M-PROJECTIVE AND STRONGLY M-PROJECTIVE MODULES

BY

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Introduction

Given a module M over a ring R, G. Azumaya [1] introduced the dual notions of M-projective and M-injective modules. These concepts have actually led M. S. Shrikhande to a study of hereditary and cohereditary modules [5]. More recently Azumaya, Mbuntum and the present author obtained necessary and sufficient conditions for the direct sum $\oplus_{\alpha \in J} A_\alpha$ of a family of modules to be M-injective [2]. While R-injective modules are the same as injective modules over R, the class of R-projective modules in the sense of Azumaya in general is larger than the class of projective R-modules. In this paper we introduce the notion of a strongly M-projective module and the associated notion of a strong M-projective cover. Next we investigate strong M-projective covers. We show that if every module possesses a strong M-projective cover then $R/\mathfrak{A}(M)$ is (left) perfect, where $\mathfrak{A}(M)$ is the annihilator of M. If $R/\mathfrak{A}(M)$ is perfect, we show that every R-module A with $t_M(A) = 0$ possesses a strong M-projective cover, where

$$t_M(A) = \{x \in A \mid f(x) = 0 \text{ for all } f \in \text{Hom}(A, M)\}.$$

Another application of the ideas here is the result that if $\mathfrak{A}(M) = 0$, then an R-module B is strongly M-projective iff B is projective. In particular if R is (left) perfect and $\mathfrak{A}(M) = 0$, then an R-module B is M-projective iff B is actually projective. Since $\mathfrak{A}(R) = 0$, we can regard this result as a generalization of the "known" result that when R is perfect an R-module is R-projective iff it is projective. It will be interesting to characterise the rings with the property that R-projective modules are the same as the projective modules over R.

1. Preliminaries

Throughout this paper R denotes a ring with $1 \neq 0$, R-mod the category of unital left modules. All the modules we deal with are unital left modules. M denotes a fixed object in R-mod. We recall briefly the concepts of M-projective and M-injective modules introduced by G. Azumaya and state two results due to him [1].

DEFINITION 1.1. A module P is called M-projective if given any eipmorphism $\phi: M \to N$ and any $f: P \to N$, there exists a $g: P \to M$ such that $\phi \circ g = f$.

Received July 2, 1975.

[1] Research done while the author was partially supported by a NRC grant.

An M-injective module is defined dually.

DEFINITION 1.2. An epimorphism $\psi: A \to B$ is called an M-epimorphism if there exists a map $h: A \to M$ such that $\ker \psi \cap \ker h = 0$.

M-monomorphisms are defined dually.

PROPOSITION 1.3. [1] *Let $P \in R$-mod. Then the following statements are equivalent.*

(1) *P is M-projective.*

(2) *Given any M-epimorphism $\psi: A \to B$ and any $f: P \to B$, there exists a $g: P \to A$ such that $\psi \circ g = f$.*

(3) *Every M-epimorphism onto P splits.*

The dual of this proposition characterises M-injective modules.

DEFINITION 1.4. $C_p(M)$ is the class of all M-projective modules, $C_i(M)$ is the class of all M-injective modules. For any $A \in R$-mod,

$$C^p(A) = \{M \in R\text{-mod} \mid A \text{ is } M\text{-projective}\}$$

and

$$C^i(A) = \{M \in R\text{-mod} \mid A \text{ is } M\text{-injective}\}.$$

PROPOSITION 1.5. [1] (1) *$C_p(M)$ is closed under the formation of direct sums and direct summands.*

(2) *$C_i(M)$ is closed under the formation of direct products and direct factors.*

(3) *$C^p(A)$ is closed under submodules, homomorphic images and formation of finite direct sums. If A has a projective cover, $C^p(A)$ is closed under the formation of arbitrary direct products (and hence arbitrary direct sums as well).*

(4) *$C^i(A)$ is closed under submodules, homomorphic images and arbitrary direct sums.*

In this paper the term R-projective module will be used to denote a module which is R-projective in the sense of Definition 1.1. As has already been pointed out in [2] the class of R-projective modules in general is larger than the class of projective R-modules.

LEMMA 1.6. *Let $A \in C_p(M)$, $K \subset A$ and $i: K \to A$ the inclusion.* If

$$i^*: \text{Hom}(A, M) \to \text{Hom}(K, M)$$

is the zero map then $A/K \in C_p(M)$.

