ERGODIC ACTIONS WITH GENERALIZED DISCRETE SPECTRUM

BY

ROBERT J. ZIMMER

The investigation of extensions in the theory of ergodic actions of locally compact groups was undertaken by the author in [26]. In particular, we considered the notion of extensions with relatively discrete spectrum, and saw how the classical von Neumann-Halmos theory of transformations with discrete spectrum could be generalized to the case of extensions. In this paper, which is a sequel to [26], we study those actions which can be built up from a point by taking extensions with relatively discrete spectrum and inverse limits. We shall say that such actions have generalized discrete spectrum.

A similar construction is well known in topological dynamics. In [4], Furstenberg introduced the notion of an isometric extension of a continuous transformation group, and called an action quasi-isometric if it could be built up from a point by taking isometric extensions and inverse limits. The main result of [4] is the striking theorem that among the minimal transformation groups, the quasi-isometric ones are precisely those that are distal. Thus, one obtains a description of the structure of an arbitrary minimal distal transformation group, and using this, one can answer a variety of questions about such groups.

The structure of extensions with relatively discrete spectrum was described in Theorem 4.3 of [26]. Examination of the conclusion of this theorem shows that extensions with relatively discrete spectrum are a reasonable measure-theoretic analogue of Furstenberg's isometric extensions. Thus, we can consider actions with generalized discrete spectrum as a measure-theoretic analogue of the quasi-isometric transformation groups. Parry has described, at least for actions of the integers, a measure-theoretic analogue of the topological notion of distallity [20]. It is not difficult to generalize Parry's definition to arbitrary group actions, and now the question arises as to whether one can prove a measure-theoretic analogue of Furstenberg's theorem. We prove such a theorem below. It asserts that among the nonatomic ergodic actions, those with a separating sieve (as Parry called his actions) are precisely those with generalized discrete spectrum. Using this theorem, one sees immediately, for example, that any minimal distal action preserving a probability measure has generalized discrete spectrum.

Though there are formal similarities between the proof of our theorem and Furstenberg's proof, the proofs are basically quite different. Our proof depends upon, among other things, generalizing the concepts of weak mixing and the

Received October 1, 1975.

Cartesian product action to extensions; this leads to the notions of relative weak mixing and the fibered product. In addition, we make use of a general existence theorem for factors proved in [26]. These notions are all of independent interest and prove useful in other circumstances. Furstenberg's proof rests heavily on topological notions that are not available in the measure-theoretic context.

Given any specific class of actions, one would, of course, like to know which members of this class have generalized discrete spectrum. One class of actions that has attracted considerable attention is the set of affine actions on compact abelian groups. An algebraic criterion for affine transformations to have some nontrivial discrete spectral part was established by Hahn [6], and was extended (along with much of the other theory of affine transformations) to affine actions of a general locally compact abelian group by Wieting [24]. We extend the Hahn-Wieting analysis to establish an algebraic criterion for an affine extension to have some nontrivial relative discrete spectrum. This has two interesting consequences. First, we are able to give an algebraic criterion for an affine action to have generalized discrete spectrum. Second, in the case of a transformation, it enables us to clarify the relationship between generalized discrete spectrum and quasi-discrete spectrum.

Transformations with quasi-discrete spectrum were introduced by von Neumann and Halmos, and first studied systematically by Abramov [1]. The inductive definition of quasi-discrete spectrum can be shown to be a special case of the definition of generalized discrete spectrum. The nilflows considered by Auslander, Green, and Hahn [2] show, however, that not every transformation with generalized discrete spectrum has quasi-discrete spectrum. However, using the algebraic criterion of the preceding paragraph, we show that every totally ergodic affine transformation with generalized discrete spectrum actually has quasi-discrete spectrum. Combined with a result of Abramov, this yields a new characterization of (totally ergodic) transformations with quasi-discrete spectrum, as precisely those that are equivalent to affine transformations with generalized discrete spectrum.

The results of this paper depend heavily on the framework established in [26]. For ease of reference, we have begun this paper with Section 7, all references to Sections 1–6 being to those in [26]. For any unexplained notation the reader is also referred to [26]. The organization of the paper is as follows. Section 7 discusses relative weak mixing and applications of the general existence theorem for factors appearing in Section 2. Section 8 contains the central result, namely the measure-theoretic analogue of the Furstenberg structure theorem. The proof depends heavily on the results of Section 7. Section 9 considers some examples and general properties of actions with generalized discrete spectrum. The connections between affine actions, quasi-discrete spectrum, and generalized discrete spectrum mentioned above are proved in Section 10. This section concludes with a new proof, based on the results of Section 6, of the Abramov-Wieting existence theorem for actions with quasi-discrete spectrum [1], [24].

Some of the main results of this paper were announced in [25].

The author wishes to thank Professor G. W. Mackey for many helpful conversations and suggestions during the preparation of this paper.

7. Relative weak mixing

In preceding sections, we have examined the properties of extensions with relatively discrete spectrum. We now turn to questions involving the appearance or nonappearance of such extensions in various situations.

THEOREM 7.1. *Suppose X is a Lebesgue G-space. Let $H \subset L^2(X)$ be the closed subspace generated by the finite-dimensional G-invariant subspaces of $L^2(X)$. Then there exists a factor G-space Y of X such that $H = L^2(Y)$ (and hence Y has discrete spectrum).*

Proof. It is well known that for $f \in L^2(X)$, $f \in H$ if and only if $\{U_g f \mid g \in G\}$ is precompact in $L^2(X)$, where U_g is the natural representation of G on $L^2(X)$. Let $B = \{A \subset X \mid A \text{ is measurable, and } \chi_A \in H\}$.

LEMMA 7.2. *B is an invariant σ-field of subsets of X.*

Proof. If $A \in B$, $\chi_{X-A} = 1 - \chi_A \in H$, so $X - A \in B$. If $A_i \in B$, and A_i are mutually disjoint, then

$$\chi_{\cup_1^n A_i} = \sum_1^n \chi_{A_i} \in H,$$

and as $\mu(\bigcup_1^\infty A_i) \leq 1$,

$$\chi_{\cup_1^\infty A_i} = \lim_{n\to\infty} \sum_1^n \chi_{A_i},$$

and so $\bigcup_1^\infty A_i \in B$ since H is closed. B is clearly G-invariant and hence it suffices to see that B is closed under finite intersections. So suppose A, $D \in B$ and g, $h \in G$. Then

$$\begin{aligned}
(A \cap D) \cdot g \,\Delta\, (A \cap D) \cdot h \\
&= [(X - Ag \cap Dg) \cap Ah \cap Dh] \cup [Ag \cap Dg \cap (X - Ah \cap Dh)] \\
&= [(X - Ag) \cap Ah \cap Dh] \cup [(X - Dg) \cap Ah \cap Dh] \\
&\quad \cup [Ag \cap Dg \cap (X - Ah)] \cup [Ag \cap Dg \cap (X - Dh)] \\
&\subset (Ag \,\Delta\, Ah) \cup (Dg \,\Delta\, Dh).
\end{aligned}$$

Thus

$$\|\chi_{(A \cap D)g} - \chi_{(A \cap D)h}\|^2 \leq \|\chi_{Ag} - \chi_{Ah}\|^2 + \|\chi_{Dg} - \chi_{Dh}\|^2.$$

As a set in a metric space is precompact if and only if every sequence has a Cauchy subsequence, to see $A \cap D \in B$, it suffices to show that if $g_i \in G$, then $\chi_{(A \cap D)g_i}$ has a Cauchy subsequence. Since $\{\chi_{Ag}\}$ and $\{\chi_{Dg}\}$ are precompact, there exists a subsequence g_{i_n} such that $\chi_{Ag_{i_n}}$ and $\chi_{Dg_{i_n}}$ are Cauchy, and it follows easily from the inequality above that $\chi_{(A \cap D)g_{i_n}}$ is also.

We now return to the proof of the theorem. It suffices to show that $f \in L^2(X)$ is measurable with respect to B if and only if $f = H$. If f is measurable with respect to B, then f is the limit in $L^2(X)$ of finite linear combinations of characteristic functions of sets in B, and hence $f \in H$. To see the converse, first note that $f \in H$ if and only if $\bar{f} \in H$, and hence it suffices to show the converse for real valued functions. Let $t \in \mathbf{R}$ and $A = \{x \in X \mid f(x) \leq t\}$. It suffices to show that $A \in B$, i.e., the orbit of χ_A is precompact. Suppose not. Then there exists $\varepsilon > 0$ and a sequence $g_i \in G$ such that

$$\|\chi_{Ag_i} - \chi_{Ag_j}\| \geq \varepsilon,$$

i.e.,

$$\mu(Ag_i \,\Delta\, Ag_j) \geq \varepsilon^2.$$

Since $\mu(A) < 1$, it is easy to see that there exists some $\delta > 0$ and a set $D \subset X - A$ such that (i) $\mu(D) \geq 1 - \mu(A) - \varepsilon^2/4$, i.e., $\mu((X - A) - D) \leq \varepsilon^2/4$ and (ii) $f(x) \geq t + \delta$ for every $x \in D$. Now since $\mu(Ag_i \,\Delta\, Ag_j) \geq \varepsilon^2$, we can assume

$$\mu(Ag_i \cap (X - Ag_j)) \geq \varepsilon^2/2.$$

We have $Dg_j \subset X - Ag_j$ and $\mu((X - Ag_j) - Dg_j) \leq \varepsilon^2/4$ by the G-invariance of μ, and hence $\mu(Ag_i \cap Dg_j) \geq \varepsilon^2/4$. For $x \in Ag_i \cap Dg_j$, we have

$$(U_{g_i^{-1}}f)(x) = f(xg_i^{-1}) \leq t,$$

since $xg_i^{-1} \in A$.

Similarly, $(U_{g_j^{-1}}f)(x) \geq t + \delta$ by (ii) above. Thus

$$\int_{Ag_i \cap Dg_j} (U_{g_i^{-1}}f - U_{g_j^{-1}}f)^2 \geq \delta^2(\varepsilon^2/4),$$

and hence

$$\|U_{g_i^{-1}}f - U_{g_j^{-1}}f\| \geq \delta\varepsilon/2.$$

Since this holds for every i, j, the orbit of f is not precompact, which contradicts the assumption that $f \in H$.

This theorem was established in the case $G = Z$ by Krengel [15, Theorem 22].

The above theorem required no assumptions of ergodicity. When the action is ergodic, the following theorem provides a significant generalization; the proof gives a new proof of Theorem 7.1 in the ergodic case.

THEOREM 7.3. *Let $\phi: X \to Y$ be a G-factor map. Let $H \subset L^2(X)$ be the closed subspace generated by the G-invariant fields of finite dimensional spaces, where $L^2(X)$ is considered as a Hilbert bundle over Y. (We suppose X is ergodic.) Then there exists an essential set $X' \subset X$, a G-space Z, and a sequence of factor G-maps $X' \to Z \to Y$ whose composition is ϕ, such that $L^2(Z) = H$ (and hence, Z has relatively discrete spectrum over Y).*

We begin the proof with the following:

LEMMA 7.4. *Let X be an ergodic G-space. Suppose $H \subset G$ is a countable dense subgroup, and $f: X \to \mathbf{R}$ is a Borel function such that for each $h \in H$, $f(xh) = f(x)$ almost everywhere. Then f is constant on a conull set.*

Proof. If not, there exist $\alpha, \beta \in \mathbf{R}$ such that $A = \{x \mid \alpha \leq f(x) \leq \beta\}$ has positive measure less than 1. But $f(xh) = f(x)$ almost everywhere implies for each h, $\chi_A(x) = \chi_A(xh)$ almost everywhere. Thus, $U_h(\chi_A) = \chi_A$ in $L^2(X)$, and since U is continuous and $H \subset G$ dense, χ_A is G-invariant in $L^2(X)$. This implies A is null or conull, which is a contradiction.

LEMMA 7.5. *Let $V = \int^{\oplus} H_y$ be a G-invariant field of subspaces of the Hilbert bundle $L^2(X) = \int^{\oplus} L^2(F_y)$, such that* $\dim H_y = n < \infty$ *for almost all $y \in Y$. Suppose $f_i \in L^2(X)$ such that* (i) $\|(f_i)_y\| \leq 1$ *for every y (here, $(f_i)_y = f_i \mid F_y$);* (ii) *for almost all $y \in Y$, $\{(f_i)_y\}_{i=1,\ldots,n}$ is an orthonormal basis of H_y. Then each $f_i \in L^\infty(X)$.*

Proof. Let α be the natural cocycle representation; we can assume $\|\alpha(y, g)\| \leq 1$ for all (y, g). Let

$$a_{ij}(y, g) = \langle \alpha(y, g)(f_j)_{yg} \mid (f_i)_y \rangle_y.$$

Then $a_{ij}(y, g)$ is Borel, $|a_{ij}(yg)| \leq 1$ by (i) above, and for each g, $a_{ij}(y, g)$ is a unitary matrix for almost all y. Choose H to be a countable dense subgroup of G, and define

$$\theta_j(x) = \sup_{g \in H} \left| \sum_{i=1}^{n} a_{ij}(\phi(x), g) f_i(x) \right|.$$

This exists since $|a_{ij}(y, g)| \leq 1$ and is Borel since H is countable. Now let $h \in H$. Then

$$\theta_j(xh) = \sup_{g \in H} \left| \sum_{i=1}^{n} a_{ij}(\phi(x)h, g) f_i(xh) \right|.$$

But for almost all x,

$$\begin{aligned} f_i(xh) = (f_i)_{\phi(x)h}(xh) &= [\alpha(\phi(x), h)(f_i)_{\phi(x)h}](x) \\ &= \left[\sum_k a_{ki}(\phi(x), h)(f_k)_{\phi(x)} \right](x) \\ &= \sum_k a_{ki}(\phi(x), h) f_k(x). \end{aligned}$$

So for almost all x,

$$\begin{aligned} \theta_j(xh) &= \sup_{g \in H} \left| \sum_k \sum_i a_{ki}(\phi(x), h) a_{ij}(\phi(x)h, g) f_k(x) \right| \\ &= \sup_{g \in H} \left| \sum_{k=1}^{n} a_{kj}(\phi(x), hg) f_k(x) \right| \\ &= \theta_j(x). \end{aligned}$$

Thus, by Lemma 7.4, θ_j is constant on a conull set. For almost all x,

$$\theta_j(x) \geq \left| \sum_i a_{ij}(\phi(x), e) f_i(x) \right| = |f_j(x)|.$$

Thus for each j, $|f_j(x)|$ is bounded on a conull set, i.e., $f_j \in L^\infty(X)$.

