TOTALLY SPINELESS MANIFOLDS

BY

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Let W^m be a compact piecewise linear (P.L.) manifold with boundary. A spine of W is a P.L. embedding $\phi: M^n \to W^m$, where M is a closed P.L. manifold, that is a homotopy equivalence.

THEOREM 1. *Let M^n be a closed orientable P.L. manifold. Suppose that $\Pi_1 M$ has a quotient group that possesses a central subgroup of finite index with some nontrivial abelian quotient (e.g., $\Pi_1 M$ nontrivial abelian). Let n be even and at least four. Then there are infinitely many non P.L.-homeomorphic manifolds W^{n+2}, simple homotopy equivalent to M, that have no spine.*

This result asserts the existence of totally spineless manifolds. Naturally one conjectures its validity for any nontrivial fundamental group.

In sharp contrast to Theorem 1, it was shown in [3] that for n odd or $\Pi_1 M$ trivial, any homotopy equivalence $h: M^n \to W^{n+2}$ is homotopic to a P.L. embedding. Of course, a locally flat embedding usually will not exist.[2] For example, it is shown in [3] that the Poincaré dual of certain Hirzebruch L-classes must be carried by cycles in the set of nonlocally-flat points of any embedding homotopic to h. Depending upon the normal invariant of a homotopy equivalence $M' \to M$, the size of the nonlocally flat points can sometimes be reduced by replacing M by M' (this may also change the L-classes.) The results of Kato and Matsumoto on locally flat spines can be obtained in this way. (But they follow most directly and conceptually from the codimension two splitting principle of [2, Section 8].) In [3], examples are given of P.L. manifolds, homotopy equivalent to $T^n = S^1 \times \cdots \times S^1$, so that any spine has nonlocally flat points of dimension at least $n - 2$.

The idea of the proof is to construct an invariant of a manifold W^{n+2}, which has the homotopy type of an n-manifold. An explicit realization result for this invariant is given. The invariants that arise from W^{n+2} with spines will lie in the image of a map from the bordism of a classifying space for P.L. regular neighborhoods. The result follows by algebraically constructing obstructions not in this image.

The special case of Theorem 1 with *finite* fundamental group was announced at a conference in Utah in February, 1974. An example of a spineless 4-

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[2] In higher codimension all P.L. embeddings are locally flat and one has the existence theorem of Browder-Haefliger-Casson-Sullivan-Wall. In codimension one there is the theorem of Hollingsworth and Galewski [5].

manifold homotopy equivalent to T^2 has been given by Matsumoto [10]. Our invariants can also be used to show that there are 4-dimensional manifold regular neighborhoods of S^2 with no locally flat spines. Galewski pointed out that this gives a new type of example of the failure of Whitney's procedure in low dimensions.

1. An invariant

Let W^{n+2} be a compact, oriented P.L. manifold, and let $h: M^n \to W$ be a (simple) homotopy equivalence, M a closed oriented P.L. manifold. Let $\chi(W)$ denote the image in $H^2(W)$ of the Poincaré dual of $h_*[M]$, $[M]$ in $H_n(M)$ the orientation class. Assume that $\chi(M) = 0$. Then by [3, 1.6], there is a map

$$f: (W, \partial W) \to (M \times D^2, M \times S^1)$$

with the following properties:

(i) f has degree one and induces isomorphisms of homology groups with local coefficients in the integral group-ring, $Z\Pi_1 M$, and
(ii)[3] $f \circ h$ is homotopic to the inclusion $M \subset M \times D^2$.

If h is actually a simple homotopy equivalence, then it follows from Poincaré duality that f is a simple homology equivalence over $Z\Pi_1 M$ (see [2, Section 1]). Further, if $H_1(W; Z) = H_1(M; Z)$ is finite, then the set of homotopy classes of maps of M into $SO(2)$ (denoted $[M, SO(2)]$) is trivial and hence, by [3, 1.6] f is unique up to homotopy. In general f is unique up to composition with a bundle map of the trivial $SO(2)$ bundle over M to itself.

From the existence of the augmentation homomorphism from $Z\Pi_1 M$ to the integers Z, it follows that f induces an isomorphism of homology groups and cohomology groups with integral coefficients, and so also an isomorphism $[M \times D^2; BSPL] \to [W, BSPL]$, $BSPL$ the classifying space for stable oriented PL (or block) bundles.[4] Hence there exists a unique stable bundle map $b: \nu_W \to \xi$, ν_W the stable normal bundle of W and ξ a bundle over $M \times D^2$.