Proof. Write B for A/K and let $\eta: A \to B$ denote the canonical quotient map. Let $\phi: M \to N$ be any epimorphism and $f: B \to N$ any map. Since $A \in C_p(M)$, there exists a map $g: A \to M$ such that $\phi \circ g = f \circ \eta$. Now, $g \circ i = i^*(g) = 0$. Hence g induces a map $\bar{g}: B \to M$ satisfying $\bar{g} \circ \eta = g$. It is clear that $\phi \circ \bar{g} = f$.

Recall that an epimorphism $\alpha: A \to B$ is called minimal if Ker α is small in A.

LEMMA 1.7. *Any minimal M-epimorphism $\alpha: A \to B$ with $B \in C_p(M)$ is an isomorphism.*

Proof. By (3) of Proposition 1.3, α splits. Thus ker α is a direct summand of A. Since ker α is small in A we see that ker $\alpha = 0$.

LEMMA 1.8. *Let*

$$0 \longrightarrow K \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$$

be exact with $i(K)$ small in A. If $B \in C_p(M)$, then i^*: Hom $(A,M) \to$ Hom (K, M) *is the zero map.*

Proof. Let $f \in$ Hom (A, M). Writing L for $K \cap \ker f$ we get an exact sequence

$$0 \longrightarrow K/L \xrightarrow{\bar{\imath}} A/L \xrightarrow{\bar{\phi}} B \longrightarrow 0$$

where $\bar{\imath}$ and $\bar{\phi}$ are induced by i and ϕ respectively. If $\bar{f}: A/L \to M$ is induced by f, it is clear that $\ker \bar{f} \cap \ker \bar{\phi} = 0$. Thus $\bar{\phi}: A/L \to B$ is an M-epimorphism. Moreover $\bar{\imath}(K/L)$ is small in A/L. Lemma 1.7 now implies that $\bar{\phi}$ is an isomorphism and hence $K/L = 0$. Thus, $L = K$ and $i^*(f) = f \circ i = f/K = 0$.

2. Strongly M-projective modules

Given any set J and any $A \in R$-mod, we write A^J for the direct product $\prod_{\alpha \in J} A_\alpha$ and $A^{(J)}$ for the direct sum $\oplus_{\alpha \in J} A_\alpha$, where $A_\alpha = A$ for each $\alpha \in J$. The annihilator of A will be denoted by $\mathfrak{A}(A)$.

DEFINITION 2.1 A module A is called strongly M-projective if $A \in C_p(M^J)$ for every indexing set J.

Trivially every projective module is strongly M-projective for every $M \in R$-mod. From the second half of (3) of Proposition 1.5 we get the following as an immediate consequence.

LEMMA 2.2 *Let $A \in C_p(M)$. If A possesses a projective cover, then A is strongly M-projective.*

DEFINITION 2.3. A submodule K of A is said to be M-independent in A if given any $x \neq 0$ in K, there exists an $f \in$ Hom (A, M) such that $f(x) \neq 0$.

If $K = 0$, the condition stated in Definition 2.3 is emptily satisfied. Also if $L \subset K \subset B \subset A$ and K is M-independent in A, then trivially L is seen to be M-independent in B.

DEFINITION 2.4. A homomorphism $f: A \to B$ is called M-independent if ker f is M-independent in A.

LEMMA 2.5. *Let $\phi: A \to B$ be an M-independent epimorphism and $L =$* ker ϕ. *Then ϕ is an M^L-epimorphism.*

Proof. For any $x \neq 0$ in L let $f_x\colon A \to M$ be such that $f_x(x) \neq 0$. Let $f_0\colon A \to M$ be the zero map. Let $h\colon A \to M^L$ be defined by $h(a) = (f_x(a))_{x \in L}$. Then $\ker h \cap \ker \phi = 0$.

For any $A \in R$-mod, let $t_M(A) = \{x \in A \mid f(x) = 0 \text{ for all } f \in \operatorname{Hom}(A, M)\}$. Then $t_M(R) = \mathfrak{A}(M)$. It is clear that A is M-independent in itself if and only if $t_M(A) = 0$.

DEFINITION 2.6. An object $A \in R$-mod is called M-independent if $t_M(A) = 0$.