Proof of Theorem 7.3. Let $A = \{f \in L^\infty(X) \mid f$ is contained in a G-invariant field of finite dimensional subspaces$\}$. It is straightforward to check that A is a subspace of $L^\infty(X)$, closed under complex conjugation, and multiplication by elements of $L^\infty(Y)$. We claim it is also closed under multiplication. Let f, $h \in A$, $f \in V = \int^\oplus H_1(y)$, $h \in W = \int^\oplus H_2(y)$, V, W G-invariant, $\dim H_i(y) < \infty$. Let $f_i(x) \in L^2(X)$ such that $\{(f_i)_y\}_i$ is an orthonormal basis of $H_1(y)$ almost everywhere, and $h_j(x) \in L^2(X)$ such that $\{(h_j)_y\}_j$ is an orthonormal basis of $H_2(y)$ almost everywhere. Since $f \in V$, $h \in W$, there exist Borel functions $a_i(y)$, $b_i(y)$ such that $f(x) = \sum a_i(\phi(x)) f_i(x)$ and $h(x) = \sum b_j(\phi(x)) h_j(x)$ almost everywhere. For almost all y, $\|f_y\|^2 = \sum_i |a_i(y)|^2$, so

$$|a_i(y)|^2 \leq \|f_y\|^2 \leq \|f\|_\infty^2.$$

Thus, each $a_i \in L^\infty(Y)$, and similarly, $b_j \in L^\infty(Y)$. Now $f \cdot h = \sum_{i,j} (a_i \circ \phi) \cdot (b_j \circ \phi) f_i h_j$, so by the remarks above, to see that $fh \in A$, it suffices to see that $f_i h_j \in A$. We have $f_i h_j \in L^\infty(X)$ by Lemma 7.5, and hence (changing functions on a null set if necessary), for each $y \in Y$, $(f_i)_y (h_j)_y \in L^2(F_y)$. Let $Z(y)$ be the subspace of $L^2(F_y)$ spanned by $\{(f_i)_y (h_j)_y\}_{i,j}$. To see that $f_i h_j \in A$, it suffices to see that $Z = \int^\oplus Z(y)$ is G-invariant, and for this, to see that for a given i, j, g, $U_g(f_i h_j) \in Z$. Let

$$(U_g f_i)(x) = \sum_k a_{ki}(\phi(x)) f_k(x) \quad \text{almost everywhere,}$$

and

$$(U_g h_j)(x) = \sum_p b_{pj}(\phi(x)) h_p(x) \quad \text{almost everywhere,}$$

where a_{ki}, b_{pj} are Borel functions on Y. Since $U_g f_i \in L^\infty(X)$, we see as above that $a_{ki}(y) \in L^\infty(Y)$ and similarly that $b_{pj} \in L^\infty(Y)$. Thus,

$$U_g(f_i h_j)(x) = (U_g f_i)(U_g h_j)(x) = \sum_{k,p} a_{ki}(\phi(x)) b_{pj}(\phi(x)) f_k(x) h_p(x) \in Z.$$

Using the same technique in a somewhat simpler setting, one can prove the following companion to Theorem 7.3.

THEOREM 7.6. *Let $\phi: X \to Y$ be a factor map of ergodic G-spaces. Let $H^1 \subset L^2(X)$ be the closed subspace generated by the G-invariant fields of one dimensional subspaces. Then there exists an essential set $X' \subset X$, a G-space Z', and a sequence of factor maps $X' \to Z' \to Y$ whose composition is ϕ, such that $H^1 = L^2(Z')$. (Hence Z' has relatively elementary spectrum.) If Z is as in Theorem* 7.3, *then $B(Z') \subset B(Z) \subset B(X)$, so Z' is also a factor of Z.*

Theorem 7.3 says that if Y is a factor of X, then the subspace of $L^2(X)$ which corresponds to the discrete part of the spectrum of the natural (Y, G) cocycle is given by $L^2(Z)$, where Z is a space "between" X and Y. We now turn to the question of when the natural cocycle representation has (nontrivial) finite dimensional subrepresentations. If $Y = \{e\}$, this will be the case if and only if the space $X \times X$ is not ergodic [19, Proposition 1]. When $Y \neq \{e\}$, we shall see below that the relevant consideration is the ergodicity of the fibered product $X \times_Y X$. It will be convenient to consider a somewhat more general question, namely the ergodicity of the fibered product $X \times_Y Z$, where Y is a factor of both X and Z. [We note that $X \times_Y Z$ is a G-invariant subset of $X \times Z$ under the product action, and it is straightforward to check that the measure on $X \times_Y Z$ defined in Section 1 is G-invariant.]

LEMMA 7.7. *Let $\phi: X \to Y$ be a factor G-map of ergodic G-spaces. Then the natural cocycle representation α contains the identity one-dimensional cocycle exactly once.*

Proof. $L^2(Y) \subset L^2(X)$ is a G-invariant field of one-dimensional spaces and α restricted to $L^2(Y)$ is the identity. Now suppose $V = \int^{\oplus} V(y)$ is a one-dimensional G-invariant field, and that restricting $\alpha(y, g)$ to V gives us a cocycle equivalent to 1. Then there exist maps $U(y): V(y) \to \mathbf{C}$ such that $U(y)$ is unitary almost everywhere and for each g, $U(y)\alpha(y, g)U(yg)^{-1} = 1$ almost everywhere. Let $f = \int^{\oplus} f_y, f \in L^2(X)$, where $f_y = U(y)^{-1}(1)$ almost everywhere. Then

$$(\alpha(y, g)f_{yg})(x) = (\alpha(y, g)U(yg)^{-1}(1))(x) = (U(y)^{-1}(1))(x) = f_y(x)$$

almost everywhere. But by definition of α, $(\alpha(y, g)f_{yg})(x) = f(xg)$ almost everywhere. By the ergodicity of X, f is essentially constant so $f_y \in \mathbf{C}$ for almost all y. Since f_y is a basis of $V(y)$ almost everywhere, $V = L^2(Y)$.

THEOREM 7.8. *Let (X, μ), (Z, ν), (Y, m) be ergodic Lebesgue G-spaces and $\phi: X \to Y$, $\psi: Z \to Y$ G-factor maps. Then $X \times_Y Z$ is an ergodic G-space if and only if the natural cocycle representations (which we denote α_X and α_Z) do not have a common finite dimensional subcocycle representation other than the identity.*

Proof. (i) We suppose that α_X and α_Z have a finite dimensional cocycle in common. Then it is easy to see that there exist Borel functions $f_i(x)$, $h_i(z)$, $a_{ij}(y, g)$ such that:

(a) For almost all y, $(f_1)_y, (f_2)_y, \ldots, (f_n)_y$, 1 are mutually orthogonal; $(h_1)_y, \ldots, (h_n)_y$, 1 are mutually orthogonal.

(b) For each g, $(a_{ij}(y, g))$ is a unitary matrix for almost all y.

(c) For each g and almost all x,

$$f_j(xg) = \sum_i a_{ij}(\phi(x), g)f_i(x) \quad \text{and} \quad h_j(zg) = \sum_i a_{ij}(\psi(z), g)h_i(z).$$

Now define $\theta(x, z) = \sum_{i=1}^{n} f_i(x)\overline{h_i(z)}$. Then $\theta \in L^\infty(X \times_Y Z)$ by Lemma 7.5 and it follows from (a) that $\theta \perp 1$ in $L^2(X \times_Y Z)$. Further, for each g and almost all $(x, z) \in X \times_Y Z$, we have

$$\begin{aligned}\theta(xg, zg) &= \sum_j f_j(xg)\overline{h_j(zg)} \\ &= \sum_j \left(\sum_i a_{ij}(\phi(x), g) f_i(x)\right)\left(\overline{\sum_k a_{kj}(\psi(z), g) h_k(z)}\right) \\ &= \sum_{i,k}\left(\sum_j a_{ij}(\phi(x), g)\overline{a_{ij}(\psi(z), g)} f_i(x)\overline{h_k(z)}\right).\end{aligned}$$

Since $\phi(x) = \psi(z)$, we obtain, using (b),

$$\theta(xg, zg) = \sum_{i,k} \delta_{ik} f_i(x)\overline{h_k(z)} = \theta(x, z).$$

Thus θ is nonconstant and essentially G-invariant, which shows $X \times_Y Z$ is not ergodic.

(ii) We now show the converse. If $X \times_Y Z$ is not ergodic, choose $\theta \in L^\infty(X \times_Y Z)$ to be nonconstant and G-invariant. For each $y \in Y$, define

$$T_y: L^2(\phi^{-1}(y)) \to L^2(\psi^{-1}(y))$$

by

$$(T_y\lambda)(z) = \int_{\phi^{-1}(y)} \theta(x, z)\lambda(x)\, d\mu_y(x).$$

Then $\{T_y\}$ is a Borel field of compact linear operators, and $T = \int^{\oplus} T_y$ is a bounded linear operator, $T: L^2(X) \to L^2(Z)$. Letting U and W be the natural representations of G on $L^2(X)$ and $L^2(Z)$ respectively, we claim that T is an intertwining operator for U and W. It suffices to see that for each $g \in G$, $T_y\alpha_X(y, g) = \alpha_Z(y, g)T_{yg}$ for almost all y. If $\lambda \in L^2(\phi^{-1}(yg))$, then

$$\begin{aligned}(T_y \circ \alpha_X(y, g)\lambda)(z) &= \int_{\phi^{-1}(y)} \theta(x, z)(\alpha_X(y, g)\lambda)(x)\, d\mu_y(x) \\ &= \int_{\phi^{-1}(y)} \theta(x, z)\lambda(xg)\, d\mu_y(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\alpha_Z(y, g)T_{yg}\lambda)(z) &= (T_{yg}\lambda)(zg) \\ &= \int_{\phi^{-1}(yg)} \theta(w, zg)\lambda(w)\, d\mu_{yg}(w) \\ &= \int_{\phi^{-1}(y)} \theta(xg, zg)\lambda(xg)\, d\mu_y(x).\end{aligned}$$

Since θ is G-invariant, this becomes $\int_{\phi^{-1}(y)} \theta(x, z)\lambda(xg)\, d\mu_y(x)$ and comparing with the equation above, we see that T is an intertwining operator. Now let $A = T^*T = \int^{\oplus} T_y^* T_y$. Then

$$AU_g = T^*TU_g = T^*W_gT = U_gT^*T = U_gA.$$

Thus A is a self-intertwining operator for U, and $A = \int^{\oplus} A_y$, where $A_y = T_y^*T_y$ is compact and self-adjoint. Let $\lambda_1(y) = \sup \{c \mid c \text{ is an eigenvalue of } A_y\}$. Then $\lambda_1(y)$ is Borel and hence, if $V_1(y) = \{v \in L^2(\phi^{-1}(y)) \mid A_y(v) = \lambda_1(y)v\}$, then $\{V_1(y)\}$ is a subbundle of $L^2(X)$ over Y. Since for each g,

$$\alpha_X(y, g)^{-1}A_y\alpha_X(y, g) = A_{yg}$$

for almost all y, we have $\lambda_1(yg) = \lambda_1(y)$ (for each g and almost all y). By ergodicity of Y, this implies λ_1 is constant on a conull set. Hence $V_1 = \int^{\oplus} V_1(y)$ is G-invariant. Now let

$$\lambda_2(y) = \inf \{c \mid c \text{ is an eigenvalue of } A_y \mid V_1(y)^{\perp}\},$$

and

$$V_2(y) = \{v \in V_1(y)^{\perp} \mid A_y(v) = \lambda_2(y)v\}.$$

As above, λ_2 is essentially constant and $V_2 = \int^{\oplus} V_2(y)$ is a G-invariant subbundle. Continuing inductively, using the spectral theorem for compact self-adjoint operators, we can obtain (after suitable relabelling) the following decomposition: there exist real numbers $\lambda_0 = 0$, $\lambda_i \neq 0$, $i = 1, \ldots,$ and G-invariant subbundles of $L^2(X)$, $V_i = \int^{\oplus} V_i(y)$, $i = 0, \ldots,$ such that:

(i) $L^2(X) = \sum_{i=0}^{\infty} V_i$ and
(ii) $V_i(y) = \{v \in L^2(\phi^{-1}(y)) \mid A_y(v) = \lambda_i v\}$

for almost all y. Thus $V_0 = \ker(A)$. Since $A = T^*T$, we also have $V_0 = \ker(T)$. If $i > 0$, V_i is a finite-dimensional G-invariant field. So $T(V_i)$ will be a finite dimensional G-invariant subfield of $L^2(Z)$, and the hypothesis of the theorem together with Lemma 7.7 show that $T(V_i) \subset L^2(Y)$. Since this holds for each i, we have $T(L^2(X)) \subset L^2(Y)$, i.e., for almost all y, $T_y(L^2(\phi^{-1}(y)) = \mathbf{C} \subset L^2(\psi^{-1}(y))$. It follows that for almost all y, $\theta(x, z)$ is essentially independent of z. But then the G-invariance of θ and the ergodicity of X imply θ is essentially constant. This is a contradiction and completes the proof.