Let $\frac{1}{2}D^2 \subset D^2$ be the disk of radius $\frac{1}{2}$. We may suppose that f is transverse to M, and that $f \mid f^{-1}(M \times \frac{1}{2}D^2)$ is a bundle map. Write

$$(M \times (D^2 - \tfrac{1}{2}D^2)^-, M \times S^1) = (M \times S^1 \times [0, 1], M \times S^1 \times 0)$$

and let $V = f^{-1}(M \times S^1 \times [0, 1])$. Then $(f \mid V, b \mid V)$ is a *normal map* [1] from $(V, \partial_- V = \partial W, \partial_+ V)$ to $M \times S^1 \times ([0, 1], 0, 1)$, whose restriction to $\partial_- V$ is a homology equivalence over $Z\Pi_1 M$, and a simple one if h was a simple homotopy equivalence.

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[3] Actually (ii) implies (i) by Poincaré duality over $Z\Pi_1 M$ for $(W, \partial W)$ and $M \times (D^2, S^1)$, assuming f has degree one.

[4] Compare with [2, p. 307] for example.

Hence we may take the homology surgery obstruction of $(f \mid V, b \mid V)$, by [2, Chapter I.] More precisely, let Φ' denote the diagram

$$\begin{array}{ccc} Z[\Pi_1(M \times S^1)] & \xrightarrow{\text{id}} & Z[\Pi_1(M \times S^1)] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow \\ Z[\Pi_1(M \times S^1)] & \longrightarrow & Z\Pi_1 M \end{array}$$

where the unlabelled map is induced by projection on M. Then, by [2, Section 3], the homology surgery obstruction $\sigma(f \mid V, b \mid V) \in \Gamma^{\varepsilon}_{n+2}(\Phi')$ is defined, where $\varepsilon = s$ if f is a simple homotopy equivalence and $\varepsilon = h$ otherwise. When $\varepsilon = s$, we often omit it. Recall that $\sigma(f \mid V, b \mid V)$ vanishes if and only $(f \mid V, b \mid V)$ is *normally* cobordant, relative $\partial_- V$, to a normal map that induces a (simple) homology equivalence over $Z\Pi_1 M$ and, on the boundary component disjoint from $\partial_- V$, a (simple) homology equivalence over $Z\Pi_1(M \times S^1)$.

Let $\Pi_1 W = \Pi$. Let Φ_Π be the diagram

$$\begin{array}{ccc} Z[\Pi \times Z] & \xrightarrow{\text{id}} & Z[\Pi \times Z] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow \\ Z[\Pi \times Z] & \longrightarrow & Z[\Pi] \end{array}$$

the unlabelled maps induced by projection on Π. Then $h_*: \Pi_1 M \to \Pi_1 W$ induces a map from $\Gamma^{\varepsilon}_{n+2}(\Phi') \to \Gamma^{\varepsilon}_{n+2}(\Phi_\Pi)$, also denoted h_*.

Any element z of $H^1(\Pi) = \operatorname{Hom}(\Pi, Z)$ determines an automorphism of $\Pi \times Z$ given by $a_z(x, y) = (x, z(x) + y)$. Let $\overline{\Gamma}^{\varepsilon}_{n+2}(\Phi_\Pi)$ be the quotient obtained by setting $(a_z)_* w = w$ for all w in $\Gamma^{\varepsilon}_{n+2}(\Phi_\Pi)$ and all z in $H^1(\Pi)$. For Π finite, $\overline{\Gamma}^{\varepsilon}_{n+2}(\Phi_\Pi) = \Gamma^{\varepsilon}_{n+2}(\Phi_\Pi)$.

DEFINITION. Let $\alpha^{\varepsilon}(W)$ be the image of $h_*\sigma(f \mid V, b \mid V)$ in $\overline{\Gamma}^{\varepsilon}_{n+2}(\Phi_\Pi)$.

PROPOSITION 1. *The invariant $\alpha^{\varepsilon}(W)$ depends only upon W.*

Proof. Since $[M, SO(2)] = H^1(M) = H^1(\Pi_1 M)$, the fact that we passed to the quotient $\overline{\Gamma}_{n+2}(\Phi_\Pi)$ insures that $\alpha^{\varepsilon}(W)$ does not depend upon the choice of f. Given a (simple) homotopy equivalence $h': M' \to W$, let k be the composition of h and a homotopy inverse for h'. Then, using $(k \times \text{id}_{D^2}) \circ f$ in the definition, in place of f, one obtains by naturality the homology surgery obstruction $k_*\sigma(f \mid V, b \mid V)$. The result follows.