Remark 2.7. (a) Given $x \in A$ with $x \notin t_M(A)$, there exists an $f\colon A \to M$ with $f(x) \neq 0$. Since $f/t_M(A) = 0$, we get an induced map $\bar{f}\colon A/t_M(A) \to M$. Clearly $\bar{f}(x + t_M(A)) \neq 0$. Thus $A/t_M(A)$ is M-independent in itself. In otherwords $t_M(A/t_M(A)) = 0$. For any $g\colon A \to B$ it is clear that $g(t_M(A)) \subset t_M(B)$. Thus t_M is a radical on R-mod in the sense of Bo-Stenström [6, Chap 1]. However, t_M is neither left exact, nor idempotent. For instance consider $t = t_{Z_p}$ on Z-mod, where $Z_p = Z/pZ$. Then $t(Z) = pZ$, $t(pZ) = p^2Z$. Thus

$$t(Z) \cap pZ = pZ \neq p^2Z = t(pZ).$$

Also $t(t(Z)) = p^2Z \neq t(Z)$. This is just to impress upon the reader that M-projectivity and M-injectivity can not in general be "subsumed" under "torsion theories".

(b) When M is injective t_M is the radical associated to a hereditary torsion theory on R-mod.

It is easily seen that every $A \in R$-mod is M-projective iff M is semi-simple iff every $A \in R$-mod is M-injective. The next theorem gives conditions under which every $A \in R$-mod is strongly M-projective.

THEOREM 2.8. *The following statements are equivalent.*

(1) *Every R-module is strongly M-projective.*

(2) *Every cyclic R-module is strongly M-projective.*

(3) *$R/\mathfrak{A}(M)$ is a semisimple Artinian ring.*

(4) *M^J is a semisimple R-module for every indexing set J.*

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3). Any left ideal of $R/\mathfrak{A}(M)$ is of the form $I/\mathfrak{A}(M)$ with I a left ideal of R satisfying $I \supset \mathfrak{A}(M)$. Let $\eta\colon R/\mathfrak{A}(M) \to R/I$ denote the quotient map. Then $\ker \eta = I/\mathfrak{A}(M)$. Since $R/\mathfrak{A}(M)$ is M-independent in itself it follows that $I/\mathfrak{A}(M)$ is M-independent in $R/\mathfrak{A}(M)$. If we write K for $I/\mathfrak{A}(M)$, from Lemma 2.5 it follows that η is an M^K-epimorphism. Assumption (2) implies that $R/I \in C_p(M^K)$. An application of (3), Proposition 1.3 shows that $\eta\colon R/\mathfrak{A}(M) \to R/I$ splits in R-mod and hence in $R/\mathfrak{A}(M)$-mod. Thus $I/\mathfrak{A}(M)$ is a direct summand of $R/\mathfrak{A}(M)$ as an $R/\mathfrak{A}(M)$-module.

(3) ⇒ (4). Since $\mathfrak{A}(M)M^J = 0$ (for any indexing set J) we can regard M^J as an $R/\mathfrak{A}(M)$-module. The R-submodules of M^J are the same as the $R/\mathfrak{A}(M)$

submodules of M^J. The semisimplicity of $R/\mathfrak{A}(M)$ implies that M^J is semisimple as an $R/\mathfrak{A}(M)$-module and hence as an R-module also.

(4) $\Rightarrow$ (1) is trivial.

Remark 2.9. $M = \oplus_p Z_p$ (direct sum over all the primes p) is an example of a semisimple Z-module for which $Z/\mathfrak{A}(M) = Z$ is not semisimple.

PROPOSITION 2.10. *If every M-independent R-module is injective then $R/\mathfrak{A}(M)$ is a semisimple ring.*

Proof. Since $R/\mathfrak{A}(M)$ is M-independent, any left ideal of $R/\mathfrak{A}(M)$ being a submodule of $R/\mathfrak{A}(M)$ is M-independent, and hence injective as an R-module. Thus every left ideal of $R/\mathfrak{A}(M)$ is an R-direct summand and hence an $R/\mathfrak{A}(M)$ direct summand of $R/\mathfrak{A}(M)$.

LEMMA 2.11. *For any $A \in R$-*mod *we have $\mathfrak{A}(M)A \subset t_M(A)$.*

Proof. Trivial.