DEFINITION 7.9. If $X \to Y$ is a factor G-map of ergodic G-spaces, call X relatively weakly mixing over Y if $X \times_Y X$ is ergodic.

When $Y = \{e\}$, this is just the usual notion of weak mixing.

Theorem 7.8 has the following corollaries.

COROLLARY 7.10. *X is relatively weakly mixing over Y if and only if the natural $Y \times G$ cocycle representation contains no finite dimensional subcocycle representations other than the identity.*

COROLLARY 7.11. *X is relatively weakly mixing over Y if and only if $X \times_Y Z$ is ergodic for every ergodic extension Z of Y.*

COROLLARY 7.11. *If X is relatively weakly mixing over Y, so is $X \times_Y X$.*

Proof. If Z is an ergodic extension of Y, then $(X \times_Y X) \times_Y Z = X \times_Y (X \times_Y Z)$. Two applications of Corollary 7.11 imply $(X \times_Y X) \times_Y Z$ is ergodic, and it follows from the same corollary that $X \times_Y X$ is relatively weakly mixing over Y.

8. Generalized discrete spectrum and separating sieves

If X is an ergodic extension of Y, we have considered the notion of X having relatively discrete spectrum over Y. We now consider a more general class of extensions, which we shall call extensions with generalized discrete spectrum over Y. Loosely, these will be extensions built up from Y by the operations of taking extensions with relatively discrete spectrum, and taking inverse limits. Formally, this is done in the same way as Furstenberg's notion of quasi-isometric extension of a continuous flow is built up by isometric extensions and limits [4; Definition 2.4]. Thus, some formal aspects of what follows will be similar to those in [4]. Later, we shall discuss the relationship between the content of [4] and the content of the results of this section.

We begin with some remarks on inverse limits of G-spaces. Let η be a countable ordinal. Suppose for each ordinal $\gamma < \eta$ we have a Lebesgue G-space X_γ and for each pair of ordinals $\sigma < \gamma < \eta$ a factor G-map $\phi_{\gamma\sigma}: X_\gamma \to X_\sigma$ such that for any triple $\beta < \sigma < \eta$, the diagram

$$\begin{tikzcd} X_\gamma \arrow[r] \arrow[dr] & X_\sigma \arrow[d] \\ & X_\beta \end{tikzcd}$$

commutes. Now suppose X is a G-space, $X' \subset X$ essential, and for each γ we have a factor map $p_\gamma: X' \to X_\gamma$ such that for any σ, γ the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & X_\gamma \arrow[d] \\ & X_\sigma \end{tikzcd}$$

Then we call $\{X, p_\gamma, X_\gamma, \phi_{\gamma\sigma}\}$ an ordered system of factors of X. We say that $X = \text{inj lim } X_\gamma$ if $L^2(X) = \overline{\bigcup_{\gamma<\eta} L^2(X_\gamma)}$ or equivalently, $B(X)$ is the σ-algebra generated by $\bigcup B(X_\gamma)$. We also point out that X can be characterized in terms of $\{X_\gamma\}$ by a universal property.

PROPOSITION 8.1. *If Y is a Lebesgue G-space and there exist factor maps $q_\gamma: Y \to X_\gamma$ such that all diagrams*

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X_\gamma \arrow[d] \\ & X_\sigma \end{tikzcd}$$

commute, then there exists an essential set $Y' \subset Y$, and a factor G-map $Y' \to X'$ such that

$$\begin{array}{ccc} Y' & \to & X' \\ & \searrow & \downarrow \\ & & X_\gamma \end{array}$$

commutes. Any two such factor maps agree on a conull set. If $\bar{X}$ also (in addition to X) has this property, then X and $\bar{X}$ are essentially isomorphic.

Proof. We have maps $B(X_\gamma) \to B(Y)$. Under the metric $d(A, B) = \mu(A \Delta B)$, these spaces are complete metric spaces and the maps are isometric. Since the maps are compatible, we have an isometry $\bigcup_\gamma B(X_\gamma) \to B(Y)$ and as $\bigcup B(X_\gamma)$ is dense in $B(X)$, this extends to an isometry $B(X) \to B(Y)$. Since G acts on $B(X)$ and $B(Y)$ by isometries, the map $B(X) \to B(Y)$ is a G-map. Thus, by Proposition 2.1, there is an essential set $Y_0 \subset Y$ and a factor G-map $\theta \colon Y_0 \to X'$ inducing the Boolean G-map $B(X) \to B(Y)$. For each γ, $p_\gamma \circ \theta = q_\gamma$ on an essential $Y_\alpha \subset Y_0$. Since η is a countable ordinal, $Y' = \bigcap Y_\alpha$ is essential, and $\theta \mid Y' \colon Y' \to X'$ is the required map. The remaining assertions are straightforward.

PROPOSITION 8.2. *If $X = \text{inj lim } X_\gamma$ and each X_γ is ergodic, so is X.*

Proof. If X is not ergodic, there exists $f \in L^2(X)$, $f \perp \mathbf{C}$, $f \neq 0$ such that $U_g f = f$ for every $g \in G$. Now $L^2(X) \ominus \mathbf{C} = \overline{\bigcup L^2(X_\gamma) \ominus \mathbf{C}}$. So if P_γ is orthogonal projection onto $L^2(X_\gamma) \ominus \mathbf{C}$, then $P_\gamma f \neq 0$ for some γ. But since $L^2(X_\gamma) \ominus \mathbf{C}$ is G-invariant, P_γ commutes with all U_g and this implies $P_\gamma f$ is also G-invariant, contradicting the ergodicity of X_γ.

PROPOSITION 8.3. *If $X = \text{inj lim } X_\gamma$, then there exists a conull set $Z \subset X$ such that $x, y \in Z$ implies there exists γ such that $p_\gamma(x) \neq p_\gamma(y)$.*

Proof. We first give an alternative description of inj lim X_γ. Namely, let

$$W = \left\{ (x_\gamma) \in \prod_\gamma X_\gamma \mid \phi_{\gamma\sigma}(x_\gamma) = x_\sigma \quad \text{for all } \gamma, \sigma \right\}.$$

Then W is a standard Borel space, and by the Kolmogorov consistency theorem [21; Theorem 5.1], admits a probability measure for which $W = \text{inj lim } X_\gamma$, where $W \to X_\gamma$ is just projection on X_γ. Since W clearly satisfies the requirements from the Proposition, the result now follows from Proposition 8.1.

We now introduce extensions with generalized discrete spectrum.

DEFINITION 8.4. Let X, Y ergodic Lebesgue G-spaces, and X an extension of Y. We say that X has generalized discrete spectrum over Y if there exists a countable ordinal η, and an ordered system of factors $(X_\gamma, \gamma < \eta)$ of X such that, calling $X = X_\eta$,

(i) $X_0 = Y$,

(ii) For each $\gamma < \eta$, $X_{\gamma+1}$ has relatively discrete spectrum over X_γ (and is a nontrivial extension of X_γ),

(iii) If $\gamma \le \eta$ is a limit ordinal, then $X_\gamma = \text{inj lim } X_\sigma\ \sigma < \gamma$.

If the factors X_γ can be chosen so that $X_{\gamma+1}$ has relatively elementary spectrum over X_γ, we shall say that X has simple generalized discrete spectrum over Y. If $Y = \{e\}$, we shall omit the phrase "over $\{e\}$."

In light of the structure theorem (4.3) and Corollary 4.6, one has a description of the structure of any action with generalized or simple generalized discrete spectrum. The question now arises as to what conditions on a G-space will imply that it has generalized discrete spectrum, or more generally, generalized discrete spectrum over a given factor. We will show that there is a very satisfactory answer to this question. The following definition generalizes a notion due to Parry [20]. It was originally introduced by him as a measure-theoretic analogue of a distal transformation.

DEFINITION 8.5. Let $\phi: X \to Y$ a factor G-map of ergodic G-spaces, and let $S_1 \supset S_2 \supset \cdots$ be a sequence of Borel sets in X such that $\mu(S_n) > 0$, $\mu(S_n) \to 0$. Then $\{S_n\}$ is called a separating sieve over Y if for every countable set $N \subset G$, there exists a conull set $A \subset X$ such that $x, y \in A$, $\phi(x) = \phi(y)$, and for each n, $xg_n, yg_n \in S_n$ for some $g_n \in N$, implies $x = y$. $\{S_n\}$ will be called a separating sieve if it is a separating sieve over $\{e\}$.

An immediate but important property of separating sieves is the following.

PROPOSITION 8.6. *Suppose $X \to Y \to Z$ are factor G-maps, and that $\{S_n\}$ is a separating sieve for X over Z. Then it is also a separating sieve for X over Y.*

We now state the main result of this section.

THEOREM 8.7. *If X is an ergodic extension of Y, then X has generalized discrete spectrum over Y if and only if X is either atomic or has a separating sieve over Y.*

Before proving this theorem, we make some remarks on the relationship of this theorem to Furstenberg's work in topological dynamics. There are numerous analogies between topological dynamics and ergodic theory. (See [5], for example.) An extension of a Lebesgue G-space Y of the form $Y \times_\alpha K/H$ can be considered a measure-theoretic analogue of Furstenberg's topological notion of isometric extension (see [4] for this and other related concepts mentioned below), and in light of the structure theorem (Theorem 4.3), an action with generalized discrete spectrum is analogous to a quasi-isometric flow. The main theorem of [4] asserts that among the minimal flows, the quasi-isometric ones are exactly the distal flows. As Theorem 8.7 asserts, when $Y = \{e\}$, that the actions with generalized discrete spectrum are the actions with a separating sieve (trivial cases aside), we can view this theorem as the measure-theoretic analogue of Furstenberg's structure theorem. Despite some formal similarities, the proofs are basically quite different. The difficult part of Furstenberg's proof makes heavy use of the Ellis semigroup and its properties for minimal distal flows, which is not available in the measure-theoretic situation. Our proof makes use of the results of Section 7.

We begin the proof of Theorem 8.7 with some lemmas.

LEMMA 8.8. *If Y is nonatomic and has relatively discrete spectrum over an atomic factor Z, then Y has a separating sieve.*

Proof. Since Z is atomic and ergodic, it is essentially transitive, and hence we can assume $Z = G/G_0$ for some closed subgroup $G_0 \subset G$. Any $G/G_0 \times G$ cocycle is cohomologous to a strict one [23, Lemma 8.26], and for a strict cocycle α into a compact group K, $(z, [k])g = (zg, [k]\alpha(z, g))$ defines not only a near action of G on $Z \times K/H$, but an action. Thus by the structure theorem, discarding invariant null sets, we can assume $Y = Z \times_\alpha K/H$ for a strict cocycle α, and that the factor map $Y \to Z$ is given by projection of $Z \times K/H$ onto Z. Since Y is nonatomic, so is K/H, and we can choose a decreasing sequence of open neighborhoods U_i of $[e]$ in K/H such that $\bigcap_i U_i = \{[e]\}$ and $\mu(U_i) \to 0$. Choose an atom $z_0 \in Z$; we claim $\{z_0\} \times U_i$ is a separating sieve for $Z \times K/H$. If

$$(z_1, [k_1])g_n,\ (z_2, [k_2])g_n \in \{z_0\} \times U_n \quad \text{for some } g_n \in G,$$

then $z_1 g_n = z_2 g_n = z_0$ implies $z_1 = z_2$. Further,

$$[k_1]\alpha(z_1, g_n) \in U_n, \qquad [k_2]\alpha(z_1, g_n) = [k_2]\alpha(z_2, g_n) \in U_n$$

implies $[k_1] = [k_2]$, by the existence of a K-invariant metric on K/H.