The invariant of Proposition 1 can be thought of as an obstruction to the existence of a locally flat spine. It is actually defined for W^{n+2} with the homotopy type of an n-dimensional Poincaré complex.

PROPOSITION 2. *Let $\phi: M^n \to W^{n+2}$ be a P.L. embedding of M as a spine of W. Let W' be a regular neighborhood of $\phi(M)$ in the interior of W. Then $\alpha^{\varepsilon}(W') = \alpha^{\varepsilon}(W)$.*

Proof. Let T be the closure of $W' - W$, so that $\partial T = \partial W' \cup \partial W$. By excision

$$H_i(T, \partial W'; Z\Pi_1 W) \cong H_i(W, W', Z\Pi_1 W),$$

and the second group vanishes as $W' \subset W$ is a homotopy equivalence. Thus $(T; \partial W', \partial W)$ is a h-cobordism with coefficients in $Z\Pi_1 W \cong Z\Pi_1 M$, and will be an s-cobordism with these coefficients if $W' \subset W$ is a simple homotopy equivalence, i.e., if ϕ is, by excision for Whitehead torsion.

Let $f'\colon (W', \partial W') \to (M \times D^2, M \times S^1)$ be the canonical homology equivalence defined above, satisfying (i) and (ii). Write

$$f'(x) = (f_1(x), f_2(x)), f_1(x) \in M, f_2(x) \in S^1, \quad \text{for } x \in \partial W'.$$

Since $f' \circ \phi$ is homotopic to the inclusion $M \subset M \times D^2$ and since ϕ is a homotopy equivalence, it follows easily using the homotopy extension property that f_1 extends to T. Since T is a homology h-cobordism, it follows from obstruction theory that f_2 also extends. Thus we obtain $f\colon (W, \partial W) \to (M \times D^2, M \times S^1)$, with $f \mid W' = f'$. Clearly property (ii) is satisfied by f, and f is easily seen to have degree one, as f' does, so that f also satisfies (i). Let b be as above.

Now, $f \mid \partial T$ is a homology or simple homology equivalence over $Z\Pi_1 M$; hence by [2, Chapter I],

$$\sigma(f \mid T, b \mid T) \in \Gamma^{\varepsilon}_{n+2}(Z[\Pi_1(M \times S^1)] \to Z\Pi_1 M)$$

is defined. Let

$$j_*\colon \Gamma^{\varepsilon}_{n+2}(Z[\Pi_1 M \times S^1] \to Z\Pi_1 M) \to \Gamma^{\varepsilon}_{n+2}(\Phi').$$

be the natural map. Then, in the notation of the definition of our invariant, $V = V' \cup T$, so that

$$\sigma(f \mid V, b \mid V) = \sigma(f' \mid V', b' \mid V') + j_*\sigma(f \mid T, b \mid T),$$

by additivity of homology surgery obstructions. But T is an h- or s-cobordism over $Z\Pi_1 M$; hence $\sigma(f \mid T, b \mid T) = 0$; thus $\alpha^{\varepsilon}(W) = \alpha^{\varepsilon}(W')$.

To conclude this section, recall from [3] the classifying space $BSRN_2$ for codimension two regular neighborhoods and the fiber G_2/RN_2 of the natural map $\chi\colon BSRN_2 \to BSO_2$ for the associated $SO(2)$-bundle. Thus, a homotopy class of mappings $g\colon M \to G_2/RN_2$ gives a concordance class of embeddings, $\phi_g\colon M^2 \subset W_g^{n+2}$ a representative, say, so that W_g is actually a regular neighborhood of M and $\chi(W) = 0$. Thus $\alpha(W)$ is well defined, as well as $\alpha^{\varepsilon}(W)$. Further, given a map $\tau\colon M \to K(\Pi, 1)$, we may consider

$$\tau_* \circ (\phi_g)_*^{-1}\colon \Gamma^{\varepsilon}_{n+2}(\Phi_{\Pi_1 W}) \to \Gamma^{\varepsilon}_{n+2}(\Phi_{\Pi}).$$

Let Ω_* denote the oriented bordism functor.