Remark 2.12. If A is any M-independent R-module, from Lemma 2.11 we see that $\mathfrak{A}(M)A = 0$. Hence A can be regarded as an $R/\mathfrak{A}(M)$-module in a natural way. If $R/\mathfrak{A}(M)$ is semisimple Artin (as a ring) then A is injective as an $R/\mathfrak{A}(M)$-module. But in general A need not be injective as an R-module. Thus the converse of Proposition 2.10 is not true. For instance let $M = Z_p$ in Z-mod and $A = Z_p$. Then $\mathfrak{A}(M) = pZ$ and $Z/\mathfrak{A}(M) = Z_p$ is a field. Also $t_M(Z_p) = t_{Z_p}(Z_p) = 0$. However Z_p is not injective as a Z-module.

When M is an injective R-module the converse of Proposition 2.10 is valid.

PROPOSITION 2.13. *Let M be an injective R-module such that $R/\mathfrak{A}(M)$ is a semisimple ring. Then any M-independent R-module is injective*

Proof. Let A be any M-independent R-module. Let I be any left ideal in R and $f: I \to A$ any map. We will show that $f(I \cap \mathfrak{A}(M)) = 0$ using the fact that M is an injective R-module. Suppose on the contrary $f(\lambda) \neq 0$ for some $\lambda \in I \cap \mathfrak{A}(M)$. Since $t_M(A) = 0$ we can find a $g: A \to M$ with $g(f(\lambda)) \neq 0$. Since M is injective, there exists an $h: R \to M$ such that $h \mid I = g \circ f$. Then $0 \neq g(f(\lambda)) = h(\lambda) = h(\lambda \cdot 1) = \lambda h(1) = 0$ since $\lambda \in \mathfrak{A}(M)$ and $h(1) \in M$. This contradiction shows that $f(I \cap \mathfrak{A}(M)) = 0$.

Thus f induces a map $\bar{f}: I/I \cap \mathfrak{A}(M) \to A$. Clearly $\bar{f}$ is an $R/\mathfrak{A}(M)$-map. The semisimplicity of $R/\mathfrak{A}(M)$ implies that $\bar{f}$ can be extended to an $R/\mathfrak{A}(M)$ homomorphism $\theta: R/\mathfrak{A}(M) \to A$. If $\eta: R \to R/\mathfrak{A}(M)$ is the canonical quotient map, then it is clear that $\theta \circ \eta: R \to A$ is an R-homomorphism extending $f: I \to A$. Thus A is an injective R-module.

Combining Propositions 2.10 and 2.13 we get the following:

COROLLARY 2.14. *When M is injective, each of the statements* (1), (2), (3), (4) *of Theorem* 2.8 *is equivalent to* (5) *stated below:*

(5) *Every M-independent R-module is injective.*

3. Strong M-projective covers

DEFINITION 3.1. A minimal epimorphism $\alpha: A \to B$ is called a strong M-projective cover if

(1) A is strongly M-projective and

(2) α is M-independent (in the sense of Definition 2.4)

As in the case of projective covers, strong M-projective covers do not exist in general. Conditions for existence will be investigated presently. But before that we will prove the essential uniqueness of a strong M-projective cover when it exists.

LEMMA 3.2. *Suppose $\alpha: A \to B$ is a strong M-projective cover and $\pi: P \to B$ an epimorphism with P strongly M-projective. Then there exists an epimorphism $h: P \to A$ satisfying $\alpha \circ h = \pi$.*

Proof. Let $L = \ker \alpha$. Since α is M-independent, from Lemma 2.5 we see that α is an M^L-epimorphism. Since $P \in C_p(M^L)$, by (2) of Proposition 1.3 we get a map $h: P \to A$ satisfying $\alpha \circ h = \pi$. Since π is onto, we get $Imh + L = A$. The smallness of L in A gives $Imh = A$.

PROPOSITION 3.3. *Suppose $\alpha_1: A_1 \to B$, $\alpha_2: A_2 \to B$ are any two strong M-projective covers of B. Then there exists an isomorphism $h: A_1 \to A_2$ such that $\alpha_2 \circ h = \alpha_1$.*

Proof. By Lemma 3.2, there exists an epimorphism $h: A_1 \to A_2$ satisfying $\alpha_2 \circ h = \alpha_1$. If $K_1 = \ker \alpha_1$, $K = \ker h$ from $\alpha_2 \circ h = \alpha_1$ we immediately get $K \subset K_1$. Hence K is M-independent in A_1 and is also small in A_1. Lemma 2.5 now implies that h is a minimal M^K-epimorphism. Since $A_2 \in C_p(M^K)$, an application of Lemma 1.7 yields that h is an isomorphism.