LEMMA 8.9. *Suppose $\phi\colon X \to Y$, $\theta\colon Y \to Z$ are factor G-maps of ergodic G-spaces such that* (i) *X has relatively discrete spectrum over Y, and* (ii) *Y has a separating sieve $\{S_n\}$ over Z. Then X has a separating sieve over Z.*

Proof. By the structure theorem, we can sssume $X = Y \times_\alpha K/H$ and that $\phi(y, [k]) = p(y, [k])$ almost everywhere (here $p(y, [k]) = y$). Choose a decreasing sequence of open neighborhoods U_i of $[e]$ in K/H such that $\bigcap_i U_i = \{[e]\}$, and let $\bar{S}_n = S_n \times U_n$. Then $\bar{S}_n$ is decreasing and $\mu(\bar{S}_n) \to 0$, $\mu(\bar{S}_n) > 0$. We claim that $\bar{S}_n$ is a separating sieve for X over Z. Let N be a countable subset of G, and $A \subset Y$ as in the definition of a separating sieve for Y over Z (see Definition 8.5). For each $g \in N$, let

$$A_g = \{(y, [k]) \mid (y, [k])g = (yg, [k]\alpha(y, g))\},$$

and

$$\bar{A} = \{(y, [k]) \mid \phi(y, [k]) = y\}.$$

Then A_g and $\bar{A}$ are conull, and hence so is

$$A' = \bigcap_{g \in N} A_g \cap \bar{A} \cap p^{-1}(A).$$

Now suppose $(y_1, [k_1]), (y_2, [k_2]) \in A'$ with $\theta \circ \phi(y_1, [k_1]) = \theta \circ \phi(y_2, [k_2])$. Since $A' \subset \bar{A}$, $\theta(y_1) = \theta(y_2)$. If we also have $(y_i, [k_i])g_n \in \tilde{S}_n$, $i = 1, 2$ where $g_n \in N$, then $y_i \cdot g_n \in S_n$ since $A' \subset \bigcap A_g$. As $A' \subset p^{-1}(A)$, $y_1, y_2 \in A$, and since S_n is a separating sieve for Y over Z, it follows that $y_1 = y_2$. Furthermore, we have $[k_1]\alpha(y_1, g_n) \in U_n$, and $[k_2]\alpha(y_1, g_n) = [k_2]\alpha(y_2, g_n) \in U_n$. Since K/H admits a K-invariant metric, $[k_1] = [k_2]$. This completes the proof.

LEMMA 8.10. *Suppose $X, Y, X_1, X_2, \ldots$ are ergodic G-spaces, and that there exist factor maps $p_n: X \to X_n$ and $\theta_n: X_n \to Y$ such that*

$$\begin{tikzcd} & X_n \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & X_p \arrow[ur] & \end{tikzcd}$$

commutes for each n, p. Suppose further that there exists a conull set $Z \subset X$ such that $x, y \in Z$, $x \neq y$ implies there exists n_0 such that $p_{n_0}(x) \neq p_{n_0}(y)$. Then if each X_n has a separating sieve over Y, so does X.

Proof. We recall that if A, B are sets of positive measure in an ergodic G-space, then there exists $g \in G$ such that $Ag \cap B$ has positive measure.

Let $\{S_n^i\}_n$ be a separating sieve for X_i over Y. Because X is ergodic, we can choose $g_n^i \in G$ such that if

$$A_n = \bigcap_{i=1}^{n} p_i^{-1}(S_n^i) \cdot g_n^i,$$

then $\mu(A_n) > 0$. Now let $S_1 = A_1$ and define S_n inductively as follows: choose $h_n \in G$ such that $\mu(S_{n-1} \cap A_n \cdot h_n) > 0$, and let $S_n = S_{n-1} \cap A_n h_n$. Then S_n is a decreasing sequence of sets of positive measure, and $\mu(S_n) \to 0$, since

$$S_n \subset A_n h_n \subset p_1^{-1}(S_n^1 g_n^1 h_n),$$

and $\mu_1(S_n^1) \to 0$ (where μ_1 is the measure on X_1). We now claim that $\{S_n\}$ is a separating sieve for X over Y. Let $N \subset G$ be countable, and for each i, let $N^i = \bigcup_{n=i}^{\infty} N h_n^{-1}(g_n^i)^{-1}$. Let $B^i \subset X_i$ be the corresponding null set for the separating sieve over Y, $\{S_n^i\}$ (i.e., given N^i). Now let $B \subset X$ be $B = \bigcap_i p_i^{-1}(B^i) \cap Z$. Then B is conull. Suppose $x, y \in B$, with $p(x) = p(y)$ (where $p = \theta_n p_n$, which is independent of n), and that $xg_n, yg_n \in S_n$ for some $g_n \in N$. When $i \leq n$, let $h_n^i = g_n h_n^{-1}(g_n^i)^{-1}$, and $h_n^i = h_i^i$ when $i > n$. So $h_n^i \in N^i$ for each i, n. Now for each i, and $n \geq i$,

$$xh_n^i = xg_n h_n^{-1}(g_n^i)^{-1} \in S_n h_n^{-1}(g_n^i)^{-1} \subset A_n(g_n^i)^{-1} \subset p_i^{-1}(S_n^i),$$

i.e., $p_i(x)h_n^i \in S_n^i$ when $n \geq i$; from this it follows that $p_i(x)h_n^i \in S_n^i$ for all (i, n). Similarly, $p_i(y)h_n^i \in S_n^i$ for all (i, n). But since $x, y \in B$, $p_i(x), p_i(y) \in B^i$, we have $p_i(x) = p_i(y)$, and this holds for all i. Hence, since $x, y \in Z$, $x = y$, and this completes the proof.

We are now ready to prove half of Theorem 8.7.

Proof of Theorem 8.7 (Part One). We suppose that X has generalized discrete spectrum over Y, and that X is not atomic. We claim it has a separating sieve over Y. We consider two cases.

Case 1. Y is not atomic. Then consider the set $S = \{\gamma \leq \eta \mid X_\gamma$ has a separating sieve over $Y\}$. Since Y is not atomic, $0 \in S$. If $\gamma \in S$, and $\gamma < \eta$, then $\gamma + 1 \in S$ by Lemma 6.9. If γ is a limit ordinal, then $X_\gamma = \operatorname{inj\,lim} X_\sigma$, $\sigma < \gamma$. If each $\sigma \in S$, it follows by Lemma 8.10 and Proposition 8.3 that $\gamma \in S$. (Recall that η is a countable ordinal.) Thus $\eta \in S$ by transfinite induction.

Case 2. Y is atomic. Let $T = \{\gamma \leq \eta \mid X_\gamma$ is atomic$\}$. Let $\eta_0 = \sup T$. We consider two subcases:

(a) $\eta_0 \in T$. Then $\eta_0 < \eta$, X_{η_0+1} is not atomic, and it follows from Lemma 8.8 that X_{η_0+1} has a separating sieve. Following the argument of Case 1, we conclude that X does also.

(b) $\eta_0 \notin T$. Then η_0 is a limit ordinal and $X_{\eta_0} = \operatorname{inj\,lim} X_\gamma$, $\gamma \in T$. Since each X_γ is atomic, it has discrete spectrum and hence so does X_{η_0}. Thus X_{η_0} has a separating sieve by Lemma 8.8 (take $Z = \{e\}$), and again one can use the argument of Case 1 to complete the proof.

Given the results of Section 7, we shall see that the essence of what remains to prove the converse assertion of Theorem 8.7 is the following lemma. This lemma generalizes a result of Parry [20, Theorem 3] by adapting his argument to the case at hand.

LEMMA 8.11. *If $\phi: X \to Y$ is a (nontrivial) factor map of ergodic G-spaces, and X has a separating sieve over Y, then X is not relatively weakly mixing over Y.*

Proof. Let H be a countable dense subgroup of G. Let $\{S_n\}$ be a separating sieve for X over Y, and let $A \subset X$ be a conull set for the sequence H as in Definition 8.5. It is easy to check that

$$\{(x, y) \mid x, y \in A,\ \phi(x) = \phi(y), \text{ and } xg_n, yg_n \in S_n \text{ for some sequence } g_n \in H\}$$
$$= \bigcap_{n=1}^{\infty} \left(\bigcup_{g \in H} (S_n \times_Y S_n) g^{-1} \right) \cap A \times_Y A.$$

Saying that $\{S_n\}$ is a separating sieve over Y means, given the choice of A, that this set is contained in the diagonal $D \subset X \times_Y X$. Since ϕ is not an essential isomorphism, $(\mu \times_Y \mu)(D) \neq 1$. If $(\mu \times_Y \mu)(D) > 0$, then D is a nonnull nonconull invariant set, so $X \times_Y X$ is not ergodic. In the case when $(\mu \times_Y \mu)(D) = 0$, we must have

$$\lim_{n \to \infty} (\mu \times_Y \mu) \left(\bigcup_{g \in H} (S_n \times_Y S_n) g^{-1} \right) = 0,$$

since $A \times_Y A$ is conull. Since $\mu(S_n) > 0$, we have $(\mu \times_Y \mu)(S_n \times_Y S_n) > 0$, and thus for some n, we must have

$$0 < (\mu \times_Y \mu)\left(\bigcup_{g \in H} (S_n \times_Y S_n)g^{-1}\right) < 1.$$

But this is an H-invariant Borel set. Since the natural representation of G on $L^2(X \times_Y X)$ is continuous and H is dense, $\bigcup_{g \in H} (S_n \times S_n)g^{-1}$ must be essentially G-invariant, which shows that $X \times_Y X$ is not ergodic.

Proof of Theorem 8.7 (Part Two). If X is atomic, it has discrete spectrum, so we suppose that X has a separating sieve over Y. We consider the collection $\mathscr{C}$ of factor spaces Z of X with generalized discrete spectrum over Y, together with an ordered system of factors of Z, $\{p_\gamma, Z_\gamma, \sigma_{\gamma\sigma}\}$, satisfying Definition 8.4. We identify systems which are isomorphic modulo invariant null sets. Let η_Z be the ordinal such that $Z = Z_{\eta_Z}$. We define an ordering on the set $\mathscr{C}$ as follows. Given $\{Z, p_\gamma, Z_\gamma, \phi_{\gamma\sigma}\}$ and $\{Z', p'_\gamma, Z'_\gamma, \phi'_{\gamma\sigma}\}$, define $Z < Z'$ if $\eta_Z \leq \eta_Z$ and for all γ, $\sigma \leq \eta_Z$, $Z_\gamma = Z'_\gamma$, $p_\gamma = \phi_{\eta_Z\gamma}$, $\phi_{\gamma\sigma} = \phi'_{\gamma\sigma}$ modulo G-invariant null sets. We claim any totally ordered subset $T \subset \mathscr{C}$ has an upper bound. Let $S = \{\eta_Z \mid Z \in T\}$. If S has a maximal element, clearly T does also. If not, let $\eta = \sup S$. Since for each $\sigma < \gamma < \eta$ we have closed subspaces $L^2(Z_\sigma) \subsetneqq L^2(Z_\gamma)$ of $L^2(X)$, and $L^2(X)$ is separable, η must be a countable ordinal. Let $Z_\eta = \text{inj lim } Z$, $Z \in T$. It is clear that Z_η is in $\mathscr{C}$ and is an upper bound for T. By Zorn's lemma, there exists a maximal element $Z \in \mathscr{C}$. We claim $Z = X$ (modulo invariant null sets). Suppose not. Then by Proposition 8.6, X has a separating sieve over Z. It follows from Lemma 8.11 that X is not relatively weakly mixing over Z. By Corollary 7.10 the natural (Z, G) cocycle on $L^2(X)$ has nontrivial finite dimensional subcocycle representations, and by Theorem 7.3, there exists a factor space Z' of X such that Z' has relatively discrete spectrum over Z. But then we clearly have $Z' \in \mathscr{C}$ and $Z \lneqq Z'$, contradicting the maximality of Z. Thus $Z = X$, and X has generalized discrete spectrum over Y.

We remark that the above proof shows the following:

COROLLARY 8.12. *Suppose Y is a factor G-space of an ergodic space X, and that for any G-space Z for which there is a sequence of factor G-maps $X' \to Z \to Y$ ($X' \subset X$ essential, and the composition the original factor map) we have X not relatively weakly mixing over Z. Then X has generalized discrete spectrum over Y.*

9. Examples and further properties

An ergodic extension with a relative separating sieve can be viewed as a measure-theoretic analogue of the topological notion of distal extension. We recall the definition of the latter. Let X and Y be compact metric spaces on which G acts continuously, and $\phi: X \to Y$ a continuous surjective G-map.

X is called a distal extension of Y if $x, y \in X$, $\phi(x) = \phi(y)$, and $d(xg_n, yg_n) \to 0$ for some sequence $g_n \in G$ implies $x = y$. The following is immediate.

PROPOSITION 9.1. *Suppose X is a distal extension of Y, that X is minimal* [4], *and that X has nonatomic G-invariant ergodic probability measure μ. Then X has a separating sieve over Y, and hence generalized discrete spectrum over Y.*

Proof. X minimal implies every open set has positive measure; if U_i is a decreasing sequence of open sets whose intersection is a point, it is trivial to check that $\{U_i\}$ is a separating sieve over Y. The remaining statement is just Theorem 8.7.

COROLLARY 9.2. *A minimal distal action preserving an ergodic probability measure has generalized discrete spectrum.*

We consider a specific example to illustrate this corollary. For assertions not proven below, see [2].

