibit some geographically good configuration in N. As stated in the introductory sections, 5 and 6, we shall essentially reduce this task to the consideration of plugged clusters of V_5-neighborhoods (or in the case that Q is a 5-vertex, of a single plugged V_5-neighborhood). Certainly we may ignore such neighborhoods if they contain configurations described in previous lemmas and we have previously proved that they contain geographically good configurations. The remaining q_3-neighborhoods will be called *arranged q_3-neighborhoods* (Section 21). For each arranged q_3-neighborhood N we can exhibit (Section 22) some plugged cluster N' and some modification $\bar{N}$ of N' such that the core of $\bar{N}$ is identical to the core N^* of N and such that $\bar{N}$ contains all those preliminary discharging situations which are contained in N and weakly overlap N^*. Moreover, for each major vertex of N' there is a one-to-one correspondence between its 5-neighbors in N' and those of its 5-neighbors in N which are not main V_5's of weakly overlapping *integral* discharging situations. We shall call N' a *demodification* of N. We know from previous lemmas and theorems that if N' contains a geographically good configuration then so does $\bar{N}$ and thus also N. The crucial question at this point is whether $q_3^N(Q) > 0$ always implies $q_3^{N'}(Q) > 0$. In Sections 23 and 24 we shall show that the answer "no" occurs only in a few exceptional cases where complications are caused by some strongly overlapping preliminary discharging situations in N which are not present in N'. In some of these exceptional cases a geographically good configuration can immediately be exhibited; in all of the remaining cases an *augmented plugged cluster* $N^{\sim}$ can be derived from N' either by a simple modification or by attaching a strongly overlapping preliminary discharging situation, so that $q_3^N(Q) > 0$ implies $q_3^{N^{\sim}}(Q) > 0$. With this, the consideration of q_3-neighborhoods is reduced to the consideration of arranged q_3-overcharging augmented plugged clusters (which will be described in Sections 25, 26 and 27). The case that Q is a 5-vertex will have been completely treated in Section 23. We shall describe (in Section 21, Tables 7 and 8) 49 classes of arranged q_3-neighborhoods such that every q_3-overcharging arranged q_3-neighborhood N is contained in (at least) one of these classes and such that a convenient treatment of all regular and exceptional cases is possible.

21. Arrangements

In this section we describe those q_3-neighborhoods which will require more detailed consideration.

DEFINITION W. By an *arrangement* we mean one of the 49 configurations drawn in Tables 7 and 8. The (interior 7- or 8-) vertex marked Q is called the *pivot* of the arrangement. The eight *simple arrangements* of Table 7 do not

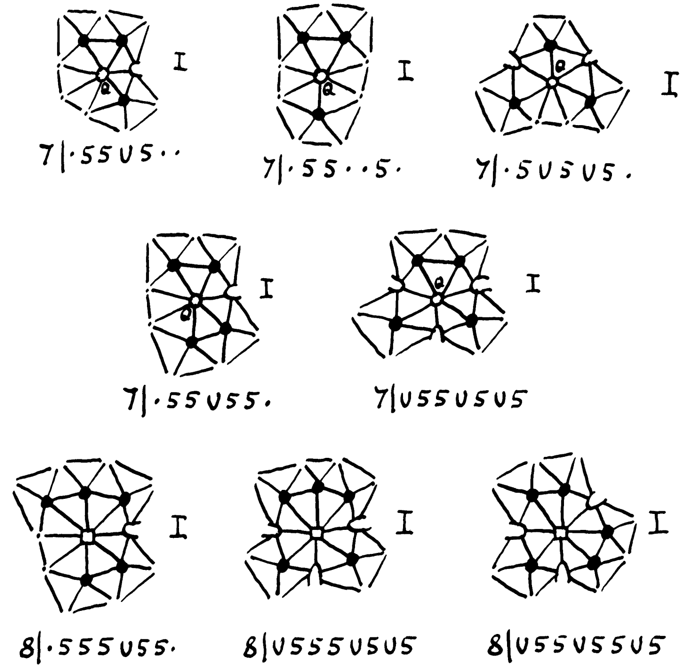


TABLE 7. The simple arrangements

contain any discharging tracks while the forty-one *augmented arrangements* of Table 8 contain specializations of integral or partial discharging situations. For the simple arrangements we use the obvious notation $7 \mid \cdot 55 \cup 5\cdot\cdot$, etc. of Table 7. For the other arrangements we write $8 \mid 65 \downarrow 5 \cup 55\cdot\cdot$ (A1), $7 \mid \cup \overrightarrow{55} \sqcup \cdot 55$ (G1), $7 \mid \cdot\vec{5} \cup 5 \cup 5\cdot$ (L2r), etc. where the largest discharging situation a specialization of which is contained in the arrangement is indicated in parentheses and is called the *augmenting configuration* of the arrangement. The arrow indicates the discharging V_5, and the letter *r* indicates that the discharging situation is reflected (i.e., contained with orientation opposite to that in the drawing in Table 3, 4, or 5).

Each arrangement X contains a subconfiguration X^*, a specialization of which is a cluster of V_5-neighborhoods. X^* is called the *core of* X. The core X^* of X together with all V_5's of X which are adjacent to vertices of X^* (and the corresponding edges and triangles) is called the *plugged core of* X. In Tables 7 and 8, the plugged cores are indicated by heavy drawings and the plug-V_5's are marked p; those arrangements which are identical with their plugged cores are marked I. The arrangements marked II and III can be obtained from their plugged cores by attaching one preliminary discharging situation which weakly or strongly overlaps the core X^*, respectively. If $X^\vee$ is a specialization of an arrangement X then a vertex V of $X^\vee$ is called *originally specified* if it is fully

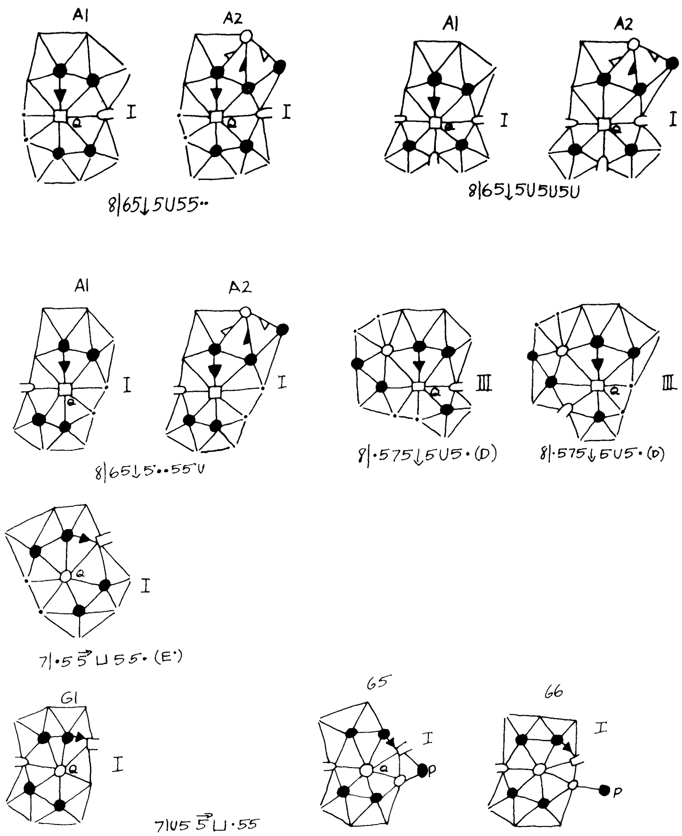


TABLE 8. Part 1: The augmented arrangements

specified in X. A specialization of an arrangement which contains an augmenting configuration and in which all neighbors of the central V_5's are fully specified is called an *augmented cluster* (*of* V_5-*neighborhoods*); an *augmented plugged cluster* is obtained from an augmented cluster by attaching plug-V_5-neighbors to vertices of the core. An augmented or nonaugmented plugged cluster is called *moderately plugged* if it does not contain any discharging situation J1 or L1 all 5-vertices of which are plug-V_5's of the (augmented) cluster.

DEFINITION X. A q_3-neighborhood N of a vertex Q is called *arranged* if it has the following properties.

(1) Q is not the receiving vertex of any partial discharging track system in N.

(2) If V_k is a fully specified vertex of N with degree $k \geq 7$ then the following conditions are satisfied.

(2.1) $q_2^N(V_k) \leq 0$.