We next show that any $B \in R$-mod which possesses a projective cover automatically admits a strong M-projective cover. We will actually indicate a method of constructing a strong M-projective cover of B from a given projective cover of B.

THEOREM 3.4. *Suppose B has a projective cover $\pi: P \to B$. Let $L = \ker \pi$ and*

$$T = \{x \in L \mid f(x) = 0 \text{ for all } f \in \text{Hom}(P, M)\}.$$

Let $\alpha: P/T \to B$ be the map induced by π. Then $\alpha: P/T \to B$ is a strong M-projective cover of B.

Proof. If $i: T \to P$ denotes the inclusion of T in P, from the very definition of T we have $i^*: \text{Hom}(P, M) \to \text{Hom}(T, M)$ to be the zero homomorphism. By Lemma 1.6 we see that $P/T \in C_p(M)$. Clearly T is small in P. Hence the canonical quotient map $\eta: P \to P/T$ is a projective cover of P/T. Lemma 2.2 now yields $P/T \in C_p(M^J)$ for every set J. It is easily seen that L/T is M-independent in P/T. In addition L/T is small in P/T. This proves that $\alpha\colon P/T \to B$ is a strong M-projective cover of B.

COROLLARY 3.5. *If R is left perfect (resp. semiperfect) every module (resp. cyclic module) over R possesses a strong M-projective cover.*

PROPOSITION 3.6. *Suppose $M \in R$-mod satisfies $\mathfrak{A}(M) = 0$. Then $B \in R$-mod is strongly M-projective iff B is projective.*

Proof. The implication $\Leftarrow$ is trivial. As for the implication $\Rightarrow$, let B be strongly M-projective. Let

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{\phi} B \longrightarrow 0$$

be an exact sequence in R-mod with F free. Let $\{e_\alpha\}_{\alpha \in J}$ be as basis for F. Suppose $0 \neq x \in k$. Then $x = \sum \lambda_\alpha e_\alpha$ with at least one $\lambda_\alpha \neq 0$. Since $\mathfrak{A}(M) = 0$ there exists a $g_\alpha\colon R \to M$ with $g_\alpha(\lambda_\alpha) \neq 0$. Then $h\colon F \to M$ given by $h \mid Re_\alpha = g_\alpha$, $h \mid Re_\beta = 0$ for $\beta \neq \alpha$ clearly satisfies $h(x) \neq 0$. Thus K is M-independent in F. By Lemma 2.5, ϕ is an M^K-epimorphism. Since $B \in C_p(M^K)$, by (3) of Proposition 1.3 we see that ϕ splits. Hence B is projective.

COROLLARY 3.7. *Let $M \in R$-mod be such that $\mathfrak{A}(M) = 0$. Suppose B is an R-module possessing a projective cover. Then B is projective iff B is M-projective.*

Proof. We have only to prove the implication $\Leftarrow$. This is immediate from Lemma 2.2 and Proposition 3.6.

Any R-module B satisfying $\mathfrak{A}(M)B = 0$ can be regarded as an $R/\mathfrak{A}(M)$-module. In particular this is the case if $t_M(B) = 0$ by Lemma 2.11.

LEMMA 3.8. *Suppose $B \in R$-mod satisfies $\mathfrak{A}(M)B = 0$. Then B is strongly M-projective iff as an $R/\mathfrak{A}(M)$-module B is projective.*

Proof. From $\mathfrak{A}(M)M^J = 0$ we see that M^J is an $R/\mathfrak{A}(M)$-module, (whatever be the indexing set J). Also it is clear that for any $A \in R$-mod satisfying $\mathfrak{A}(M) = 0$, the R-submodules of A are the same as the $R/\mathfrak{A}(M)$-submodules of A. It follows from this comment that B is strongly M-projective in R-mod iff B is strongly M-projective in $R/\mathfrak{A}(M)$-mod. The annihilator $\mathfrak{A}_{R/\mathfrak{A}(M)}(M)$ of M as an $R/\mathfrak{A}(M)$-module is clearly seen to be zero. Lemma 3.8 now follows from Proposition 3.6.