Example 9.3. Let N be the nilpotent Lie group consisting of matrices of the form

$$M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbf{R}$. For notational convenience, we denote M by $[x, y, z]$. Let $D \subset N$ be the discrete subgroup consisting of matrices M such that x, y, z are integers. Then N/D is compact and has an N-invariant probability measure. The commutator subgroup is $[N, N] = \{M \in N \mid x = y = 0\}$, and the quotient $N/D[N, N]$ is a torus. It is easy to see that the functions $g_n(M) = \exp(2\pi i n x)$ and $h_n(M) = \exp(2\pi i n y)$ factor to functions on $N/D[N, N]$ when n is an integer, and $\{h_n g_j\}_{(n, j) \in Z^2}$ is an orthonormal basis of $L^2(N/D[N, N])$. Now let $A \in N$ be a matrix of the form $[a, b, 0]$ where $a, b, 1$ are rationally independent. Then $A(t) = [ta, tb, \frac{1}{2}abt^2]$ is the 1-parameter subgroup in N with $A(1) = A$ and $[M]t = [MA(t)]$ defines an action of $\mathbf{R}$ on N/D which is measure preserving. By [2, Theorem IV 3, Theorem V 4.2, and Corollary V 4.5] this action is ergodic, minimal and distal. It is immediate that for each $(j, n) \in Z^2$, $g_j h_n$ is an eigenfunction of the flow, and by [2, Theorem V 4.2], constant multiples of these are the only eigenfunctions. Thus $L^2(N/D[N, N]) \subset L^2(N/D)$ is the closed subspace generated by the finite dimensional $\mathbf{R}$-invariant subspaces. Now let $f_n(M) = \exp(2\pi i n(z - y[x]))$ where $[x]$ is the largest integer $\leq x$. Then the closed subspace of $L^2(N/D)$ generated by $\{g_j h_n f_k\}_{j,n,k \in Z}$ is all of $L^2(N/D)$. Now

$$[x, y, z]A(t) = [ta + x, tb + y, \tfrac{1}{2}abt^2 + tbx + z].$$

So

$$(g_j h_n f_k)([x, y, z]A(t)) = 0(t, x, y) \exp(2\pi i j x) \exp(2\pi i n y) \exp(2\pi i k(z - y[x]))$$

where

$$\theta(t, x, y) = \exp(2\pi i(jta + ntb + kabt^2/2 + ktbx + ky[x] - k(tb + y)[ta + x])).$$

Thus

$$(g_j h_n f_k)([x, y, z]A(t)) = \theta(t, x)(g_j h_n f_k)([x, y, z]).$$

But for each t, $(x, y) \mapsto \theta(t, x, y)$ is in $L^\infty(N/D[N, N])$. Thus for each $(j, n, k) \in Z^3$, $g_j h_n f_k$ is contained in a 1-dimensional field of subspaces over $N/D[N, N]$ that is **R**-invariant. Since these functions generate $L^2(N/D)$, N/D has relatively elementary spectrum over $N/D[N, N]$, and N/D has simple generalized discrete spectrum. In this case, the ordinal $\eta = 2$.

Example 9.4. We remark that for continuous G-actions, the condition of having a separating sieve is more general than being distal. This follows from an example of Kolmogorov of a continuous, ergodic, measure-preserving flow on the torus with discrete spectrum, but no continuous eigenfunctions [see 27]. This flow thus has a separating sieve, but we claim it is not distal. Since every open set has positive measure, and the flow is ergodic, it is also regionally transitive [2, p. 57]. If it were distal, it would be pointwise almost periodic [3, Theorem 1], and hence minimal [2, p. 57]. But minimal distal flows have continuous eigenfunctions [4].

Example 9.5. Another class of actions with generalized discrete spectrum are those with quasi-discrete spectrum. In the case where $G = Z$, these were first studied systematically by Abramov [1]. Subsequently, Wieting has considered these actions when G is an arbitrary locally compact abelian group. We review Wieting's definition. Let G be a locally compact abelian group and X a Lebesgue G-space. We suppose that X is totally ergodic, i.e., that $\{\chi \in G^* \mid \chi$ is a subrepresentation of U_g on $L^2(X)\}$ is torsion free. Let $E_0 = S^1$ (=circle) and define E_n, $n \geq 1$, inductively by

$$E_n = \{f \in L^\infty(X) \mid |f(x)| = 1 \quad \text{and} \quad U_g f/f \in E_{n-1} \quad \forall\, g \in G\}.$$

If $E = \bigcup_{n=0}^{\infty} E_n$ generates $L^2(X)$, then X is said to have quasi-discrete spectrum. We show that this implies that X has generalized discrete spectrum. It is clear that E_n is an increasing sequence of G-invariant multiplicative subgroups of the group of functions of absolute value 1 on X. Generalizing a result of Abramov [1, 7°], Wieting showed [24, Theorem P] that if $f, g \in E$ are not constant multiples of one another, they are perpendicular. Now the finite linear combinations of elements of E_n form a G-invariant *-subalgebra of $L^\infty(X)$. By Corollary 2.2, there exists a sequence of factors X_n of X

$$\to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \{e\}$$

such that E_n is an orthonormal basis, together with constant multiples, of $L^2(X_n)$, and

$$L^2(X) = \overline{\bigcup L^2(X_n)}.$$

If $f \in E_n$, then $(U_g f)/f \in E_{n-1}$ and thus $(U_g f)/f$ is a function on X_{n-1} for each $g \in G$. Therefore, each $f \in E_n$ is contained in a 1-dimensional G-invariant field over X_{n-1}, and since E_n generates $L^2(X_n)$, X_n has relatively elementary spectrum over X_{n-1}. Thus, X has simple generalized discrete spectrum, and the ordinal $\eta_X \leq \omega$, the first infinite ordinal.

We remark that even when $G = Z$ and the ordinals are finite, not every G-space with simple generalized discrete spectrum has quasi-discrete spectrum. If we restrict the **R**-action of Example 9.3 to the integers, the resulting Z-action is still ergodic (this follows from [2, Theorem V 4.2]) and has generalized discrete spectrum. However, it cannot have quasi-discrete spectrum since it embeds in an **R**-action [8, Theorem 4.1]. We shall examine in Section 10 the question of how one distinguishes the transformations with quasi-discrete spectrum within the class of transformations with generalized discrete spectrum.

When $G = Z$ or **R**, any transitive action (preserving a probability measure) has discrete spectrum. For more general groups, this statement is, of course, no longer true. The following proposition describes when a transitive action has generalized discrete spectrum.

PROPOSITION 9.6. *Let $H \subset G$ be a closed subgroup such that G/H has finite invariant measure. Then the action of G on G/H has generalized discrete spectrum if and only if there exists a countable ordinal η, and a collection of closed subgroups of G, $H_\gamma \supset H$, $\gamma \leq \eta$, such that:*

(i) *$H_0 = G$, $H_\eta = H$; if $\sigma \lneqq \gamma$, then $H_\gamma \subsetneqq H_\sigma$.*
(ii) *The action of H_γ on $H_\gamma/H_{\gamma+1}$ has discrete spectrum.*
(iii) *If γ is a limit ordinal, $H_\gamma = \bigcap_{\sigma<\gamma} H_\sigma$.*

Proof. As every factor of a transitive action is transitive, and is determined by a (conjugacy class of a) closed subgroup, the proof is readily reduced to demonstrating the following statement: If $H \subset K \subset G$ (so $G/H \to G/K$), then G/H has relatively discrete spectrum over G/K if and only if the action of K on K/H has discrete spectrum. Now $(G/K, G)$ cocycles correspond to representations of K [23, Theorem 8.27], and the natural G/K cocycle representation on the Hilbert bundle $L^2(G/H)$ will correspond to the representation of K on the fiber over $[e]$ in $L^2(G/H)$, i.e., to the natural representation of K on $L^2(K/H)$. Under this correspondence, the $(G/K, G)$ cocycle has discrete spectrum if and only if the representation of K does also.

Example 9.7. An example of a transitive action with generalized discrete spectrum is the action of a connected, simply-connected nilpotent Lie group on a nilmanifold. If N is such a group, and $D \subset N$ a uniform, discrete subgroup, the proof of [2, Theorem IV.3] shows that N acts distally on N/D. ([2, Theorem IV.3] states that a one-parameter subgroup of N acts distally on N/D, but an examination of the proof shows that the assumption that the elements of N considered lie in a 1-parameter subgroup was never used.) By Corollary 9.2, this action has generalized discrete spectrum.

We remark that the structure of N/D given by Definition 8.4 (or Proposition 9.6) gives a corresponding decomposition of $L^2(N/D)$ into mutually orthogonal G-invariant subspaces. A thorough study of the decomposition of $L^2(N/D)$ has been made by Moore [18], Richardson [22], and Howe [13]. It would be interesting to see how the decomposition above fits into their scheme.

Another question that arises is to describe which subgroups of a given group, say in particular, which lattice subgroups, define homogeneous spaces with generalized discrete spectrum. One might then try to obtain an understanding of the decomposition of L^2 of the homogeneous space, based upon the L^2-decomposition defined via Definition 8.4.

If G is an abelian group, and X is a transitive G-space, every irreducible (X, G) cocycle representation is one-dimensional. This is because such cocycles correspond to the representations of the stability group of the action. It is thus perhaps somewhat surprising to find that if X is not transitive, there may exist irreducible cocycle representations of dimension greater than one, even if G is abelian. In [17] (see also [14]), Mackey gives an example of an ergodic G-space X, with G abelian, and a minimal cocycle $\alpha\colon X \times G \to K$, where $K = K_\alpha$, and K is compact but not abelian. Thus, by Proposition 3.12, there exist irreducible (X, G) cocycle representations that have dimension greater than one. Equivalently, there exist extensions of X with relatively discrete but not relatively elementary spectrum over X. In virtual group terms, a virtual subgroup of an abelian group can have nonabelian "homomorphic images."

In topological dynamics, there is another example of nonabelian phenomena arising from an abelian situation. If G acts continuously on a compact metric space X, let $\phi(g)$ denote the homeomorphism of X corresponding to $g \in G$. Let $E(G, X)$ be the closure of $\phi(G)$ in X^X under the topology of pointwise convergence. $E(G, X)$ can be shown to be a semigroup (under composition) [4, p. 484] and is called the Ellis semigroup of the action. Now even if G (and hence $\phi(G)$) is abelian, $E(G, X)$ may not be.

We now point out in the consideration of distal actions, the occurrence of these types of nonabelian phenomena are related.

PROPOSITION 9.8. *Suppose G is a locally compact abelian group, and X is a compact metric space, minimal and distal under a continuous G-action, and supporting an ergodic probability measure. If $E(G, X)$ is abelian, then X has simple generalized discrete spectrum.*

Proof. This follows from [28, Theorem 1.2], once one notices that restricted to each fiber, Image (P_λ) is one-dimensional. (Notation as in [28].) One can see this from the definition of P_λ ([28, p. 18]), and the fact that $I_y \subset E(G, X)$ is abelian.

We now turn to consideration of the properties of factors of actions with generalized discrete spectrum.

LEMMA 9.10. *Suppose $X \to Y \to Z$ are factor G-maps and that X has generalized discrete spectrum over Z. Then Y is not relatively weakly mixing over Z.*

Proof. Let $\{X_\gamma \mid \gamma \leq \eta\}$ be the factors of X showing that X has generalized discrete spectrum over Z. Consider

$$S = \{\gamma \leq \eta \mid (L^2(X_\gamma) \ominus L^2(Z)) \perp (L^2(Y) \ominus L^2(Z))\}.$$

Assuming that Y is a nontrivial extension of Z, $\eta \notin S$. Let σ be the first ordinal not in S. It follows from property (iii) of Definition 8.4 that σ is not a limit ordinal. Hence, $\sigma - 1$ exists, and is a maximal element of S. Let us denote $X_{\sigma-1}$ by W. By Proposition 1.5, we have a factor map $X \to W \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} & W \arrow[dr] & \\ X \arrow[ur] \arrow[r] \arrow[dr] & W \times_Z Y \arrow[u] \arrow[d] & Z \\ & Y \arrow[ur] & \end{tikzcd}$$

Now projection of $L^2(W \times_Z Y)$ into $L^2(X_\sigma)$ is a G-map commuting with multiplication by $L^\infty(W)$. Furthermore, since $\sigma \notin S$, the image of $L^2(W \times_Z Y)$ in $L^2(X_\sigma)$ is not contained in $L^2(W)$. Since X_σ has relatively discrete spectrum over W, it follows that $W \times_Z Y$ must have some discrete spectrum over W; i.e., $(W \times_Z Y) \times_W (W \times_Z Y)$ is not ergodic. But it is easy to see that this space is isomorphic to $W \times_Z Y \times_Z Y$, which is thus not ergodic. If Y is relatively weakly mixing over Z, then so is $Y \times_Z Y$ (Corollary 7.12), and then $W \times_Z Y \times_Z Y$ would be ergodic (Corollary 7.11). Thus Y is not relatively weakly mixing over Z.

THEOREM 9.11. *If $X \to Y \to Z$ are factor G-maps, and X has generalized discrete spectrum over Z, so does Y.*

Proof. By Corollary 8.12, it suffices to show that if $Y \to Y_0 \to Z$ are factor maps, then Y is not relatively weakly mixing over Y_0. But if X has generalized discrete spectrum over Z, it also has generalized discrete spectrum over Y_0 by Theorem 8.7 and Proposition 8.6. The result now follows by Lemma 9.10.

COROLLARY 9.12. *A factor of an action with generalized discrete spectrum also has generalized discrete spectrum.*

We remark that the analogous result holds for distal actions [4, Theorem 3.3], and for transformations with quasi-discrete spectrum [8, Corollary 2.8].

COROLLARY 9.13. *The fibered product of two extensions of a G-space Y does not have generalized discrete spectrum over Y if one of the factors is relatively* *weakly mixing over Y. In particular, the product of two G-spaces will not have generalized discrete spectrum if one of them has continuous spectrum.*

PROPOSITION 9.14. *If T is an invertible transformation such that the associated Z-action has generalized discrete spectrum, then T has entropy* 0. *More generally, a* Z^n *action with generalized discrete spectrum has* 0 *joint entropy* [24].

Proof. By Theorem 8.7, if the space is not atomic, the action has a separating sieve, and an argument of Parry [20] (see also [24, Theorem N]) shows that the entropy is zero.