(2.2) If $\mathrm{cap}^N(V_k) < 45$ and if C is a configuration as described in the hypothesis of the Load Lemma (Lemma 8) such that N contains a specialization of C with pivot at V_k then C is a critical combination.

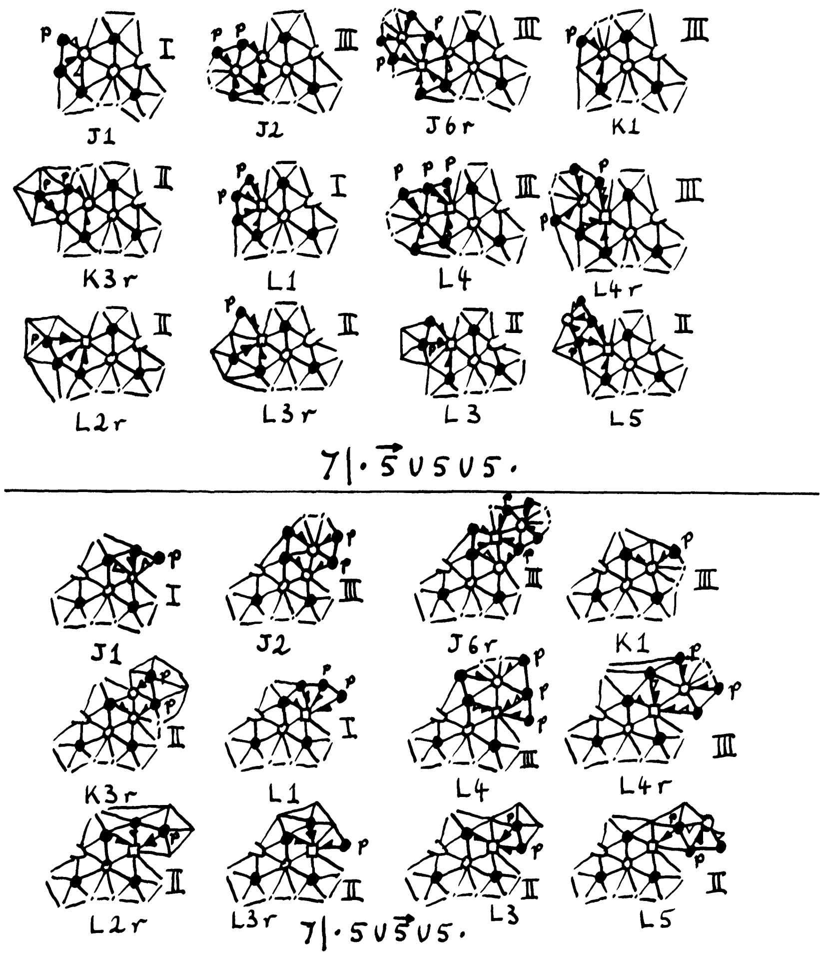


TABLE 8. Part 2

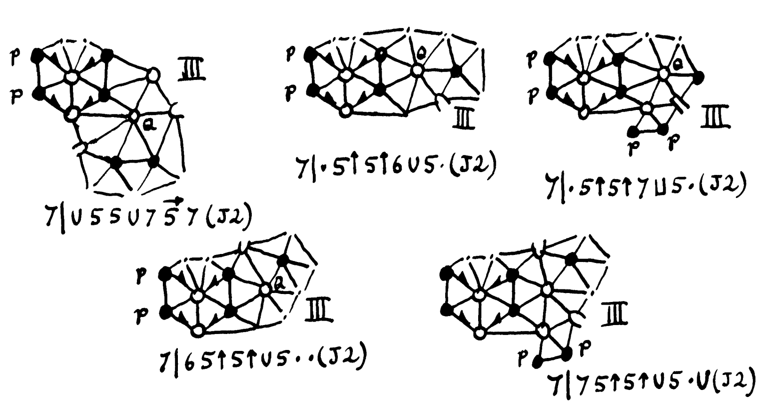


TABLE 8. Part 3

(3) If N contains any specialization of a configuration C which is obtained by merging from two preliminary discharging situations as described in the hypothesis of Lemma 9 then one of Cases (α), (β) of Lemma 9 applies.

(4) If N contains a specialization $C_0^\vee$ of a configuration C_0 which is obtained from some primary preliminary discharging situation C_0' by 6-to-5 degree lowering, subsequent modification, and attachment of integral discharging situations as described in Theorem 4 then one of two cases applies.

(4.1) C_0 contains a subconfiguration which is a preliminary discharging situation with the same discharging tracks and pivot as C_0'.

(4.2) C_0' is $H1$ and C_0 is the configuration of Figure 15d.

(5) If Q is a major vertex then the following hold.

(5.1) N contains a specialization of some arrangement with pivot of the arrangement at Q. We say that N *corresponds to the arrangement* X (X in Table 7 or 8) if N contains some specialization of X with pivot at Q.

(5.2) If X_1 and X_2 are arrangements and if $x_1\colon |X_1| \to |N|$ and $x_2\colon |X_2| \to N$ are immersions which respect the leg-specifications of some specializations $X_1^\vee$, $X_1^\vee$ of X_1, X_2 and of N so that the pivots of X_1, X_2 are mapped to Q then there exists an embedding $\bar{x}\colon |X_i| \to |X_j|$ (either $i = 1, j = 2$ or $i = 2, j = 1$) so that $x_i = x_j \circ \bar{x}_i$.

(6) If N contains a specialization C of a *partial* discharging situation such that some partial discharging track system of (the image of) C leads to some vertex V_k in the core N^* of N then at least one of the following four cases applies:

(6.a) V_k is the image of the pivot C, and C weakly overlaps N^* (see Definition L); but if N contains a specialization, say $X^\vee$, of some arrangement X with pivot at Q, then V_k is *not* (the image of) an *originally specified* vertex of $X^\vee$.

(6.b) V_k is not the image of the pivot of C, but C contains a proper subconfiguration, say $C^\vee$, which is (a specialization of) a partial discharging situation so that V_k is the image of the pivot of $C^\vee$.

(6.c) Every 5-neighbor of V_k in N^* is a discharging vertex of (the image of) C.

(6.d) There is an arrangement X and an immersion $x\colon |X| \to |N|$ which respects the leg-specifications of some specialization $X^\vee$ of X and of N so that the following condition is satisfied. X contains a subconfiguration, say $C^\sim$, which is a partial discharging situation such that $x(|C^\sim|)$ is identical to the image of $|C|$ in N.

(7) If N contains a specialization C of an *integral* discharging situation with pivot in the core N^* of N then one of the following three cases applies.

(7.a) Q is a 5-vertex and C weakly overlaps N^*.

(7.b) Q is a major vertex and C weakly overlaps N^* so that the following holds. If N contains a specialization, say $X^\vee$, of some arrangement X with pivot at Q then the main discharging edge of C is *not* identified with any edge of the plugged core of $X^\vee$ which is incident to an originally specified V_5 of $X^\vee$; moreover, the pivot of C is *not* identified to any originally specified vertex of $X^\vee$.

(7.c) There is an arrangement X and an immersion $x\colon |X| \to |N|$ which respects the leg-specifications of some specialization $X^\vee$ of X and of N so that

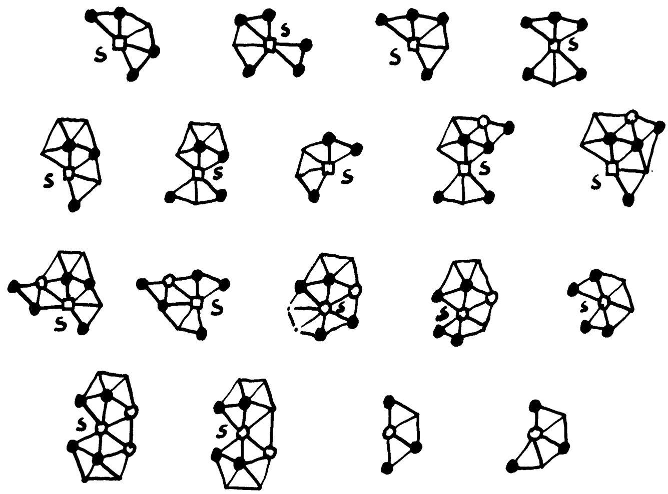


FIGURE 27

the following condition holds. X contains a subconfiguration, say C, which is a specialization of an integral discharging situation such that $x(|C^{\sim}|)$ is identical to the image of $|C|$ in N.