PROPOSITION 3. *The assignment of* $(\tau_* \circ (\phi_g)_*^{-1})(\alpha^\varepsilon(W_g))$ *induces a homomorphism, natural in* Π:

$$\sigma_\Pi^\varepsilon\colon \Omega_n(G_2/RN_2 \times K(\Pi, 1)) \to \overline{\Gamma}_{n+2}^\varepsilon(\Phi_\Pi).$$

Proof. Suppose given $g\colon K \to G_2/RN_2$ and $\tau\colon K \to K(\Pi, 1)$, where K is a connected smooth manifold with boundary components M_0 and M_1. Let $\tau_i = \tau \mid M_i$, $g_i = g \mid M_i$, $\phi_i = \phi_{g_i}$, $i = 0, 1$. Then we wish to show that the elements $(\tau_i)_*(\phi_i)_*^{-1}(\alpha^\varepsilon(W_{g_i}))$ are equal, $i = 0, 1$.

More precisely, let $W = W_g$. For $W_{g_i} = W_i$ we may choose disjoint regular neighborhoods of M_i in the boundary of W; see [3]. By [3, 1.6], there exists a map,

$$f\colon (W, (\partial W - W_0 \cup W_1)^-, W_i, \partial W_i) \\ \to (K \times D^2, K \times S^1, M_i \times D^2, M_i \times S^1),$$

of degree one, whose composition with ϕ_g is homotopic to the zero-section $(K, M_0, M_1) \subset (K, M_0, M_1) \times D^2$. Further, f will be a simple homotopy equivalence over $Z\Pi_1 K$ and

$$f \mid W_i\colon (W_i, \partial W_i) \to (M_i \times D^2, M_i \times S^1)$$

will be a simple homology equivalence over $Z\Pi_1 M_i$, $i = 0, 1$. In particular, f will be an integral homology equivalence and so will be covered by a bundle map $b\colon \nu_K \to \xi$.

We may suppose that f, $f \mid W_0$, and $f \mid W_1$ are transverse to the zero-sections K, M_0, and M_1, respectively, and that

$$f \mid f^{-1}(K \times \tfrac{1}{2}D^2)$$

is a bundle map. As above, let

$$V = f^{-1}(K \times S^1 \times [0, 1]),$$

and let $V_i = f^{-1}(M_i \times S^1 \times [0, 1])$. Then $(f \mid V, b \mid V)$ is a normal cobordism of $(f \mid V_0, b \mid V_0)$ and $(f \mid V_1, b \mid V_1)$. Hence it follows from cobordism invariance of homology surgery obstructions [2] that if ξ_0 and ξ_1 are the respective images of $(f \mid V_0, b \mid V_0)$ and $(f \mid V_1, b \mid V_1)$ under the inclusion induced maps of $\Gamma_{n+2}^\varepsilon(\Phi_{\Pi_1 M_i})$ to $\Gamma_{n+2}^\varepsilon(\Phi_{\Pi_1 K})$, then $\xi_1 = \xi_0$. By functoriality

$$\tau_*\xi_i = (\tau_i)_*\sigma(f \mid V_i, b \mid V_i), \quad i = 0, 1.$$

By definition, the right side represents $(\tau_i)_* \circ (\phi_i)_*^{-1}\alpha^\varepsilon(W_i)$ in $\overline{\Gamma}_{n+2}^\varepsilon(\Phi_\Pi)$, as desired.

Thus σ_Π^ε is a well-defined map. Additivity of homology surgery obstructions over disjoint unions [2, Section 3] implies easily that σ_Π^ε is a homomorphism. Naturality follows from naturality of homology surgery obstructions.

Let $\tilde{\Gamma}^{\varepsilon}_{n+2}(\Phi_{\Pi})$ denote the cokernel of the natural map

$$\Gamma_{n+2}(\Phi_e) \to \bar{\Gamma}^{\varepsilon}_{n+2}(\Phi_{\Pi});$$

by naturality, $\sigma^{\varepsilon}_{\Pi}$ induces

$$\tilde{\sigma}^{\varepsilon}_{\Pi}\colon \Omega_n(G_2/RN_2 \times (K(\Pi, 1), \mathrm{pt})) \to \tilde{\Gamma}^{\varepsilon}_{n+2}(\Phi_{\Pi}).$$

PROPOSITION 4. *If Π is finite, $\Omega_n(G_2/RN_2 \times (K(\Pi, 1), \mathrm{pt}))$ is a torsion group.*

Proof. To any finite skeleton, apply the Kunneth formula for homology and the spectral sequence relating homology and bordism.