10. Applications to affine actions and quasi-discrete spectrum

We now consider how the above theory applies to a special class of actions, namely affine actions on compact abelian groups. Affine transformations have been studied by various authors [6], [11], and much of this theory has been generalized to affine actions of arbitrarily locally compact abelian groups by Wieting [24]. Theorem 10.7 below gives an algebraic criterion for an affine action to have generalized discrete spectrum. When $G = Z$, we go on to show in Theorem 10.10 that every totally ergodic affine transformation on a compact connected abelian group with generalized discrete spectrum actually has quasi-discrete spectrum. In light of Abramov's results [1], this enables us to distinguish the class of totally ergodic transformations with quasi-discrete spectrum as the class (up to isomorphism) of totally ergodic affine transformations on compact, connected, abelian groups with generalized discrete spectrum.

We recall the central notions of the theory of affine actions [6], [24]. Let G be a locally compact abelian group and X a compact abelian group. A homeomorphism $\phi: X \to X$ is called affine if it is of the form $\phi(x) = x_0 A(x)$, where $A: X \to X$ is an automorphism, and $x_0 \in X$. If G acts continuously on X, by affine homeomorphisms, we will call X an affine G-space. Then, for each $x \in X$, $g \in G$, we can write $x \cdot g = x_0(g) \cdot A(g)(x)$, where $x_0(g) \in X$ and $A(g) \in \mathrm{Aut}(X)$. The map $A: G \to \mathrm{Aut}(X)$ is a continuous homomorphism, and $x_0: G \to X$ is a continuous crossed homomorphism with respect to A; i.e.,

$$x_0(gh) = x_0(g) \cdot [A(g)(x_0(h))].$$

Conversely, given A and x_0, satisfying the above, they define an affine action. We will thus identify affine actions with pairs (x_0, A). If X and Y are affine G-spaces, we shall call X an affine extension of Y if there exists a surjective G-homomorphism $\phi: X \to Y$.

PROPOSITION 10.1. *Suppose the affine actions of G on X and Y are given by* (x_0, A), (y_0, B) *respectively. Then a surjective homomorphism* $\phi: X \to Y$ *is a G-map if and only if* $\phi(x_0(g)) = y_0(g)$ *and* $\phi \circ A(g) = B(g) \circ \phi$.

If $A: X \to X$ is an automorphism, let $A^*: X^* \to X^*$ be the induced automorphism of dual groups.

PROPOSITION 10.2. *Suppose* (x_0, A) *is an affine action of* G *on* X, *and suppose* $D \subset X^*$ *is an* $A(G)^*$*-invariant subgroup. Let* $\phi\colon X \to Y = D^*$ *be the map induced by inclusion. Let* $\tilde{A}\colon G \to \operatorname{Aut}(Y)$ *be the map* $\tilde{A}(g) = (A(g)^* \mid D)^*$ *and* $y_0(g) = \phi(x_0(g))$. *Then* $(y_0, \tilde{A})$ *is an affine action of* G *on* Y, *and* ϕ *is then a* G*-map.*

A criterion for determining when a given ergodic affine G-space is weakly mixing was established for Z-actions by Hahn [6, Corollary 3], and subsequently extended to arbitrary abelian group actions by Wieting [24, Theorem H]. An extension of this analysis will enable us to determine when an ergodic affine extension is relatively weakly mixing, and more generally, when it has relatively generalized discrete spectrum. We begin with some preparatory lemmas.

If $\phi\colon X \to Y$ is a surjective homomorphism, let $K = \ker \phi$. Let μ_K, μ_X, μ_Y be the Haar measures. Choose a Borel section $\theta\colon Y \to X$ for ϕ. The following lemma is then straightforward.

LEMMA 10.3. $y \mapsto \mu_y = \mu_K \cdot \theta(y)$ *is a decomposition of* μ_X *with respect to* μ_Y *over the fibers of* ϕ.

We have an induced map $\phi^*\colon Y^* \to X^*$ that is injective, and we shall identify Y^* with its image in X^*. Then the inclusion $K \to X$ induces an isomorphism $X^*/Y^* \to K^*$.

LEMMA 10.4. *If* $f, g \in X^*$ *and* $f \not\equiv g$ *in* X^*/Y^*, *then* $f_y \perp g_y$ *for each* $y \in Y$ (*where* $f_y = f \mid \phi^{-1}(y)$).

Proof. Let $f_0 = f \mid K$. Since $f \not\equiv g$ in X^*/Y^*, $f_0 \neq g_0$ which implies $f_0 \perp g_0$. Hence $f_0 \cdot \theta(y)^{-1} \perp g_0 \cdot \theta(y)^{-1}$ in $L^2(\phi^{-1}(y), \mu_y)$, for each $y \in Y$. But for $x \in \phi^{-1}(y)$,

$$(f_0 \cdot \theta(y)^{-1})(x) = f(x\theta(y)^{-1}) = f_y(x)f(\theta(y)^{-1}).$$

Similarly,

$$(g_0 \cdot \theta(y)^{-1})(x) = g_y(x)g(\theta(y)^{-1}).$$

It follows immediately that $f_y \perp g_y$.

With ϕ as above, $\psi\colon X \times X \to Y$ defined by $\psi(x, z) = \phi(x)\phi(z)^{-1}$ is a surjective homomorphism, and $\ker \psi = X \times_Y X$. Thus, $X \times_Y X$ is a compact abelian group. We give a realization of its dual.

LEMMA 10.5. *Let* $s\colon X^*/Y^* \to X^*$ *be a section* (*not necessarily homomorphic*) *of the natural projection* $p\colon X^* \to X^*/Y^*$, *with* $s([1]) = 1$. *Then the map*

$$T\colon Y^* \times s(X^*/Y^*) \times s(X^*/Y^*) \to (X \times_Y X)^*$$

defined by $T(h, f, g) = h(f \times_Y g)$ *is a bijection.*

Proof. We note first that the range of T is clearly contained in $(X \times_Y X)^*$. From Lemma 10.4, it follows that the T-images of distinct elements are orthogonal, and hence that T is injective. We now claim T is surjective. Any character of $X \times_Y X$ is of the form $\lambda \times \beta \mid X \times_Y X$ where $\lambda, \beta \in X^*$ [10, 24.12]. We can write $\lambda = h_1 s(p(\lambda))$ and $\beta = h_2 s(p(\beta))$ where $h_i \in Y^*$. Then we have

$$\lambda \times \beta \mid X \times_Y X = h_1 h_2 (s(p(\lambda)) \times_Y s(p(\beta)))$$

showing surjectivity.

Now suppose that the homomorphism $\phi: X \to Y$ is an extension of ergodic affine G-spaces, where G is locally compact and abelian. The following result is a partial generalization of [6, Corollary 3] and [24; Theorem H].

THEOREM 10.6. *X is relatively weakly mixing over Y if and only if every nonidentity element in X^*/Y^* has an infinite orbit under $A(G)^*$ (where the action on X is given by (x_0, A)).*

Proof. (i) Suppose $f \in X^*/Y^*$, $f \not\equiv 1$ and that f has a finite orbit, say $f = f_1, f_2, \ldots, f_n$ under $A(G)^*$. Then the closed subspace of $L^2(X)$ generated by

$$\left\{ \sum_{i=1}^{n} h_i f_i \mid h_i \in L^\infty(Y) \right\}$$

is a G-invariant finite dimensional subbundle of $L^2(X)$ over Y, that is not equal to $L^2(Y)$. Thus X is not relatively weakly mixing by Corollary 7.10.

(ii) Conversely, suppose every nonidentity element in X^*/Y^* has infinite orbit. We claim $X \times_Y X$ is ergodic. Let the affine action of G on Y be given by (y_0, B). The action of G on $X \times_Y X$ is also affine, say (z_0, C). If every nontrivial orbit in $(X \times_Y X)^*$ under $C(G)^*$ is infinite, $X \times_Y X$ is ergodic by [24; Theorem C]. (See also [6, Theorem 1] when $G = Z$.) So suppose $\lambda \in (X \times_Y X)^*$ has a finite orbit. By Lemma 10.5, $\lambda = h \cdot (f \times_Y k)$, for $h \in Y^*, f, k \in s(X^*/Y^*)$. Now for $g \in G$,

$$(*) \qquad C(g)^*(\lambda) = C(g)^*(h \cdot (f \times_Y k)) = B(g)^* h(A(g)^* f \times_Y A(g)^* k).$$

We claim that λ having a finite orbit implies that f and k have finite orbits in X^*/Y^*. To see this, suppose $C(g)^*\lambda = \lambda$. Now $A(g)^*f = \alpha f_0$ and $A(g)^*k = \beta k_0$ where $\alpha, \beta \in Y^*$ and $f_0, k_0 \in s(X^*/Y^*)$. Equation $(*)$ implies

$$C(g)^*(\lambda) = \alpha\beta B(g)^* h(f_0 \times_Y k_0),$$

and since $C(g)^*\lambda = \lambda$, we have

$$\alpha\beta B(g)^* h(f_0 \times_Y k_0) = h(f \times_Y k).$$

By Lemma 10.5, $f = f_0$, $k = k_0$. Thus, $C(g)^*\lambda = \lambda$ implies $A(g)^*f \equiv f$ and $A(g)^*k \equiv k$ in X^*/Y^*, showing that f and k have finite orbits in X^*/Y^*. By the hypothesis of the theorem, $f \equiv k \equiv 1$ in X^*/Y^*, and since $s(1) = 1$, $f = k = 1$. This is turn, via equation $(*)$ implies that h has a finite orbit under $B(G)^*$. By [24, Theorem C] (see also [6, Theorem 4]), the ergodicity of Y implies that there is $g \in G$ such that $h(y_0(g)) \neq 1$. Since $\lambda = h$, it readily follows that $\lambda(z_0(g)) \neq 1$, and by [24, Theorem C] ([6, Theorem 4]), that $X \times_Y X$ is ergodic.

Theorem 10.6 provides an algebraic criterion for determining when an affine extension is relatively weakly mixing. We now establish an algebraic criterion for determining when an affine extension has relatively generalized discrete spectrum.

THEOREM 10.7. *Suppose $\phi: X \to Y$ is an affine extension. Then the following are equivalent.*

(a) *X has generalized discrete spectrum over Y.*

(b) *There exists a countable ordinal ξ, a collection of compact abelian groups X_η, $\eta \leq \xi$, and for each $\eta < \sigma \leq \xi$, a surjective, noninjective homomorphism $\phi_{\sigma\eta}: X_\sigma \to X_\eta$ such that:*

- (i) *For $\rho < \eta < \sigma$, $\phi_{\sigma\rho} = \phi_{\eta\rho}\phi_{\sigma\eta}$.*
- (ii) *$X_0 = Y$, $X_\xi = X$ and $\phi_{\xi 0} = \phi$.*
- (iii) *Each X_η is an affine G-space and $\phi_{\sigma\eta}$ is a G-map.*
- (iv) *Every element of $X^*_{\eta+1}/X^*_\eta$ has a finite orbit under $A(G)^*$. (Here dual groups are identified with their images under the induced embeddings.)*
- (v) *If η is a limit ordinal, $X_\eta = \operatorname{inj\,lim}_{\sigma<\eta} X_\sigma$.*

(c) *There exists a countable ordinal ξ and a collection of $A(G)^*$ invariant subgroups D_η of X^*, $\eta \leq \xi$ such that:*

- (i) *$D_0 = Y^*$, $D_\xi = X^*$.*
- (ii) *$\sigma < \eta$ implies $D_\sigma \subsetneqq D_\eta$.*
- (iii) *Every element of $D_{\eta+1}/D_\eta$ has finite orbit under $A(G)^*$.*
- (iv) *If η is a limit ordinal, $D_\eta = \bigcup_{\sigma<\eta} D_\sigma$.*

Proof. (b) $\Rightarrow$ (a) It suffices to see that $X_{\eta+1}$ has relatively discrete spectrum over X_η. This follows from condition (b)(iv), as in the proof of (i) of Theorem 10.6.

(b) $\Rightarrow$ (c) Let $D_\eta = \phi^*_{\xi\eta}(X^*_\eta)$.

(c) $\Rightarrow$ (b) Let $X_\eta = D^*_\eta$, and for $\sigma < \eta$, let $\phi_{\eta\sigma}: X_\eta \to X_\sigma$ be the map induced by the inclusion $D_\sigma \to D_\eta$. X_η is an affine G-space by Proposition 10.2, and the remaining assertions follow easily.

(a) $\Rightarrow$ (c) Let $\{D_\eta \mid \eta \leq \xi\}$ be a maximal collection of subgroups satisfying the conditions of (c), with the possible exception of the condition $D_\xi = X^*$. This exists by Zorn's lemma, as in the proof of Theorem 8.7. Let $Z = D^*_\xi$. Then by Proposition 10.2, Z is an affine G-space, and there are G-homomorphisms

$$X \xrightarrow{\phi_1} Z \xrightarrow{\phi_2} Y$$

such that $\phi_2\phi_1 = \phi$. We claim $Z = X$. Suppose not. Now X has relatively generalized discrete spectrum over Y, and hence also over Z by Theorem 8.7 and Proposition 8.6. In particular X is not relatively weakly mixing over Z.

By Theorem 10.6, there exists $f \in X^*, f \notin Z^*$, such that the orbit of f in X^*/Z^* under $A(G)^*$ is finite. Let $D_{\xi+1} = \{h \in X^* \mid \text{orbit of } h \text{ under } A(G)^* \text{ in } X^*/Z^* \text{ is finite}\}$. Then $D_{\xi+1}$ is an $A(G)^*$-invariant subgroup of X^*, and $D_{\xi+1} \supsetneqq D_\xi$. By the definition of $D_{\xi+1}$, we see that $\{D_\eta \mid \eta \leq \xi\}$ is not maximal, which is a contradiction. Thus $Z = X$.