(8) N does *not* contain any modification of any one of the eighteen configurations drawn in Figure 27 (where all V_5's are specified to be nonplug).

LEMMA 12. *Let N be a q_3-neighborhood of a vertex Q such that $q_3^N(Q) > 0$. Then either N contains a geographically good subconfiguration in Class $\langle 10, 23\rangle$ or N is arranged (or both).*

22. Demodification

DEFINITION Y. Let N be an arranged q_3-neighborhood of a vertex Q which is obtained by merging from configurations $C^0, \ldots, C^u$ (according to immersions $f^i\colon |C^i| \to |N|$ as described in Definition V). Assume that every subconfiguration of N which is a specialization of any preliminary discharging situation has a preimage among $C^1, \ldots, C^u$ and if it is a specialization of one of $L1$, L_4 that it has at least two such preimages. Further assume that the enumeration of $C^1, \ldots, C^u$ is chosen so that for some v, $0 \leq v \leq u$, the following hold.

(i) The pivots of $C^1, \ldots, C^v$ are identified to vertices of the core N^* of N, and $C^1, \ldots, C^v$ weakly overlap N^*.

(ii) If $v < i \leq u$ then either the pivot of C^i is not identified to a vertex of N^*, or C^i strongly overlaps N^*.

Let $\overline{N}$ be a configuration which is obtained by merging from $C^0, \ldots, C^v$, compatible with the merging of $C^0, \ldots, C^u$ to N, such that the pivots of $C^1, \ldots, C^v$ are identified with vertices in C^0. Denote the corresponding immersions by

$\bar{f}^i\colon |C^i| \to |\bar{N}|$ and assume that $\bar{f}^0$ and f^0 agree on the core of C^0 (i.e., that $\bar{N}$ has the "same core" N^* as N).

Then a configuration N' (in particular, a specialization of a plugged cluster if Q is a major vertex, and a plugged V_5-neighborhood if Q is a V_5) is called a *demodification of* $\bar{N}$ and also a *demodification corresponding to* N if $\bar{N}$ is a modification of N' such that (in the notation of Definition M) the following hold.

(a) The $C_1^{(\alpha)}$ $(\alpha = 1, \ldots, a; i = 1, \ldots, d_\alpha)$ are some (or all) of $C^1, \ldots, C^v$; the $f_i^{(\alpha)}$ are the corresponding immersions among $\bar{f}^1, \ldots, \bar{f}^v$; (and, trivially $C' = N'$, $C = \bar{N}$, and $C^{\#\#} = N^*$).

(b) If $V^{(\alpha)}$ $(1 \leq \alpha \leq a)$ is a *major* vertex *of* C^* and V_5 is a 5-vertex of N which is joined to $V^{(\alpha)}$ by an edge, say E, of N then there are a corresponding 5-vertex, say V_5'', and a corresponding edge, say E'' in C'' so that $E = f''(E'')$ and $V_5 = f''(V_5'')$.

(c) N' does not contain any weakly overlapping preliminary discharging situation.

THEOREM 13. *Let N be an arranged q_3-neighborhood of a vertex Q. Then there exists a demodification N' corresponding to N. In particular, one of the following four cases applies.*

Case 0. *Q is a 5-vertex.*

Case I. *N corresponds to some arrangement, say X, of the twenty-four arrangements marked* I *in Tables* 7 *and* 8, *but not to any of the arrangements marked* II *or* III. *Then N' corresponds to the same arrangement X. In particular, N' can be obtained from some specialization of X by attaching* 5*-vertices.*

Case II. *N corresponds to some arrangement, say X, marked* II *in Table* 8. *Then there exists a simple modification, say $N^\#$, of N' by a discharging situation of Class* (A) *such that $\bar{N}$ (notation as in Definition* Y) *is an extension of $N^\#$ and so that $N^\#$ corresponds to X and can be obtained from a specialization of X by attaching V_5's.*

Case III. *N corresponds to some arrangement, say X, marked* III *in Table* 8. *Then there exists a configuration, say N'^+, which contains N' as a proper sub-configuration and which can be obtained from N' by attaching one preliminary discharging situation (D, $J2$, $K1$, $J6$, or $L4$) so that N'^+ corresponds to X, and can be obtained from a specialization of X by attaching V_5's.*

23. q_3-positive 5-vertices

In this section we prove Main Lemma 2 for the case that Q is a 5-vertex.

LEMMA 14. *Suppose that N is an arranged q_3-neighborhood of a 5-vertex Q such that $q_3^N(Q) > 0$. Then N contains a geographically good configuration in Size Class $\langle 10, 10\rangle$.*

24. q_3-overcharging

We still must prove Main Lemma 2 for the case of arranged, q_3-overcharging, q_3-neighborhoods N. By Theorem 13, there is a demodification N' corresponding to N; and, in case of arrangements labelled II or III there exists a configuration $N^{\#}$ or N'^{+}, respectively, as described in Cases II and III of Theorem 13. In this section we consider all cases in which N', $N^{\#}$, or N'^{+}, respectively, is *not* q_3-overcharging and thus we essentially reduce the question of q_3-overcharging to the consideration of the relatively simple configurations N', $N^{\#}$, and N'^{+}.

THEOREM 14. *Let $X^{\vee}$ be a specialization of an arrangement X with pivot Q, and let $N^{\sim}$ be an augmented or nonaugmented plugged cluster which is obtained from $X^{\vee}$ by attaching plug-V_5's to its core N^*. Suppose that C is a preliminary discharging situation such that a subconfiguration $C^{\vee}$ of $N^{\sim}$ is a specialization of C. Then one of the following cases applies* (*see Figure* 34).

Case (i) *$C^{\vee}$ is a subconfiguration of the augmenting configuration M of $X^{\vee}$.*

Case (ii) *The degree of the pivot of $C^{\vee}$ is* 7 *or* 8 *and $C^{\vee}$ together with all* 5-*neighbors of its pivot in $N^{\sim}$ is not a critical combination* (*see Definition* T).

Case (iii) *C is a primary situation, and $C^{\vee}$ is contained in the plugged core of $N^{\sim}$ so that either $q_2^{N^{\sim}}(Q) = 0$ or* $\mathrm{cap}^{N^{\sim}}(Q) \geq 30$.

Case (iv) *$C = C^{\vee}$ is $J1$ or $L1$ and weakly overlaps N^*.*

Case (v) *$C = C^{\vee}$ is one of $H1$, $J1$, $L1$, $(H1)_{J1}$, $(H1)_{L1}$, $(H2)_{J1}$, $(H2)_{L1}$ and strongly overlaps N^* so that Case* (iii) *does not apply.*

LEMMA 15. *Let N be a q_3-overcharging, arranged q_3-neighborhood of a major vertex Q. Let $X^{\vee}$ be a specialization of an arrangement X which is a subconfiguration of N with pivot at Q. In case X is simple suppose that N does not contain any specialization of any augmented arrangement with pivot at Q. If X is augmented we denote the augmenting subconfiguration of X by M, and its specialization in $X^{\vee}$ by $M^{\vee}$.*

Let N' be a demodification corresponding to N, and, in case X is of type II *or* III, *let $N^{\#}$ or N'^{+} be derived from N' as described in Case* II *or* III *of Theorem* 13. *Further suppose that none of the central V_5's of N^* is q_3^N-positive. According to whether Case* I, II, *or* III *applies, we denote N', $N^{\#}$, or N'^{+} by $N^{\sim}$.*

Let Z be a central V_5 of N^* which is not the main V_5 of any specialization of any integral discharging situation in N, i.e., $q_1^N(Z) = 60$. Let F be the fractional discharging value of (Z, Q) in N (*see Definition* U), *and let $F^{\sim}$ be the fractional discharging value of (Z, Q) in $N^{\sim}$. Then either $F^{\sim} \geq F$ or N contains a geographically good configuration in Size Class $\langle 10, 23\rangle$.*