2. Construction of spineless manifolds

The spineless manifolds will be constructed using the next result.

THEOREM 2. *Let M^n, $n \geq 4$, be a closed, orientable P.L. manifold. Let $\gamma \in \Gamma_{n+2}(\Phi_{\Pi})$ be an element whose image in $L_{n+1}(\Pi \times Z)$ is trivial, under the natural connecting homomorphism [2, Section 3]. Then there is a compact orientable P.L. manifold W^{n+2}, of the same simple homotopy type as M, with $\chi(W) = 0$ and $\alpha(W) = \bar{\gamma}$, $\bar{\gamma}$ the image of γ in $\bar{\Gamma}_{n+2}(\Phi_{\Pi})$.*

Notes. (1) There is a similar result for elements in $\Gamma^h_{n+2}(\Phi_{\Pi})$.

(2) The condition on the vanishing of $\partial\gamma$ is sufficient to insure that any W with $\chi(W) = 0$ has the homotopy type of an n-manifold. It could be replaced by a necessary and sufficient condition.

Proof of Theorem 2. By [2, Section 3], we have the exact sequence

$$L^s_{n+2}(\Pi \times Z) \longrightarrow \Gamma_{n+2}(Z[\Pi \times Z] \longrightarrow Z\Pi)$$
$$\xrightarrow{i^*} \Gamma_{n+2}(\Phi_{\Pi}) \xrightarrow{\partial} L^s_{n+1}(\Pi \times Z).$$

Hence $\gamma = i_*\gamma_1$. Let $j_*\colon \Gamma_{n+2}(Z[\Pi \times Z] \to Z\Pi) \to L^s_{n+2}(\Pi)$ be the natural map. Since the natural map of $L_{n+2}(\Pi \times Z]$ to $L_{n+2}(\Pi)$ is functorially surjective, we may suppose that $j_*\gamma_1 = 0$. Hence $\gamma_1 = \partial_1\gamma_2$, by [2, Section 3], where ∂_1 is the connecting homomorphism

$$\Gamma^s_{n+3}(\Phi'_{\Pi}) \xrightarrow{\partial_1} \Gamma_{n+2}(Z[\Pi \times Z] \longrightarrow Z\Pi),$$

for Φ'_{Π} the diagram

$$\begin{array}{ccc} Z[\Pi \times Z] & \to & Z\Pi \\ \downarrow & & \downarrow \\ Z\Pi & \to & Z\Pi \end{array}.$$

By the realization theorem [2, 3.4], γ_2 can be realized as the homology surgery obstruction $\sigma(F, B)$ of a normal cobordism (F, B) of the identity of $M \times D^2$ to a simple $Z\Pi$-homology equivalence

$$f\colon (W, \partial W) \to (M \times D^2, M \times S^1),$$

which induces isomorphism of fundamental groups. In particular, h induces a simple homotopy equivalence of W with M, so that $\alpha(W)$ is defined.

Write $F\colon U_1 \to M \times D^2$. We may assume that F is transverse regular to $M \times \frac{1}{2}D^2$, in a way that respects boundaries, and that $F/F^{-1}(M \times \frac{1}{2}D^2)$ is a bundle map. Let U be the closure of $F^{-1}(M \times \frac{1}{2}D^2)$. Then $(F \mid U, B \mid U)$ is a normal cobordism whose boundary is the union of the identity map on

$$M \times (D^2 - \tfrac{1}{2}D^2)^- \cup M \times \partial(\tfrac{1}{2}D^2) \times I;$$

the normal map $(F \mid \partial_0 U_1, B \mid \partial_0 U_1)$, where $F \mid \partial_0 U_1\colon \partial_0 U_1 \to M \times S^1 \times I$; and, in the notation of Section 1, $(f \mid V, b \mid V)$. Hence, by additivity and cobordism invariance of surgery obstructions, we see that $\alpha(W)$ is represented by $i_*\partial_1\gamma_2 = \gamma$.

To complete the proof of Theorem 1, one has the next result.