We now use Theorem 10.6 to prove that when $G = Z$, any totally ergodic affine transformation with generalized discrete spectrum actually has quasi-discrete spectrum. We begin with a lemma that is a small modification of Abramov's uniqueness theorem for transformations with quasi-discrete spectrum [1]. (See also [7; Theorem 3]).

LEMMA 10.8. *Let K and X be compact metric spaces, each with a probability measure, positive on open sets. Let S and T be totally ergodic measure-preserving homeomorphisms of K and X respectively. Suppose that S and T have quasi-discrete spectrum, with quasi-eigenfunction groups E_S and E_T such that:*

(i) $E_S \subset C(K)$, $E_T \subset C(X)$.
(ii) *The linear spans $[E_S]$ and $[E_T]$ are uniformly dense in $C(K)$ and $C(X)$ respectively.*
(iii) *The system of quasi-eigenvalues of S and T are equivalent* [1], [7].

Then there exists a homeomorphism $\phi\colon X \to K$ such that $S\phi = \phi T$.

Proof. Under the assumption that the systems of quasi-eigenvalues are equivalent, Abramov constructs [1; proof of uniqueness theorem] (using somewhat different notation) a unitary map $V\colon L^2(K) \to L^2(X)$, such that:

(i) $V \mid E_S$ is a group isomorphism $E_S \to E_T$.
(ii) $T^*V = VS^*$, where T^*, S^* are the induced maps in L^2.

It follows as in [9; p. 47] or [24; proof of Theorem A], that $V(L^\infty(K)) = L^\infty(X)$ and that V is an isometry of these Banach spaces. In particular, $V\colon [E_S] \to [E_T]$ is an involutive, multiplicative isometry (since open sets have positive measure) of dense *-subalgebras of $C(K)$ and $C(X)$, and hence is an involutive multiplicative isometry $C(K) \to C(X)$. It follows that there is a homeomorphism $\phi\colon X \to K$ such that $\phi^* = V\colon C(K) \to C(X)$. Since $T^*\phi^* = \phi^*S^*$, we also have $\phi T = S\phi$.

COROLLARY 10.9. *Let T be a totally ergodic affine transformation of a compact abelian group X, with quasi-discrete spectrum, and whose group of quasi-eigenfunctions consists of the constant multiples of elements of X^*. Then T is totally minimal* [7].

Proof. By [7; Theorem 4 and Corollary to Theorem 7], there exist S and K such that all the hypotheses of Lemma 10.8 are satisfied and such that S is totally minimal. The conclusion of Lemma 10.8 implies that T is totally minimal.

THEOREM 10.10. *A totally ergodic transformation has quasi-discrete spectrum if and only if it is isomorphic to a totally ergodic affine transformation on a compact, connected abelian group with generalized discrete spectrum.*

Proof. (i) Any totally ergodic transformation with quasi-discrete spectrum is isomorphic to an affine transformation on a compact, connected abelian group by a theorem of Abramov [1], and it has generalized discrete spectrum by Example 9.5.

(ii) Conversely, let $T = (a, A)$ be a totally ergodic affine transformation of a compact connected abelian group X, and assume that T has generalized discrete spectrum. Consider the set $\mathscr{A}$ of all quotient groups Y of X on which T induces an affine transformation T_Y on Y, such that T_Y is minimal. We can order $\mathscr{A}$ by setting $Y > Z$ if Z is in turn a quotient of Y. If $\{Y_\alpha\}$ is a totally ordered collection in $\mathscr{A}$, we claim inj lim Y_α is again in $\mathscr{A}$. T will induce a transformation on inj lim Y_α by Proposition 10.2, since the dual (inj lim $Y_\alpha)^* = \bigcup_\alpha Y_\alpha^*$ is A^*-invariant. Furthermore, $T_{(\text{inj lim } Y_\alpha)}$ is minimal by the remark of Furstenberg [5; p. 28] that the inverse limit of minimal transformations is minimal. By Zorn's lemma, $\mathscr{A}$ has a maximal element Y. Now T_Y is a minimal affine transformation on a compact connected abelian group Y. Since Y is connected, T_Y is actually totally minimal [7; p. 310]. It follows from the theorem of [12] that for each n, T_Y^n has quasi-discrete spectrum, and by [11; Theorem 3] that the quasi-eigenfunctions of T_Y^n are exactly the multiples of elements of Y^*.

To prove the theorem, it suffices to show $X = Y$. Suppose not. Since X has generalized discrete spectrum, it has generalized discrete spectrum over Y, and hence is not relatively weakly mixing over Y. By Theorem 10.6, there exists $f \in X^*, f \notin Y^*$, such that f has a finite orbit under A^* in X^*/Y^*. Hence, there is an integer n such that $(A^*)^n(f) \equiv f$ in X^*/Y^*. Let B be the subgroup of X^* generated by $Y^*, f, A^*f, \ldots, (A^*)^{n-1}f$. Then B is invariant under A^*, and for each element $h \in B$, $(A^n)^*h \equiv h$ in X^*/Y^*. Let $Z = B^*$. By Proposition 10.2, we have an induced affine transformation T_Z on Z. Furthermore, for $h \in B$, $(T^n)^*h = c\lambda h$, where $c \in S$ and $\lambda \in Y^*$ (here S is the unit circle). Since T_Y^n has quasi-discrete spectrum with quasi-eigenfunctions $S \cdot Y^*$, this implies that T_Z^n also has quasi-discrete spectrum. Since each element of B is a quasi-eigenfunction, and T_Z^n is totally ergodic (being a factor of the totally ergodic transformation T^n), the quasi-eigenfunctions of Z are exactly the elements of $S \cdot B$ [1, 1.7]. By Corollary 10.9, T_Z^n is totally minimal, which implies that T_Z is totally minimal. Since $Z \neq Y$, this contradicts the maximality of Y. Hence $X = Y$. Therefore, T is minimal on X, and by the theorem of [12], T has quasi-discrete spectrum.

We conclude this section with another application, in a somewhat different direction, to quasi-discrete spectrum. Namely, we show how the existence theorem (6.4) can be applied to give a new proof of Abramov's existence theorem for transformations with quasi-discrete spectrum [1, paragraph 3]. This theorem has been generalized by Wieting [24, Theorem S] to actions of a locally compact abelian group, and it is in this context that we shall work. We begin by recalling in more detail (see Example 9.5) Wieting's definition of quasi-discrete spectrum [24, 3.1]. Let G be locally compact and abelian, and X a Lebesgue G-space. We assume the action is totally ergodic; i.e., one of the following equivalent [24, p. 83] statements holds:

(i) The point spectrum is torsion free.

(ii) If $H \subset G$ is a subgroup such that G/H is finite, then H acts ergodically on X.

Let $F = F(X) = \{f\colon X \to \mathbf{C} \mid f \text{ Borel}, |f(x)| = 1\}$, with functions identified if they agree almost everywhere. F is an abelian group under pointwise multiplication, and G acts naturally on F by $(f \cdot g)(x) = f(xg)$, for $f \in F$, $g \in G$. If $\gamma\colon G \to F$, γ is called a crossed-homomorphism if $\gamma(g_1 g_2) = \gamma(g_1)(\gamma(g_2)g_1)$. Let

$$\Gamma = \Gamma(X) = \{\gamma\colon G \to F(X) \mid \gamma \text{ is a crossed homomorphism}\}.$$

Then Γ is an abelian group under pointwise multiplication, and $h \in G$ acts on Γ by $(\gamma \cdot h)(g) = \gamma(g) \cdot h$, where the right side is the action of h on F. Define the map $Q\colon F \to \Gamma$ by $Q(f)(g) = (f \cdot g)/f$. It is easy to check that Q is a G-homomorphism. The kth order quasi-eigenfunctions are defined inductively as follows:

$$E_0 = \{f \in F \mid f \text{ is constant}\}, \quad E_k = \{f \in F \mid Q(f)(g) \in E_{k-1} \text{ for each } g\}.$$

A totally ergodic G-space X is said to have quasi-discrete spectrum if $E = \bigcup E_k$ generates $L^2(X)$ as a closed subspace. The order of X is the first integer k such that $E_k = E_{k+1}$ if this exists, and is ∞ otherwise.

$B_k = Q(E_k)$ is called the group of kth order quasi-eigenvalues. $B = \bigcup B_k$ is a G-invariant subgroup of $\Gamma(X)$ and the existence theorem is meant to answer the following question: Given an abelian group A on which G acts by homomorphisms, and an increasing sequence of G-invariant subgroups $A_0 \subset A_1 \subset \cdots \subset A$ such that $A = \bigcup A_n$, when does there exist a totally ergodic G-space with quasi-discrete spectrum such that A_k is (up to compatible isomorphisms) the group of kth order quasi-eigenvalues?

Before answering this, we need one more concept. Because G acts ergodically on X, $\ker(Q) = E_0$, which is divisible. Thus, there exists a homomorphic section of Q, i.e., a homomorphism $\phi\colon B \to E$ such that $Q(\phi(\gamma)) = \gamma$ for all $\gamma \in B$. Now $Q(\phi(b \cdot g)) = Q(\phi(b) \cdot g)$ for all b, g, so there is a constant $c(b, g)$ such that

$$c(b, g) = \frac{\phi(b) \cdot g}{\phi(bg)}.$$

It is easy to check that c is a cocycle, and that it is multiplicative, i.e.,

$$c(b_1 b_2, g) = c(b_1, g)c(b_2, g).$$

We call c the cocycle defined by the section ϕ.

Wieting's generalization of Abramov's theorem is:

THEOREM 10.11. (Wieting, Abramov). *Let A be a torsion free abelian group on which G acts by homomorphisms. For each n, let*

$$A_n = \{a \in A \mid N(g_1)\cdot \ldots \cdot N(g_n)(a) = 1, \text{ for all } g_1, \ldots, g_n \in G\},$$

where $N(g): A \to A$ is defined by $N(g)(a) = (a \cdot g)/a$. Suppose further that there exists a multiplicative cocycle $c: A \times G \to U(1)$ (= circle) such that the corresponding map $c_0: A_1 \to G^*$ (= dual of G) defined by $c_0(a)(g) = c(a, g)$ is injective. Then there exists a totally ergodic G-space X with quasi-discrete spectrum, and a G-isomorphism $\psi: A \to B$ such that $\psi(A_n) = B_n$. Moreover, the cocycle

$$d: B \times G \to U(1), \quad d(b, g) = c(\psi^{-1}(b), g)$$

is the cocycle defined by a section.

Proof. We claim it suffices to construct a sequence of G-spaces $\{X_n\}$ so that the following conditions are satisfied.

(1) There exists a factor G-map $p_n: X_n \to X_{n-1}$ ($n \geq 2$). We note that this induces maps $F(X_{n-1}) \to F(X_n)$ and $\Gamma(X_{n-1}) \to \Gamma(X_n)$, both of which we denote by p_n^*.

(2) X_n has quasi-discrete spectrum of order n. Let B^n be the group of all quasi-eigenvalues on X_n, E^n the eigenfunctions. We further suppose that $p_n^*: B^{n-1} \to B^n_{n-1}$ is an isomorphism, where B^n_{n-1} is the group of $(n-1)$st order quasi-eigenvalues on X_n.

(3) There exist G-isomorphisms $\psi_n: A_n \to B^n$ such that

$$\begin{tikzcd} A_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d] & B^{n-1} \arrow[d, "p_n^*"] \\ A_n \arrow[r, "\psi_n"] & B^n \end{tikzcd}$$

commutes.

(4) There exists a homomorphism $\phi_n: B^n \to E^n$ such that:

(a) $Q \circ \phi_n$ is the identity.

(b)

$$\begin{tikzcd} B^{n-1} \arrow[r, "\phi_{n-1}"] \arrow[d, "p_n^*"'] & E^{n-1} \arrow[d, "p_n^*"] \\ B^n \arrow[r, "\phi_n"] & E^n \end{tikzcd}$$

commutes.

(c) If $d_n: B^n \times G \to U(1)$ is defined by $d_n(b, g) = c(\psi_n^{-1}(b), g)$, then d_n is the cocycle defined by the section ϕ_n.

In the case where the order of A is $n < \infty$ then X_n is the space, ϕ_n, ψ_n the maps, required in the theorem. (By the order of A, we mean the first integer n such that $A_n = A_{n+1}$.) If order $(A) = \infty$, let $X = \text{inj lim } X_n$, and $q_n\colon X \to X_n$ the associated factor map. It follows easily that X is totally ergodic, has quasi-discrete spectrum, and using [24, Theorem P], it is clear that the group of nth order quasi-eigenvalues on X is $q_n^*(B^n)$. The compatibility conditions (3) and (4b) allow one to construct a suitable isomorphism and section. Thus, it remains to show that such a sequence of spaces X_n exists. We proceed inductively and begin by constructing X_1. From the hypothesis of the theorem, $c_0(A_1)$ is a countable subgroup of G^*, and by Corollary 6.5, there exists a G-space X_1 with discrete spectrum, and this spectrum is $c_0(A_1)$. We can naturally identify $c_0(A_1)$ with B^1, and let $\psi_1\colon A_1 \to B^1$ the corresponding isomorphism. It is easy to check that for any section $\phi_1\colon B^1 \to E^1$. The cocycle defined by ϕ_1 is just $c(\psi_1^{-1}(b), g)$. We now assume $X_1, \dots, X_n$ have been constructed satisfying the above conditions. We let

$$\Gamma_n = \{\gamma \in \Gamma(X_n) \mid \gamma(g) \in E^n \quad \text{for each } g \in G\}.$$

Step 1. We begin by defining a homomorphism $\psi\colon A_{n+1} \to \Gamma_n$. We first note that if $a \in A_{n+1}$, $((a \cdot g)/a) \in A_n$ for each $g \in G$. For $a \in A_{n+1}$, define $\psi(a) \in \Gamma_n$ by

$$(*) \qquad \psi(a)(g) = c(a, g)\phi_n\left(\psi_n\left(\frac{a \cdot g}{a}\right)\right).$$

It is immediate that ψ is a homomorphism. We now derive some other properties of ψ that we will need.