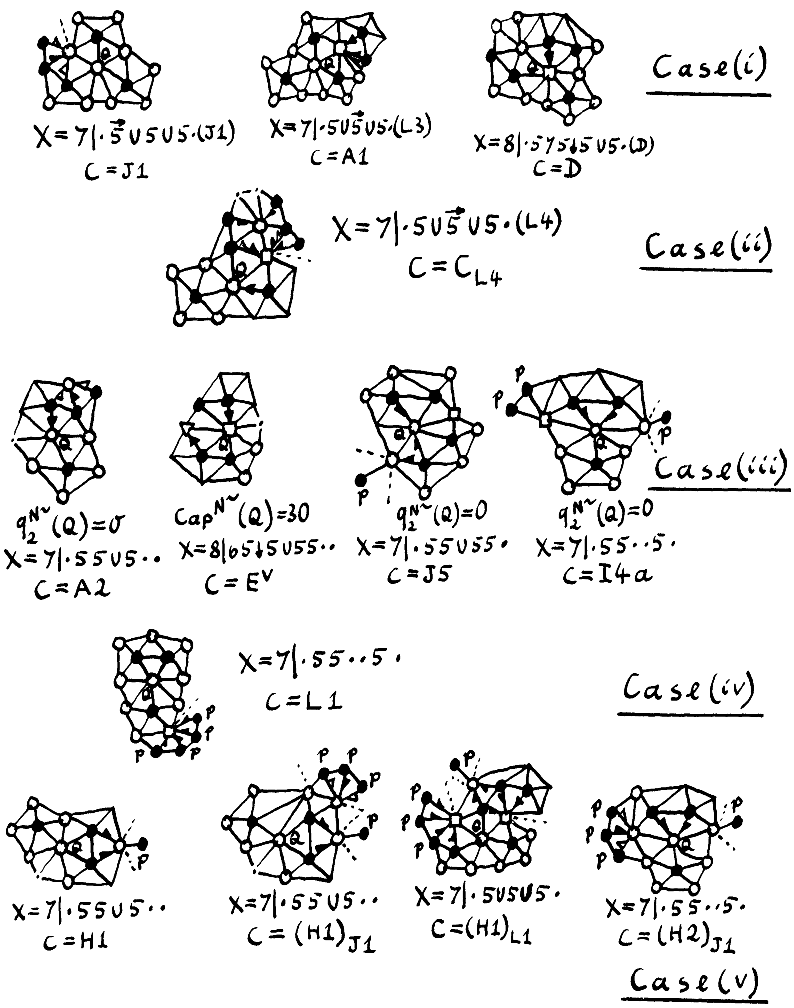


FIGURE 34

COROLLARY. *If N and $N^{\sim}$ are as in Lemma* 15 *then either N is q_3-overcharging or N contains a geographically good configuration as described in Lemma* 15.

Supplement. If N and $N^{\sim}$ are as in Lemma 15 then either $N^{\sim}$ is arranged (and moderately plugged) or N contains a geographically good configuration as described in Lemma 15.

25. Critical plugged clusters

The remaining task is to examine all q_3-overcharging arranged (augmented or nonaugmented) moderately plugged clusters $N^{\sim}$ for geographically good subconfigurations. This appears to be the largest case-enumeration problem of the procedure, and extensive computer work was required in order to find the most suitable fractional discharging algorithm (Section 19) and the most suitable list (Tables 3 and 4) of primary preliminary discharging situations. It is essential that all disastrous cases (in which a q_3-overcharging $N^\sim$ does not contain any geographically good subconfiguration) are eliminated by the preliminary discharging situations (each such $N^\sim$ now contains some primary preliminary discharging situation which prevents $N^\sim$ from being q_3-overcharging and, in particular, makes $N^\sim$ nonarranged). Moreover, many cases which would yield excessively many or excessively large geographically good configurations have been eliminated in the same way. This has reduced the complexity of the enumeration problem by an order of magnitude. For treating those cases in which $N^\sim$ corresponds to one of the four simple arrangements $7 \mid \cdot 55 \cup 5\cdot\cdot$, $7 \mid \cdot 55\cdot\cdot 5\cdot$, $7 \mid \cdot 55\cdot 55\cdot$, $8 \mid \cdot 555 \cup 55\cdot$ we have used a computer program to print the relevant cases and exhibit the geographically good subconfigurations (for more details see Section 27). For all other arrangements, relatively simple theoretical arguments are sufficient for proving the existence of geographically good subconfigurations.

In view of the fact that relatively few of the q_3-overcharging $N^\sim$'s have cores N^* with 5- or more-legger vertices, we found it most convenient to first enumerate only those basic N^*'s which do not contain any 5- or more-legger vertices. Then, in each case, we discuss those $N^\sim$'s which can be derived from N^* by adding plug-V_5's so as to obtain q_3-overcharging. Afterwards we consider those nonbasic cores which can be derived from N^* by raising the degrees of 4-legger vertices. In this section we further reduce the enumeration problem by taking care of all those q_3-overcharging $N^\sim$'s which contain some plugged V_5-neighborhood say C, such that C contains a geographically good subconfiguration. We do this by determining for each type C^* of V_5-neighborhood (which can occur in a basic cluster N^*) the maximal charge-contribution its central V_5 can make (during the fractional discharging) to the pivot Q (of any $N^\sim$ whose core contains C^*), if plug-V_5's are added to C^*. We need consider only those cases in which the added V_5's do not create a geographically good subconfiguration (see Table 9). Each basic N^* contains 3, 4, or 5 V_5-neighborhoods, and we may ignore all those N^*'s for which the sum of the maximal contributions of those V_5-neighborhoods is not greater than $|q_2(Q)|$ (which is 60 or 120). It turns out that most basic clusters N^* are noncritical in this sense and can be ignored; in fact, for some arrangements, *all* cases are noncritical.

Remark. While the argument of maximal contributions described above is extremely effective in the present context, it is expected to be much less effective in the case that we permit only geographically good configurations *without hanging pairs*. In that case, most maximal contributions will be greater than in Table 9, and more V_5-neighborhoods (namely those which contain 5-5-5 triangles) will enter into consideration. For this reason, we expect the enumeration problem in that case to be about ten times larger than in the present case (but we expect the number of configurations in the unavoidable set to increase by a much smaller factor).

DEFINITION Z. Let C^* be a V_5-neighborhood with central 5-vertex Z such that one vertex, Q, of C^* of degree 7 or 8 is specified to be the pivot of C^*. Then a rational number $\bar{F}$ is called the maximal 72-, 73-, *or* 83-*contribution of* C^* to Q if the following conditions hold.

(a) There exists an arranged, moderately plugged, V_5-neighborhood C with core C^* so that $\bar{F} = F^C(Z, Q)$ (see Definition U) and either

$$\mathrm{Deg}^C(Q) = 7, \quad L_1^C(Q) = 2$$

or

$$\mathrm{Deg}^C(Q) = 7, \quad L_1^C(Q) = 3$$

or

$$\mathrm{Deg}^C(Q) = 8, \quad L_1^C(Q) = 3,$$

respectively.

(b) There does not exist any arranged, moderately plugged, V_5-neighborhood C with the properties described in (a) except that $F^C(Z, Q) > \bar{F}$.

Let N^* be an arranged, nonaugmented cluster of V_5-neighborhoods with pivot Q. Then N^* is called *basic* if it does not contain any 5- or more-legger vertex. Suppose that $C^1, \ldots, C^v$ $(3 \le v \le 5)$ are the V_5-neighborhoods in N^* and that $Z^1, \ldots, Z^v$ are their central V_5's. Let $\bar{F}^i$ $(1 \le i \le v)$ be the maximal 72-, 73-, or 83-contribution of C^i, Q according to whether $(\deg^{N^*}(Q), L_0^{N^*}(Q))$ is (7, 2), (7, 3), or (8, 3). Then N^* is called *critical* if it wipes out and if

$$\text{(F)} \qquad \sum_{i=1}^{v} \bar{F}^i > \begin{cases} 60 & \text{if } \deg^{N^*}(Q) = 7 \\ 120 & \text{if } \deg^{N^*}(Q) = 8. \end{cases}$$

THEOREM 15. *Let N^* be a basic, arranged cluster of V_5-neighborhoods with pivot Q. Let C^* be any V_5-neighborhood in N^*. Then C^* is (degree-equivalent to) some V_5-neighborhood in that part of Table* 9 *which is marked* 72, 73, *or* 83, *according to* $(\deg^{N^*}(Q), L_0^{N^*}(Q))$ *and to which either a number* $\bar{F}_{72}$, $\bar{F}_{73}$, $\bar{F}_{83}$, *or the entry* g.g. *is assigned.*

The entry—in Table 9 means that the V_5-neighborhood is not embeddable into any *arranged* cluster; · means that it is not embeddable into any *basic, arranged* cluster. Row 1 of a V_5-neighborhood determines its *type*, Row 2 its *subtype*. Thus, for example, we may speak of a neighborhood of type 857, subtype 77.