PROPOSITION 5. *Let Π be a nontrivial finite group, and let $\Pi' \to \Pi$ be a surjective homomorphism. Assume that Π has a central subgroup with nontrivial abelian quotient. Let n be even. Then there is an element γ in $\Gamma^s_{n+2}(\Phi_{\Pi'})$, with $\partial\gamma = 0$, whose image in $\Gamma^s_{n+2}(\Phi_\Pi)$ has infinite order.*

Assuming Proposition 5, we may prove Theorem 1. For Π' we take $\Pi_1 M$. Let Π'' be a quotient of Π' with a central subgroup, A say, of finite index, as provided by the hypotheses of Theorem 1. Let A/B be a nontrivial finite abelian quotient of A. Then Π''/B will serve as the finite quotient of Π' required for Proposition 5.

Now, by Theorem 2 we may construct W_k^{n+2}, of the same simple homotopy type as M^n, with $\chi(W_k) = 0$ and with $\alpha(W_k)$ the image $k\bar{\gamma}$ of $k\gamma$ in $\overline{\Gamma}_{n+2}(\Phi_{\Pi'})$, k an integer. Here Π' is identified with $\Pi_1 W_k$ via the inclusion of M in W_k.

If W_k had a spine, then by Proposition 2, the image of $k\bar{\gamma}$ in $\overline{\Gamma}^h_{n+2}(\Phi_{\Pi'})$ would be in the image of $\sigma^h_{\Pi'}$. By naturality the image of $k\gamma$ in $\tilde{\Gamma}^h_{n+2}(\Phi_\Pi)$ would lie in the image of $\tilde{\sigma}^h_\Pi$ and hence would have finite order. Because Π is finite, $\tilde{\Gamma}^\varepsilon_{n+2}(\Phi_\Pi)$ is just the quotient of $\Gamma^\varepsilon_{n+2}(\Phi_\Pi)$ by $\Gamma_{n+2}(\Phi_e)$. It follows easily from [2, Appendix I] that the natural map of $\tilde{\Gamma}^s_{n+2}(\Phi_\Pi)$ to $\tilde{\Gamma}^h_{n+2}(\Phi_\Pi)$ is an isomorphism modulo 2-groups. Thus the image of γ in $\tilde{\Gamma}^s_{n+2}(\Phi_\Pi)$ would have finite order, a contradiction if γ is as in Proposition 5.

Proof of Proposition 5. Recall the exact sequence

$$L_{n+2}(\Pi \times Z) \longrightarrow \Gamma_{n+2}(Z[\Pi \times Z] \longrightarrow Z\Pi)$$
$$\xrightarrow{i^*} \Gamma_{n+2}(\Phi_\Pi) \xrightarrow{\partial} L_{n+1}(\Pi \times Z)$$

and let j_* be the natural map from the middle group to $L_{n+2}(\Pi)$. We will construct $\rho \in \Gamma_{n+2}(Z[\Pi \times Z] \to Z\Pi)$ with the following properties:

(i) ρ has infinite order,
(ii) $j_*\rho = 0$,
(iii) $\beta_*\rho = 0$, and
(iv) ρ is in the image of the natural map from $\Gamma_{n+2}(Z[\Pi' \times Z] \to Z\Pi')$.

Here β_* is induced by the obvious map $\beta\colon \Phi_\Pi \to \Phi_e$. To prove Proposition 5, it suffices to see that $i_*(\rho)$ has infinite order modulo $\Gamma_{n+2}(\Phi_e)$. Suppose that $i_*(k\rho)$ does lie in $\Gamma_{n+2}(\Phi_e)$, viewed as a subgroup of $\Gamma_{n+2}(\Phi_\Pi)$, k a nonzero integer. We have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \Gamma_{n+2}(Z[Z] \longrightarrow Z) \arrow[r, "i'_*"] \arrow[d, shift right] & \Gamma_{n+2}(\Phi_e) \arrow[d, shift right] \\
L_{n+2}(\Pi \times Z) \arrow[r] & \Gamma_{n+2}(Z[\Pi \times Z] \longrightarrow Z\Pi) \arrow[r, "i_*"] \arrow[u, shift right, "\beta_*"'] & \Gamma_{n+2}(\Phi_\Pi) \arrow[u, shift right, "\beta'_*"']
\end{tikzcd}$$

The rows are exact and the maps from top to bottom are induced by the inclusion $Z \subset \Pi \times Z$ and, in the other direction, by projection. Now, $\beta'_* i_*(k\rho) = i'_*\beta_*(k\rho) = 0$. Since β'_* is the identity on $\Gamma_{n+2}(\Phi_e)$, we have $i_*(k\rho) = 0$. Therefore $k\rho$ itself comes from an element of $L_{n+2}(\Pi \times Z)$; by (ii), this element maps trivially to $L_{n+2}(\Pi)$ under the natural map. By [7], the kernel of this natural map is the image of the map $L^h_{n+1}(\Pi) \to L^s_{n+2}(\Pi \times Z)$ given by crossing with a circle. But $L^h_{n+1}(\Pi)$ is a torsion group for n odd ([9], for example), contradicting (i).