THEOREM 3.9. *The following statements are equivalent.*

(1) *Every $B \in R$-mod satisfying $\mathfrak{A}(M)B = 0$, possesses a strong M-projective cover (in R-mod).*

(2) *$R/\mathfrak{A}(M)$ is left perfect.*

Proof. (1) $\Rightarrow$ (2). Let $B \in R/\mathfrak{A}(M)$-mod. Then B regarded as an R-module satisfies $\mathfrak{A}(M)B = 0$. Let $\alpha\colon A \to B$ be a strong M-projective cover of B in R-mod. Let $K = \ker \alpha$. From $\alpha(\mathfrak{A}(M)A) \subset \mathfrak{A}(M)B = 0$ we see that

$\mathfrak{A}(M)A \subset K$. Hence α induces a map $\bar{\alpha}$: $A/\mathfrak{A}(M)A \to B$. Now, $A/\mathfrak{A}(M)A$ is an $R/\mathfrak{A}(M)$-module and ker $\bar{\alpha}$: $K/\mathfrak{A}(M)A$ is small in $A/\mathfrak{A}(M)A$. Thus $\bar{\alpha}$ is a minimal epimorphism in $R/\mathfrak{A}(M)$-mod. If i: $\mathfrak{A}(M)A \to A$ denotes the inclusion, it is clear that

$$i^*: \text{Hom}\,(A, M) \to \text{Hom}_R\,(\mathfrak{A}(M)A, M)$$

is zero. Hence for any indexing set J, the map i^*: $\text{Hom}_R\,(A, M^J) \to \text{Hom}_R\,(\mathfrak{A}(M)A, M^J)$ is zero. Since A is strongly M-projective as an R-module, applying Lemma 1.6 we see that $A/\mathfrak{A}(M)A$ is strongly M-projective in R-mod. Now Lemma 3.8 implies that $A/\mathfrak{A}(M)A$ is a projective $R/\mathfrak{A}(M)$-module. Thus $\bar{\alpha}$: $A/\mathfrak{A}(M)A \to B$ is a projective cover of B in $R/\mathfrak{A}(M)$-mod. This proves that $R/\mathfrak{A}(M)$ is left perfect.

(2) $\Rightarrow$ (1). Let $B \in R$-mod be such that $\mathfrak{A}(M)B = 0$. Let π: $P \to B$ be a projective cover of B in $R/\mathfrak{A}(M)$-mod. Then P is an $R/\mathfrak{A}(M)$-direct summand and hence an R-direct summand of $\oplus_{\alpha \in S} R/\mathfrak{A}(M)$ for some set S. If i: $\mathfrak{A}(M) \to R$ denotes the inclusion, clearly i^*: $\text{Hom}_R\,(R, M) \to \text{Hom}_R\,(\mathfrak{A}(M), M)$ is zero and hence

$$i^*: \text{Hom}_R\,(R, M^J) \to \text{Hom}_R\,(\mathfrak{A}(M), M^J)$$

is zero for every set J. Since R is free it is strongly M-projective in R-mod. By Lemma 1.6 we see that $R/\mathfrak{A}(M)$ is strongly M-projective in R-mod. From (1) of Proposition 1.5 it follows that P is strongly M-projective in R-mod.

Now $R/\mathfrak{A}(M)$ is M-independent. From this it follows immediately that $\oplus_{\alpha \in S} R/\mathfrak{A}(M)$ and hence P are M-independent. If $K = \ker \alpha$, then K is M-independent in P (by the comments following Definition 2.3). Thus π: $P \to B$ is a strong M-projective cover of B in R-mod.

Obvious modifications in the proof of Theorem 3.9 yield:

THEOREM 3.10. *The following statements are equivalent.*

(1) *Every cyclic* $B \in R$-mod *satisfying* $\mathfrak{A}(M)B = 0$ *possesses a strong M-projective cover as an R-module.*

(2) $R/\mathfrak{A}(M)$ is *semiperfect.*

PROPOSITION 3.11. *The following statements are equivalent.*

(1) *The direct product* $\prod_{\alpha \in J} B_\alpha$ *of any family* B_α *of strongly M-projective R-modules with* $\mathfrak{A}(M)B_\alpha = 0$ *for all* $\alpha \in J$ *is strongly M-projective.*

(2) $(R/\mathfrak{A}(M))^J$ *is strongly M-projective for every indexing set J.*

(3) $R/\mathfrak{A}(M)$ *is left perfect, and any finitely generated right ideal of* $R/\mathfrak{A}(M)$ *is finitely related.*

Proof. Immediate consequence of Theorem 3.3 of [4] and Lemma 3.8.

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