LEMMA 16. *Let N be an arranged, moderately plugged cluster of V_5-neighborhoods with core N^* and pivot Q. Let C^* be a V_5-neighborhood in N^* with central vertex Z, and let C be the plugged V_5-neighborhood in N which has core C^* and contains all V_5-neighbors of C^* in N. Suppose that for $d = \deg^N(Q)$ and $L = L_0^N(Q)$, there is an entry* g.g. *or a number $\bar{F}_{dL}$ in the dL-part of Table* 9. *Then the following hold.*

(a) *If the entry is* g.g. *then* $C \cap N^*$ *contains a geographically good subconfiguration. Moreover, if* N^{**} *is derived from* N^* *by degree-raising of four-or more-legger vertices, then* N^{**} *contains the same geographically good configuration.*

(b) *If the entry is a number* $\bar{F}_{dL}$ *then either* $F^N(Z, Q) \leq \bar{F}_{dL}$ *or* N *contains a geographically good subconfiguration in Size Class* $\langle 5, 9\rangle$ (*or both*). *Moreover, if* N^{**} *is derived from* N^* *by degree-raising of four- or more-legger vertices, and if* $N^\wedge$ *is an arranged, moderately plugged cluster with core* N^{**} *then either* $F^{N^\wedge}(Z, Q) \leq \bar{F}_{dL}$ *or* $N^\wedge$ *contains a geographically good subconfiguration in Size Class* $\langle 5, 14\rangle$ (*or both*).

TABLE 9

$\bar{F}_{72}$, $\bar{F}_{73}$, $\bar{F}_{83}$ for

	z w		Row 3
x	5	y	Row 2
	Q		Row 1

Row 2	Row 3	55	65	56	66	75	57	76	67	77	58	68	78	88
72	655	—	g.g.	g.g.	—	g.g.	*30	30	30	$22\frac{1}{2}$	30	30	$18\frac{1}{2}$	·
	755	—	g.g.	g.g.	30	g.g.	*15	20	$22\frac{1}{2}$	15	20	$18\frac{1}{2}$	$13\frac{1}{3}$	·
	855	—	g.g.	g.g.	30	g.g.	*10	15	15	$13\frac{1}{3}$	15	10	7	·
	955	—	g.g.	g.g.	24	g.g.	*0	12	12	10	10	8	4	·
	656	g.g.	g.g.	g.g.	—	30	30	30	30	15	·	·	·	·
	756	g.g.	30	30	30	20	$22\frac{1}{2}$	15	15	10	·	·	·	·
	856	g.g.	20	30	20	15	$18\frac{1}{2}$	10	10	7	·	·	·	·
	956	g.g.	15	24	15	12	15	$7\frac{1}{2}$	$7\frac{1}{2}$	5	·	·	·	·
	757	g.g.	20	20	15	15	15	10	10	$7\frac{1}{2}$	·	·	·	·
	857	g.g.	15	15	10	$13\frac{1}{3}$	$13\frac{1}{3}$	7	7	5	·	·	·	·
	957	g.g.	12	12	$7\frac{1}{2}$	10	10	5	5	$3\frac{3}{4}$	·	·	·	·
	858	g.g.	4	4	4	7	7	4	4	$1\frac{1}{3}$	·	·	·	·
	958	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
	959	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
73	655	—	g.g.	g.g.	—	g.g.	g.g.	30	30	15	g.g.	30	15	·
	755	—	g.g.	g.g.	30	g.g.	*15	15	15	15	15	15	10	·
	855	—	g.g.	g.g.	20	g.g.	*0	10	15	10	10	4	2	·
	955	—	g.g.	g.g.	15	g.g.	*0	$7\frac{1}{2}$	12	8	8	0	0	·
	757	g.g.	g.g.	g.g.	g.g.	g.g.	g.g.	10	10	$7\frac{1}{2}$	·	·	·	·
	857	g.g.	g.g.	g.g.	10	g.g.	10	$6\frac{2}{3}$	$6\frac{2}{3}$	5	·	·	·	·
	957	g.g.	g.g.	g.g.	$7\frac{1}{2}$	g.g.	8	5	5	$3\frac{3}{4}$	·	·	·	·
	858	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
	958	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
	959	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
83	555	—	g.g.	g.g.	g.g.	g.g.	g.g.	30	30	25	g.g.	30	$22\frac{1}{2}$	15
	655	—	g.g.	g.g.	—	g.g.	g.g.	30	30	23	g.g.	30	20	·
	755	—	g.g.	g.g.	30	g.g.	*20	20	23	20	$22\frac{1}{2}$	20	16	·
	855	—	g.g.	g.g.	20	g.g.	*0	16	20	16	15	10	10	·
	955	—	g.g.	g.g.	15	g.g.	*0	15	16	16	12	$7\frac{1}{2}$	$7\frac{1}{2}$	·
	757	g.g.	g.g.	g.g.	g.g.	g.g.	g.g.	16	16	16	·	·	·	·
	857	g.g.	g.g.	g.g.	10	g.g.	16	16	16	16	·	·	·	·
	957	g.g.	g.g.	g.g.	$7\frac{1}{2}$	g.g.	16	15	15	15	·	·	·	·
	858	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
	958	g.g.	0	0	0	0	0	0	0	0	·	·	·	·
	959	g.g.	0	0	0	0	0	0	0	0	·	·	·	·

Note that in Table 9 those V_5-neighborhoods marked $*$ contain $J1$, and consequently, most of them have smaller $\bar{F}$-values than some other V_5-neighborhoods which can be obtained from them by degree-raising. But these degree-raisings cannot be induced by degree-raisings at *four- or more-legger* vertices *of* N^* and thus Lemma 16 holds without exceptions.

COROLLARY. *A basic, arranged cluster* N^* *of* V_5*-neighborhoods* $C^1, \ldots, C^v$ *with pivot* Q, $\deg^{N^*}(Q) = d$, $L_0^{N^*} = L$, *is critical only if in the dL-part of Table* 9 *there are numerical entries* $\bar{F}^i_{dL}$ *for all* C^i $(1 \leq i \leq v)$ *so that*

$$\sum_{i=1}^{v} \bar{F}^i_{dL} > \begin{cases} 60 & \text{if } d = 7 \\ 120 & \text{if } d = 8. \end{cases}$$

An arbitrary arranged cluster is critical only if it can be derived from a critical, basic cluster.

26. Discussion of the arrangements

It remains to prove the following lemma.

LEMMA 17. *Let* N^* *be an arranged, augmented or nonaugmented cluster of* V_5*-neighborhoods with pivot* Q. *If* N^* *is nonaugmented then assume that* N^* *fulfills* ($\bar{\text{F}}$) *in Definition Z. Let* N *be a moderately plugged, augmented or non-augmented cluster which contains* N^* *and suppose that* N *is* q_3*-overcharging. Then* N *contains a geographically good subconfiguration in Size Class* $\langle 7, 14\rangle$.

Proof. Let X be the arrangement of which N^* is a specialization. If X is augmented, denote the augmenting configuration by M. If X is nonaugmented and N^* is not critical then N^* contains a geographically good subconfiguration by Definition Z. In what follows we assume that N^* is critical (in case it is nonaugmented).