To construct ρ, we consider two cases:

Case I. $n \equiv 2$ (4). We first assume that Π actually has a homomorphism ω onto a cyclic group Z_p of order $p \neq 0$; this will be the case if Π has nontrivial abelian quotient. Let ζ be primitive pth root of unity, and consider the (multiplicative) homomorphism $\bar\omega\colon \Pi \times Z \to C$ so that $\bar\omega \mid \Pi$ is obtained by composing ω with a map sending a generator of Z_p to ζ, and $\bar\omega(t) = \zeta$ for t a generator of Z. This induces a homomorphism $\omega\colon Z[\Pi \times Z] \to C$ which actually factors through the semisimple ring $Q[\Pi \times Z_p]$. Using semisimplicity, it is quite easy to show that a Hermitian form over $Q[\Pi \times Z_p]$ on a free module that has a presubkernel [2, Section 1], with respect to the augmentation $Q[\Pi \times Z_p] \to Q$, actually has a presubkernel that is a direct summand. From this, and using the natural map $Z[\eta] \to Z_p$, η a primitive pth root of unity, to compute ranks, we easily obtain an induced homomorphism

$$\omega_*\colon \Gamma_{n+2}(Z[\Pi \times Z] \to Z\Pi) \to \tilde W(C),$$

where $\tilde W(C)$ denotes the Witt group (using orthogonal sum), of nonsingular Hermitian forms over C, reduced by requiring hyperbolic forms to be trivial. Let $I\colon \tilde W(C) \to Z$ be the signature homomorphism (diagonalize and subtract the number of negative entries from the number of positive ones.)

Let $g \in \Pi$ with $\bar\omega(g) = \zeta$, and consider the matrix

$$\begin{pmatrix} N(t + t^{-1} - 2) & 1 \\ 1 & g + g^{-1} - 2 \end{pmatrix}$$

where N is an integer with $(\zeta + \zeta^{-1} - 2)^2 N > 1$. This matrix represents an element ρ of $\Gamma_{n+2}(Z[\Pi \times Z] \to Z[\Pi])$ that obviously satisfies (ii) and (iii).

Since $\Pi' \to \Pi$ is onto, (iv) is also satisfied. Clearly $I\omega_*(\rho) = -2$. Hence ρ has infinite order.

In case Π has only a central subgroup $\Pi'' \subset \Pi$ with some abelian quotient, let $\rho \in \Gamma_{n+2}(Z[Z \times \Pi''] \to Z\Pi'')$ satisfy (i), (ii), (iii). Suppose that ρ also satisfies (iv) with respect to the inverse image of Π'' in Π'. Then the image of ρ in $\Gamma_{n+2}(Z[Z \times \Pi] \to Z\Pi)$ under the natural map obviously satisfies (ii)–(iv). However, because Π'' is central the composition consisting of this inclusion, followed by the transfer homomorphism back to $\Gamma_{n+2}(Z[\Pi'' \times Z] \to Z)$, is just multiplication by the index of Π'' in Π. Hence (i) is also satisfied.

Case II. $n \equiv 0$ (4). For $p \neq 2$, the argument is essentially the same. We use a form of the type

$$\begin{pmatrix} (g - g^{-1})N & 1 \\ -1 & t - t^{-1} \end{pmatrix}.$$

The homomorphism $\bar{\omega}$ then gives a skew-Hermitian form over C, which becomes a Hermitian form, upon multiplication of all entries by $\sqrt{-1}$; whose signature will be nonzero for suitable N. In the general case we again appeal to the transfer.

For $p = 2$, we use

$$\begin{pmatrix} N(g - g^{-1})(t^{-1} - t) & 1 \\ -1 & t - t^{-1} \end{pmatrix}.$$

In this case, one uses the homomorphism $\Pi \times Z \to C$ sending g to (-1) and t to $\cos 2\pi/3 + i \sin 2\pi/3$; the map on group rings factors through the semisimple ring $Q[Z_6]$. We leave the details to the reader.

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