(i) If $a \in A_n$, $\psi(a) = \psi_n(a)$.

Proof. By the inductive assumption (4c), for $a \in A_n$, we have

$$c(a, g) = \frac{\phi_n(\psi_n(a)) \cdot g}{\phi_n(\psi_n(a) \cdot g)}.$$

Thus,

$$\psi(a)(g) = \frac{\phi_n(\psi_n(a)) \cdot g}{\phi_n(\psi_n(a))} = Q(\phi_n\psi_n(a))(g) = \psi_n(a)(g)$$

so $\psi(a) = \psi_n(a)$.

(ii) ψ is a G-map, i.e., $\psi(a \cdot h) = \psi(a) \cdot h$ for $a \in A_{n+1}$, $h \in G$.

Proof. It suffices to see that

$$\frac{\psi(a \cdot h)}{\psi(a)} = \frac{\psi(a) \cdot h}{\psi(a)},$$

and since $((a \cdot h)/a) \in A_n$, it suffices to see by (i) that

$$\psi_n\left(\frac{a \cdot h}{a}\right)(g) = \frac{\psi(a) \cdot h}{\psi(a)}(g) \quad \text{for all } g \in G.$$

Now

$$\frac{\psi(a)\cdot h}{\psi(a)}(g) = \frac{\psi(a)(g)\cdot h}{\psi(a)(g)}$$
$$= Q(\psi(a)(g))(h)$$
$$= Q\left(\phi_n\psi_n\left(\frac{a\cdot g}{a}\right)\right)(h)$$
$$= \psi_n\left(\frac{a\cdot g}{a}\right)(h).$$

Thus it suffices to show that

$$(**)\qquad \psi_n\left(\frac{a\cdot h}{a}\right)(g) = \psi_n\left(\frac{a\cdot g}{a}\right)(h).$$

The left side is

$$\frac{\phi_n\left(\psi_n\left(\frac{a\cdot h}{a}\right)\right)\cdot g}{\phi_n\left(\psi_n\left(\frac{a\cdot h}{a}\right)\right)} = \frac{c\left(\frac{a\cdot h}{a}, g\right)\phi_n\left(\psi_n\left(\frac{a\cdot h}{a}\right)\cdot g\right)}{\phi_n\psi_n\left(\frac{a\cdot h}{a}\right)}$$
$$= c\left(\frac{a\cdot h}{a}, g\right)\phi_n\psi_n\left(\frac{(a\cdot hg)(a)}{(a\cdot h)(a\cdot g)}\right).$$

A similar expression can be derived for the right side of equation (**), and thus, it suffices to see that

$$c\left(\frac{a\cdot h}{a}, g\right) = c\left(\frac{a\cdot g}{a}, h\right).$$

Since c is multiplicative, this means $c(a, h)c(ah, g) = c(a, g)c(ag, h)$. But this follows from the cocycle identity and the commutativity of G.

(iii) ψ is injective.

Proof. If $\psi(a) = 1$, then

$$\phi_n\left(\psi_n\left(\frac{a\cdot g}{a}\right)\right)$$

is constant for each g. This implies

$$\psi_n\left(\frac{a\cdot g}{a}\right) = 1.$$

Since ψ_n is injective, $(a\cdot g)/a = 1$, which implies $a \in A_1$. So $\psi(a) = \psi_n(a) = 1$, and thus $a = 1$.

(iv) Let $\Gamma_{n-1} = \{\gamma \in \Gamma(X_n) \mid \gamma(g) \in E^n_{n-1}$ (= group of quasi-eigenfunctions of X_n of order $n - 1$)$\}$. Then $\psi(A_{n+1}) \cap \Gamma_{n-1} = B^n$.

Proof. For each $g \in G$, let $M(g)\colon \Gamma \to \Gamma$ be defined by $M(g)(\gamma) = (\gamma \cdot g)/\gamma$. It is easy to check that for $\gamma \in \Gamma_{n-1}$ and $g_1, \ldots, g_n \in G$, $M(g_1), \ldots, M(g_n)(\gamma) = 1$. Since ψ is an injective G-map, it is clear that if $\psi(a) \in \Gamma_{n-1}$, then $a \in A_n$. The result follows.

Step 2. We now construct the space X_{n+1} and verify the inductive conditions. Let S be the set of equivalence classes of one-dimensional cocycle representations of (X_n, G). For each $\gamma \in \Gamma(X_n)$, we have an associated element $[\alpha_\gamma] \in S$, defined by $\alpha_\gamma(x, g) = \gamma(g)(x)$. Furthermore, this map $\alpha\colon \Gamma(X_n) \to S$ is a homomorphism. By Corollary 6.5, there exists an ergodic extension $X_{n+1}, p_{n+1}\colon X_{n+1} \to X_n$ with relatively elementary spectrum such that the natural (X_n, G) cocycle representation on $L^2(X_{n+1})$ is equivalent to $\sum^{\oplus}_{\beta \in \alpha(\psi(A_{n+1}))} \beta$.

(i) We now claim $p^*_{n+1}(\psi(A_{n+1})) \subset Q(F(X_{n+1}))$. Let $\gamma \in \psi(A_{n+1})$, and choose any $f \in L^2(X_{n+1})$ such that $|f(x)| = 1$, and f is in the subspace of $L^2(X)$ corresponding to α_γ. Defining $\beta\colon X_n \times G \to \mathbf{C}$ by

$$\beta(p_{n+1}(x), g) = \frac{f \cdot g}{f}(x),$$

we see that β is a cocycle cohomologous to α_γ. Thus, there exists a Borel function $\theta\colon X_n \to U(1)$ such that $\beta(x, g) = \theta(x)\alpha_\gamma(x, g)\theta(xg)^{-1}$ (all g, almost all $x \in X_n$). Then a simple calculation shows

$$Q((\theta \circ p_{n+1}) \cdot f) = p^*_{n+1}(\gamma).$$

(ii) Let $D = Q^{-1}(p^*_{n+1}(\psi(A_{n+1})))$. It is clear that D is a G-invariant subgroup of $F(X_{n+1})$. Thus, the finite linear combinations of elements of D form a G-invariant $*$-subalgebra of $L^\infty(X_{n+1})$. By Corollary 2.2, there exists a factor G-space Z of X_{n+1} such that the closed subspace spanned by D in $L^2(X_{n+1})$ is $L^2(Z)$. Clearly, $p^*_{n+1}(E^n) \subset D$, so we have a sequence of factor maps $X_{n+1} \to Z \to X_n$ (modulo invariant null sets) whose composition is p_{n+1}. For each $\gamma \in \psi(A_{n+1})$, the function $(\theta \circ p_{n+1})(f)$ constructed above is in both D and the one-dimensional field (over X_n) corresponding to $[\alpha_\gamma]$. From this it follows that this field must be contained in $L^2(Z)$. Since the union of these fields spans $L^2(X_{n+1})$, we have $L^2(Z) = L^2(X_{n+1})$. We thus know that D spans $L^2(X_{n+1})$. Since $\psi(A_{n+1}) \subset \Gamma_n$, it follows that X_{n+1} has quasi-discrete spectrum. Because $\psi(A_{n+1}) \cap \Gamma_{n-1} = B^n$ (see (iv) above), it follows that elements of

$$Q^{-1}(p^*_{n+1}(\psi(A_{n+1} - A_n)))$$

are $(n + 1)$st order but not nth order quasi-eigenvalues. By [24, Theorem P], D contains all quasi-eigenfunctions. Defining $\psi_{n+1} = p^*_{n+1} \circ \psi$, it is easy to see that the inductive assumptions (1), (2), (3) hold. It remains only to construct a section $\phi_{n+1}\colon B^{n+1} \to E^{n+1}$ satisfying (4).

$Q\colon E^{n+1} \to B^{n+1}$ has a divisible kernel, and $p_{n+1}^* \phi_n (p_{n+1}^*)^{-1}\colon B_n^{n+1} \to E^{n+1}$ is a homomorphic section of Q defined on the subgroup $B_n^{n+1} \subset B^{n+1}$. By [10, A.8], it follows that this extends to a homomorphic section $\phi_{n+1}\colon B^{n+1} \to E^{n+1}$. Thus, (a) and (b) of (4) are satisfied. To verify (c), apply p_{n+1}^* to equation (*). We obtain

$$\psi_{n+1}(a)(g) = c(a, g)\phi_{n+1}\psi_{n+1}\left(\frac{a \cdot g}{a}\right).$$

Since ϕ_{n+1} is a section of Q, we can write this as

$$\frac{\phi_{n+1}(\psi_{n+1}(a)) \cdot g}{\phi_{n+1}\psi_{n+1}(a)} = c(a, g)\,\frac{\phi_{n+1}\psi_{n+1}(ag)}{\phi_{n+1}\psi_{n+1}(a)}.$$

Hence

$$c(a, g) = \frac{\phi_{n+1}(\psi_{n+1}(a)) \cdot g}{\phi_{n+1}(\psi_{n+1}(a) \cdot g)}.$$

This completes the proof.

Bibliography

1. L. M. Abramov, *Metric automorphisms with quasi-discrete spectrum*, Amer. Math. Soc. Transl., (2), vol. 39 (1962), pp. 37–56.
2. L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, Annals of Mathematics Studies, no. 53, Princeton, 1963.
3. R. Ellis, *Distal transformation groups*, Pacific J. Math., vol. 8 (1958), pp. 401–405.
4. H. Furstenberg, *The structure of distal flows*, Amer. J. Math., vol. 85 (1963), pp. 477–515.
5. ———, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory, vol. 1 (1967), pp. 1–49.
6. F. Hahn, *On affine transformations of compact abelian groups*, Amer. J. Math., vol. 85 (1963), pp. 428–446.
7. F. Hahn and W. Parry, *Minimal dynamical systems with quasi-discrete spectra*, J. London Math. Soc., vol. 40 (1965), pp. 309–323.
8. ———, *Some characteristic properties of dynamical systems with quasi-discrete spectra*, Math Systems Theory, vol. 2 (1968), pp. 179–190.
9. P. Halmos, *Lectures on ergodic theory*, Chelsea, New York, 1956.
10. E. Hewitt and K. Ross, *Abstract harmonic analysis, I*, Springer-Verlag, Berlin, 1963.
11. H. Hoare and W. Parry, *Affine transformations with quasi-discrete spectra*, J. London Math. Soc., vol. 41 (1966), pp. 88–96.
12. ———, *Affine transformations with quasi-discrete spectra, II*, J. London Math. Soc., vol. 41 (1966), pp. 529–530.
13. R. Howe, *On Frobenius reciprocity for unipotent algebraic groups over Q*, Amer. J. Math., vol. 93 (1971), pp. 163–172.
14. A. A. Kirillov, *Dynamical systems, factors, and group representations*, Russian Math. Surveys, vol. 22 (1967), pp. 63–75.
15. U. Krengel, *Weakly wandering vectors and weakly independent partitions*, Trans. Amer. Math. Soc., vol. 164 (1972), pp. 199–226.
16. G. W. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc., vol. 85 (1957), pp. 134–165.
17. ———, "Virtual groups," in *Topological dynamics*, J. Auslander and W. Gottschalk, eds., Benjamin, New York, 1968.
18. C. C. Moore, *Decomposition of unitary representations defined by discrete subgroups of nilpotent groups*, Ann. of Math., vol. 82 (1965), pp. 146–182.

19. ———, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math., vol. 88 (1966), pp. 154–178.
20. W. PARRY, "Zero entropy of distal and related transformations," in *Topological dynamics*, J. Auslander and W. Gottschalk, eds., Benjamin, New York, 1968.
21. K. R. PARTHASARATHY, *Probability measures on metric spaces*, Academic Press, New York, 1967.
22. L. RICHARDSON, *Decomposition of the L^2-space of a general compact nilmanifold*, Amer. J. Math., vol. 93 (1971), pp. 173–190.
23. V. S. VARADARAJAN, *Geometry of quantum theory, vol. II*, Van Nostrand, Princeton, 1970.
24. T. WIETING, *Ergodic affine Lebesgue G-spaces*, doctoral dissertation, Harvard University, 1973.
25. R. ZIMMER, *Extensions of ergodic actions and generalized discrete spectrum*, Bull. Amer. Math. Soc., vol. 81 (1975), pp. 633–636.
26. ———, *Extensions of ergodic group actions*, Illinois J. Math., vol. 20 (1976), pp. 373–409.
27. W. PARRY, *A note on cocycles in ergodic theory*, Composito Math., vol. 28 (1974), pp. 343–350.
28. A. W. KNAPP, *Distal functions on groups*, Trans. Amer. Math. Soc., vol. 128 (1967), pp. 1–40.

UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MARYLAND