The main part of the work consists in the enumeration and discussion of all those cases in which X is one of the four simple arrangements

$$7 \mid \cdot 55 \cup 5 \cdot\cdot,\ 7 \mid \cdot 55 \cdot\cdot 5 \cdot,\ 7 \mid \cdot 55 \cup 55 \cdot,\ 8 \mid \cdot 555 \cup 55 \cdot;$$

this case analysis has been printed by computer as will be described in Section 27. A part of the discussion of $7 \mid \cdot 55 \cdot\cdot 5 \cdot$ has been done by hand because of some irregularities which occur in those cases in which two of the three V_5-neighborhoods C^{*1}, C^{*2}, C^{*3} in N^*, say C^{*1} and C^{*2}, have $\bar{F}$-values (from Table 9) such that $\bar{F}^1 + \bar{F}^2 = 60$. One of the most complicated cases of such an N^* is drawn in Figure 39. Here we argue as follows. N must contain plug-V_5's in three of the four positions A, B, C, D (marked in Figure 39) *or* in three of the four positions H, I, J, K, since otherwise N would not be q_3-overcharging; (recall that by hypothesis, no major vertex of N^* has load >3). But N^* with three V_5's in positions A, B, C, or D contains (in each of the four possible cases)

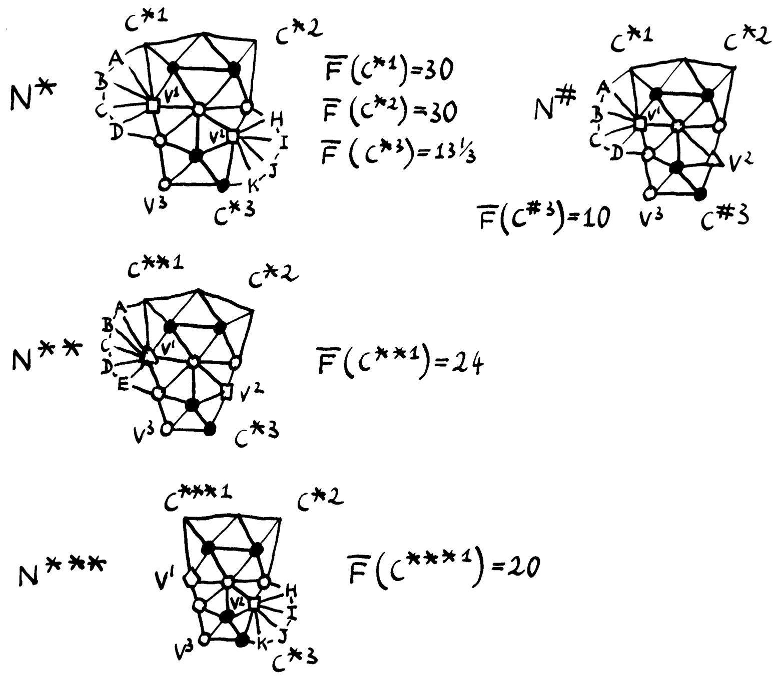


FIGURE 39

a geographically good subconfiguration, and N^* with three V_5's in positions H, I, J, or K also contains a geographically good subconfiguration. This completed the discussion of all choices of N with core N^*. Now we must discuss all critical clusters which can be derived from N^* by degree-raising at four- or more-legger vertices i.e., at V^1, V^2, or V^3 (or at any combination of these vertices). We remark that this is one of the few cases in which N^* admits infinitely many critical derivatives (the degree of V^2 may be raised arbitrarily high and a critical cluster is still obtained). The discussion is still quite simple as follows. Since none of the geographically good subconfigurations of N (as exhibited above) contains V^3, every q_3-overcharging plugged cluster, say N', with core obtained from N^* by degree-raising at V^3 contains one of the same geographically good configurations. Next we consider the cluster $N^{\#}$ of Figure 39 (degree-raising at V^2 by 1). Now any q_3-overcharging, arranged, moderately plugged cluster with core $N^{\#}$ must contain V_5's in three of positions A, B, C, D, and thus contains one of the geographically good configurations already considered above, none of which contains V^2 or V^3. Thus we need not consider any further degree-raising at V^2 or V^3. Last we must consider degree-raising at V^1. First we consider N^{**} in Figure 39 (degree raising by 1). Any q_3-overcharging, arranged, plugged cluster with core N^{**} must contain V_5's in four of the five positions A, B, C, D, E; but N^{**} with four such V_5's must always contain a geographically good subconfiguration which contains neither V^2 nor V^3. Thus the only remaining derivative is N^{***} (obtained from N^{**} by degree-raising at V^1). Here we must extend Table 9 to include the V_5-neighborhood C^{***1} (Type 10, 5, 5. Subtype 6, 6) with 72-pivot. (The $\bar{F}$-value is 20.) Any q_3-overcharging, arranged, moderately plugged cluster with core N^{***} must contain V_5's in three of the four positions H, I, J, K which yields one of the previous geographically good configurations containing neither V^1 nor V^3. Since further degree-raising at V^2 does not yield critical clusters, this completes the discussion of N^* and all its critical derivatives.

The discussion of those 45 arrangements which are different from the four mentioned above can be handled rather easily as follows.

(1) The remaining four simple arrangements

$$7 \mid \cdot 5 \cup 5 \cup 5 \cdot, \qquad 7 \mid \cup\, 55 \cup 5 \cup 5, \qquad 8 \mid \cup\, 555 \cup 5 \cup 5$$

and

$$8 \mid \cup\, 55 \cup 55 \cup 5$$

do not yield any arranged, critical clusters. This is easily seen from Table 9. For instance, suppose that $X = 7 \mid \cup\, 55 \cup 5 \cup 5$. Then (in order to be critical) N^* must contain some V_5-neighborhood, say C^{*1}, of $\bar{F}$-value >15. But in (the 73-part of) Table 9 there are only five such V_5-neighborhoods listed. Four of them yield geographically good subconfigurations in N^*; the only one which does not is Type 855, Subtype 66 with $\bar{F}^1 = 20$. But if this V_5-neighborhood is in N^* then some other V_5-neighborhood in N^*, say C^{*2}, must be of type $85x$ with $x > 5$ and thus has $\bar{F}^2 \leq 10$; therefore there must be a C^{*3} with $\bar{F}^3 > 15$ and this must not be of Type 855, Subtype 66 (since otherwise $\bar{F}^4$ would be ≤ 10 and N^* would not be critical). This completes the discussion.

(2) The twenty-four arrangements $7 \mid \cdot \vec{5} \cup 5 \cup 5 \cdot$ and $7 \mid \cdot 5 \cup \vec{5} \cup 5 \cdot$ do not yield any N whose augmented core, N^*, wipes out. The discussion uses maximal-contribution arguments; for that one V_5-neighborhood in N^* which does not contain any discharging or receiving vertex of M we may directly use the $\bar{F}_{72}$-value from Table 9. By reasoning similar to (1), it is found that every possible N^* contains a geographically good subconfiguration.

(3) The remaining seventeen augmented arrangements correspond to simple arrangements in a rather obvious way as indicated in Figure 40. If N^* is a specialization of one of these augmented arrangements then it can be transformed into the corresponding nonaugmented cluster, say $N^{*\#}$, (by a transformation which is defined as in Figure 40). Moreover, any N with core N^* can be transformed into a corresponding $N^{\#}$ with core $N^{*\#}$, such that $N^{\#}$ is also q_3-overcharging, arranged, and moderately plugged. Then $N^{\#}$ contains a geographically good subconfiguration, say $C^{\#}$, which is exhibited in the discussion of the simple arrangements. Then the "inverse transformation" from $N^{\#}$ to N transforms $C^{\#}$ into a geographically good subconfiguration of N as demanded.

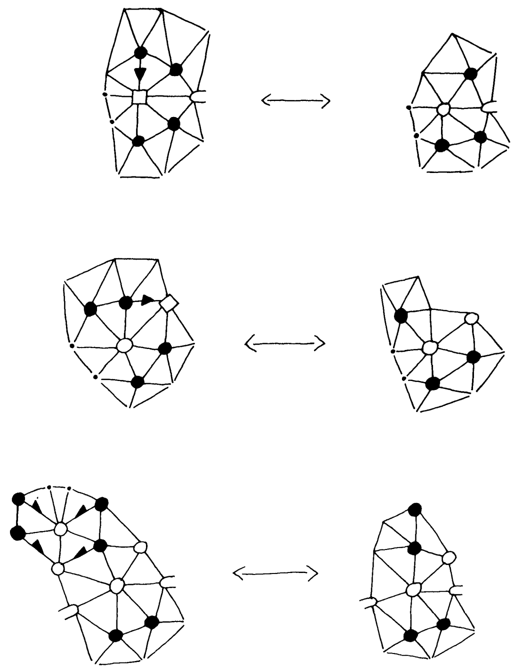


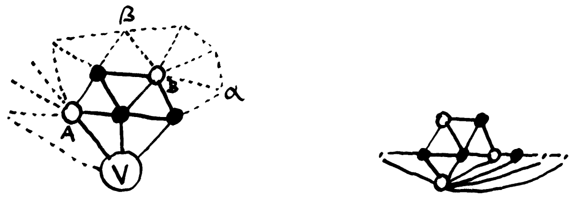
FIGURE 40

27. The use of the computer on arrangements $7 \mid \cdot 55 \cup 5\cdot$, $7 \mid \cdot 55\cdot\cdot 5\cdot$, $7 \mid \cdot 55 \cup 55\cdot$, and $8 \mid \cdot 555 \cup 55\cdot$

We restrict ourselves to those properties of the computer programs required in the proof of Lemma 17.

Description of the proof by computer. In the terminology of Lemma 17 we will show that N contains a geographically good subconfiguration. We do this by considering all possible combinations of V_5-neighborhoods in the core of such a cluster and either showing that no possible addition of plug-V_5's can lead to q_3-overcharging or examining the possible pluggings and showing that any such q_3-overcharging plugged cluster with this core contains a geographically good subconfiguration. We adopt the following terminology. The cluster is thought of as being laid out in rows (as this is most convenient for computer output). The first row consists of the pivot Q alone, the second row consists of the neighbors of the pivot (in some cyclic order) and in general the nth row consists of those neighbors of vertices in the $(n - 1)$st row which do not occur in the first $(n - 1)$ rows. Figure 43b shows the layout of the core of the cluster in Figure 43e in rows.

The V_5-neighborhoods inherit a row-arrangement in which the pivot is row 1, the central vertex and its two neighbors which are also neighbors of the pivot



Type 5-2
(a)

Type 5-2 in
Arrangement 7|·55∪5··
(b)

FIGURE 41

are on row 2 and the remaining two neighbors of the central vertex are on row 3. (This notation has previously been used in Table 9; indeed, these considerations motivate the notation in Table 9). V_5-neighborhoods fall into three natural classes according to row-2 pattern. Class 1 has three V_5's on row 2. The second V_5 in arrangement 8 | ·555 ∪ 55· must have a neighborhood in Class 1. Class 2 has two V_5's on row 2; by symmetry, we list only those of the form $X55$ (where X denotes a vertex of degree at least six) and call those of the form $55X$ mirror-image symmetries of those listed. Thus the first V_5 in arrangement 7 | ·55··5· must have a neighborhood in Class 2 while the second must have a neighborhood symmetric to one in Class 2. Class 3 neighborhoods have second rows of the form $X5X$. Hence, given an arrangement we can determine the neighborhood class of each central V_5. By a *plugged V_5-neighborhood* we mean (in what follows) that subconfiguration of the plugged cluster N consisting of the V_5-neighborhood and all plug-V_5's of N adjacent to this V_5-neighborhood. (Note that plug-neighbors cannot occur on row 2.)

Henceforth we will call a V_5-neighborhood a *type*. Given a type we can make a first calculation of the maximal possible contributions of its central V_5 to the pivot by considering all plugged V_5-neighborhoods of which it is the core and applying the fractional discharging algorithm to each, choosing the largest value so obtained. However there are some refinements we must make in this procedure. In Figure 41a, vertex B receives 30 from the central V_5 by the partial discharging procedure (Situation $J1$) and this precedes (and hence overrides) the fractional discharging algorithm. Also, if either vertex α or β were made a plug-V_5 the plugged V_5-neighborhood itself would contain a geographically good subconfiguration. Hence we may take this information into account to calculate the maximal possible contribution of the type to the pivot of N. We have done this (by hand) for each type which may occur in a basic critical cluster in each of the four arrangements (Table 9). These types and their maximal contributions are listed on the first page of the computer output corresponding to the arrangement. There are a number of simplifications to reduce the analysis which must be made by the computer. The initial simplifications concern the discarding of certain types or combinations of types in various arrangements.

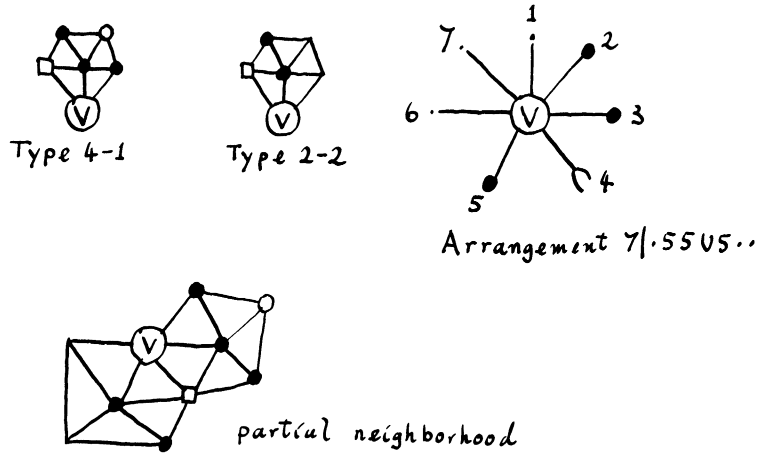


FIGURE 42

The most important such simplification is to ignore all clusters which contain preliminary discharging situations which cause the cluster to be noncritical; (recall that the preliminary discharging situations have been introduced precisely for the purpose of eliminating all undesirable clusters). We also ignore clusters which contain geographically good subconfigurations (from a previously compiled list) and clusters which are not basic. The latter must be considered eventually but their analyses are immediate consequences of the analyses of the basic clusters. The method of performing this secondary analysis will be shown in our example (compare also with (1) in Section 26).

Thus, in our analysis by computer, not all types can appear in all positions in our arrangements which their classes would appear to permit them. All types with vertices of degree greater than nine are excluded by the *l*-legger conditions. Certain types with vertices of degree nine or eight are similarly excluded if these vertices appear in positions in the arrangement in which they must be *l*-leggers. We list some other eliminations which may be made.

For example, the second V_5 in arrangement $7 \mid \cdot 55 \cup 5 \cdot \cdot$ cannot have the type symmetric to 5-2 of Figure 41. For the resulting core would have a geographically good subconfiguration (heavy in Figure 41b). Also certain pairs cannot occur in certain positions which their classes would appear to entitle them to without causing geographically good subconfigurations of the core. For example, if we make the symmetric version of Type 4-1 the neighborhood of the second V_5 in arrangement $7 \mid \cdot 55 \cup 5 \cdot \cdot$ and Type 2-2 the neighborhood of the third V_5 (see Figure 42) a nonwipeout core results. These eliminations are computed by hand and constitute part of the input to the computer program. They are also listed at the beginning of the output.

Of the many possible clusters built from the types which it considers, the program can eliminate most quite easily by considerations of total contribution

a) -6 57-- 2-
-1 5-- 5-

b) ---- 7 7 6 -- ---- 6 5-
8 5 5 7 8 5 6
7

c) Maximal possible contribution sum 65.83
Contributions 13.33 22.50 30.00

d) 25 26 27 28 29 30 31 32
9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
2 3 4 5 6 7 8
1

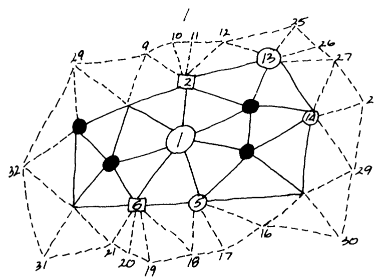
e) 

FIGURE 43

to the pivot (or more simply yet of conflicting assignments of vertex degrees). Several hundred remain critical after these initial considerations and must be treated in greater detail. It is easiest to see how the remaining cases are handled by an example. (The computer was forced to find an unnaturally complicated argument for illustrative purposes and the alert reader will probably see a simpler analysis. The purpose of the programming used for this argument is to permit the choice of small geographically good configurations when we actually construct the unavoidable sets.) Figures 43 and 44 (excluding 43e) show the printout and give some evidence that it is not necessary to know what the computer did in order to easily check the work by hand. Figure 43e is not printed but is drawn to aid the reader in interpretation of the output.

Note that we are dealing with a 7 | ·55··5· arrangement and that Type 6-1 has the first V_5 of the arrangement as central vertex, Type 7-5 in reversed position has the second V_5 as central vertex and Type 2-5 has the third V_5 as central vertex. (These may be found on the initial sheet of the printout for checking).

Figure 43b gives a diagram of the first and second neighbors of the pivot. This corresponds to the heavy part of Figure 43e. Vertex degrees are given for specified vertices. In Figure 43b the vertices on Row 2 (from the bottom) are the neighbors of the pivot (bottom row). Those on Row 3 are neighbors of vertices on Row 2 but are not the pivot or its neighbors. Adjacency is determined as follows. A vertex W on Row 3 is adjacent to a vertex V on Row 2 if W is (a) printed to the right of the left neighbor of V on Row 2 and (b) printed no further right than one place to the right of V. Vertices on a given row which are neighbors of a common vertex on a lower row are printed in clockwise order.

f) Addition of Pair 9,10 makes sub-configuration geographically good

```
g)      5 5 - -  - - -      5 -
              8 5 5 - - 5   6      m=9   n=12
                  7
```

h) Addition of Triple 9,11,12 makes sub-configuration geographically good

```
      5 - 5 5  -  -  -      -  5  -
              8  5  5  -  -  5   6   m=10  n=113
                    7
```

i) Addition of Triple 10,11,12 makes sub-configuration geographically good

```
      - 5 5 5  -  -  -      -  5  -
              8  5  5  -  -  5  6   m=10  n=113
                    7
```

j) Bound 65.83 Pivot gets 59.17

Contributions 6.67 22.50 30.00

Loads 2 1 4 3 2 2

FIGURE 44

Figure 43c gives the maximal possible contributions to the pivot of the types used and shows the configuration critical. For the remainder of the analysis it will be necessary for the program to assign vertex numbers to all vertices which either appear in Figure 43b or are neighbors of these vertices. This is printed as 43d (with the same conventions as in 43b), again see 43e for aid in interpretation.

The next observation is that (Figure 44f) if both Vertex 9 and Vertex 10 are plug-V 5's then the neighborhood contains a geographically good configuration (a smallest such is shown in Figure 44g with the same conventions as Figure 43b). The m and n are the number of vertices and of immediate neighbors of the subconfiguration—if the configuration is articulated (for reasons of technical convenience) 100 is added to n. The program records the fact that this means that Vertex 2 has load at most 2. Since this does not reduce the total contribution to the pivot below 60 (it actually reduces the pivot contribution of Vertex 3 to 10) it does not choose to print the information.

Next, it discovers that two triples of vertices (9, 11, 12 and 10, 11, 12) (Figure 44h, i) each lead to a geographically good subconfiguration and hence their joint appearances as plug-vertices can be ruled out. But this means that q_3-overcharging is no longer possible as may be seen in Figure 44j and the case is completed. (We have ignored some additional printout which is extraneous to our example).

The techniques used in our example include many of those used in the computation. Others include elimination by geographical goodness of the addition of a single vertex or of a collection of more than three vertices (no proper subset of which has otherwise been eliminated). An analysis can also terminate if additions required to satisfy the load conditions which make possible an overcharge also force the configuration to be geographically good.

The program prints the following information pertinent to the proof of Lemma 17. First, the types which are considered, the associated maximal pivot contributions and forbidden plug-neighbors, and the types or pairs of types eliminated in various position. The outputs are hand annotated to show the reasons for either total elimination of certain types or the positional elimination. Next the arrangement is given, followed, in lexicographic order by analyses of all cores determined by these types in this arrangement which are not eliminated on the basis of maximal pivot contributions or other eliminations in the input. It is not hard to check whether any case is missing and, of course, this has been done by hand. We have four distinct types of analyses. The first is the standard analysis given in our example. We call these S-analyses. A second type consists of those cores which themselves contain geographically good subconfiguration. To handle these we need only print the subconfiguration for the unavoidable set (although, when asked to try to find all elements of the unavoidable set with n bounded by a certain threshold (if possible) the program will sometimes go through a more sophisticated analysis). We call these O-analyses. In a third type of analysis the program notices reasons, not evident from the input, that q_3-overcharging is not possible. The reasons include the effects of partial dischargings or forbidden plug-neighbors when the entire core is considered. These are called E-analyses. A fourth type of analysis, called an R-analysis occurs when the program initially notes that the required plug-V_5's cause the plugged cluster to be a modification of a simpler cluster and hence unnecessary to treat separately. At the conclusion of the analyses for an arrangement, the program prints the number of analyses of each of the types, S, O, E, and R, and the number of elements added to the verifying sets obtained from previous arrangements.

For our four arrangements the figures are as follows.

	Type O	Type E	Type R	Type S	additions
$7 \mid \cdot 55 \cup 5 \cdot \cdot$	33	4	2	0	20
$7 \mid \cdot 55 \cdot \cdot 5 \cdot$	79	31	15	11	36
$7 \mid \cdot 55 \cup 55 \cdot$	14	17	0	0	9
$8 \mid \cdot 555 \cup 55 \cdot$	10	30	0	0	6

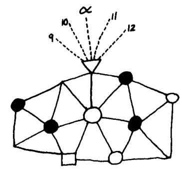


FIGURE 45

Now, to complete the proof of Lemma 17 we must justify the exclusion of l-legger vertices with $l > 4$. Basically the argument is to associate with each cluster containing such vertices another cluster without them such that the analyses will correspond. Suppose we have a critical cluster with a vertex V_k, $k > 7$, which is a $(4 + i)$-legger, $i > 0$. We consider the cluster obtained by replacing it by a V_{k-i} (which is a 4-legger) and making similar replacements at all other such vertices. For example if we are given the critical cluster of Figure 45 we note that the V_9 is a 5-legger and if it is replaced by a V_8 the cluster is that of Figure 43 and our example. Thus we perform a parallel analysis as follows.

First, in Figure 43c we note that the contribution from the first central V_5 (Position 3) is at most 10 (by Table 9) so the total contribution is at most 62.5. In Figure 43d we would add a leg-position (say between 10 and 11) and call it α (to avoid renumbering everything). Now lines corresponding to Figure 44 f, g, h are essentially unchanged (except for contributions as noted above). In 44i we may replace pair 9, 10 by triple 9, 10, α and make the appropriate change in the verifying configuration in Figure 44j. The additions in Figure 44 k and l may be replaced by quadruples 9, α, 11, 12, and α, 10, 11, 12, and 9, 10, 11, 12 with the corresponding configurations. To illustrate two other facets of the procedure, assume that the V_9 of Figure 45 was a V_{10}. Again the same type of analysis could be made but now the contribution of the first central vertex is at most 8.56 and the total contribution is 61.06. Thus after completing the part of the analysis through Figure 44 and obtaining a single quadruple, the contribution maximum drops to 59.16 and the case is completed.

Returning to Figure 44 we note that the three configurations chosen for the unavoidable set each have only one major vertex other than the pivot, namely Vertex 2 which is an 8-vertex. This means that whenever a cluster is changed to Figure 43b by replacement of l-leggers, $l > 4$ by 4-leggers, the only vertex which nontrivially enters the analysis is Vertex 2. (Note that although in Figure 44f, Vertex 5 is used, it does not nontrivially contribute to the analysis and could have been eliminated. It was not eliminated since it did not increase the size of the unavoidable set.) Thus all other major vertices which have been modified to obtain the cluster in Figure 43b may be ignored as the analyses above apply (a fortiori).

Thus the computer output constitutes a proof of Lemma 17 for the case of the four arrangements considered. ■

28. Conclusion

Let $\mathscr{V}'$ be the set of those geographically good configurations exhibited in Lemmas 16 and 17. Let $\mathscr{V}$ be a set of geographically good configurations such that every configuration, say C, which is derived from any member, say C', of $\mathscr{V}'$ by 6-to-5 degree-lowering and modification (according to Lemmas 3 and 4) (C may be identical to C') contains at least one member of $\mathscr{V}$. Let $\mathscr{U}$ be the set of those geographically good configurations exhibited in Lemmas 11, 12, 14, 15, and in $\mathscr{V}$.

If N is a q_3-neighborhood as in the hypothesis of Main Lemma 2 then by Theorems 13 and 15, N contains at least one member of $\mathscr{U}$. This completes the proof of Main Lemma 2.

Now the Discharging Lemma is a corollary of Main Lemma 2.

DISCHARGING LEMMA. *If T is a triangulation (according to the Convention in Section 4) of a 2-manifold M^2 then either the charged triangulation (T, q_3^T) (according to Definitions* A, Q, *and* U) *is completely discharged ($q_3^T(V) \leq 0$ for every vertex $V \in T$), or T contains at least one member of the set $\mathscr{U}$ of geographically good configurations described above ($\mathscr{U} \subset \langle 14, 23\rangle$), or both.*

Proof. This follows immediately from Main Lemma 2 by Theorem 12. ∎

COROLLARY. *If T is a triangulation of a 2-sphere or a projective plane then T contains at least one member of the set $\mathscr{U}$ of geographically good configurations described above. In particular, $\mathscr{U}$ is an unavoidable set for planar triangulations.*

Proof. This follows immediately from the Discharging Lemma since the Euler Characteristics of the 2-sphere and the projective plane are positive. ∎

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