SUPPLEMENT[1]

TO

Every planar map is four colorable

Part I: discharging

by K Appel and W Haken

AND

Every planar map is four colorable

Part II: reducibility

by K. Appel, W. Haken, and J. Koch

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[1] We would like to thank Dorothea Haken for her effective help in preparing and checking this supplement

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Supplement to Part I

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Every planar map is four colorable

Part I: discharging

Supplement: The details of the proof of the Discharging Theorem for $\mathcal{R}(\mathcal{T},\mathcal{S},\mathcal{L}),\mathcal{U}$

The proof of the Discharging Theorem as given in Section 3 of the paper requires the consideration of a finite but large number of cases (of configurations which are associated with vertices of positive charges q, q_{TS}, q_{TL}, etc.). It is the purpose of this supplement to exhibit a tree of case-distinctions which in a relatively convenient way covers all necessary cases. This task is carried out in two steps.

In Step 1 (pp. 2 to 197 of this supplement) we carry out the case-distinctions so far that it only remains to check certain 1-parameter or 2-parameter classes of configurations. For example, CTS#10 (Table 3) is a 1-parameter class of configurations: The class consists of 27 configurations which are obtained by attaching to the drawn configuration of Table 3 one of the 27 specified S2-situations at E (the parameter ranges over 27 values). This class must be considered in the proof of the L-Lemma: It must be checked that each member of the class contains some L-situation which discharges at least 40 (or 35, respectively) along X. Correspondingly, CTS#24 is an example of a 2-parameter class.

In Step 2, (pages C1 to C?10) we check the 1- and 2-parameter classes described in Step 1. This part of the supplement consists entirely of tables called class check lists.

(1) We begin by proving the lemmas on T-dischargings as formulated in Section 3 of the paper, i.e., Lemmas (5-6-6), (6-6-6), (5^5-7-6-6), (6-6), (T), (T2,T2), (T2,T2,T2), (T,T2,T2,T), and the three following lemmas.

Lemma (T2,T). If V_k is a major vertex of Δ^* which receives a T2- and an arbitrary T-discharging across two consecutive 6-6 edges E, F, then one of the four cases drawn below applies: Either the configuration T2T2n, or its

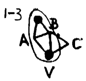
reflection T2T2r, or the configuration $\widehat{T2T1}$ occurs with V_k, E, F as indicated in the drawing, or the T-discharging across F is induced by T1#1.

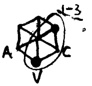
Lemma (T2,T2,T). If V_k is a major vertex of Δ^* which receives a T2-, another T2-, and an arbitrary T-discharging across three consecutive 6-6 edges E, F, G (in this order) then the T2-dischargings across E and F are induced by T2T2*r in Δ^* and the T-discharging across G is induced by T1#1 or T1#3r.

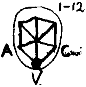
Lemma (T2,T,T2). If V_k is a major vertex of Δ^* which receives a T2-, an arbitrary T-, and another T2-discharging across three consecutive 6-6 edges E, F, G (in this order) then the T-discharging across F is T1 and is induced by T1#1.

Lemma (5-6-6) is proved in Section 3 of the paper.

Proof of Lemma (6-6-6). Assume in an arbitrary triangulation Δ, V is a 5-vertex with three consecutive neighbors A, B, C of degree 6 so that no T-discharging leaves V across the edge A-B. Then one of the following occurs.

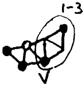


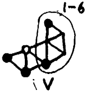


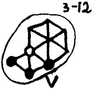


I.e., Δ contains one of the configurations 1-3, 1-6, 1-12 of $\mathcal{U}$ and thus is different from Δ^*, q.e.d. ■

Proof of Lemma $(5^5$-7-6-6). Assume in an arbitrary triangulation Δ, V is a 5-vertex as described in the statement of the lemma so that no T-discharging leaves V across the 6-6 edge. Then one of the following occurs.

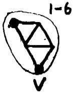




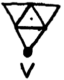


Thus $\Delta \neq \Delta^*$, q.e.d. ■

Proof of Lemma (6-6). Assume in an arbitrary triangulation Δ, V is a 5-vertex with two consecutive neighbors of degree 6 so that no T-discharging leaves V across the 6-6 edge. Then one of the following occurs.

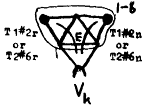


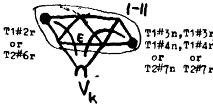
see Fig. 10

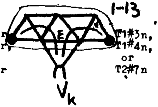


Thus, if $\Delta = \Delta^*$, then it contains the configuration of Figure 10, q.e.d. ■

Proof of Lemma (T). Assume in an arbitrary triangulation Δ, V_k is a major vertex which is adjacent to a 6-6 edge E so that V_k receives two different T-dischargings across E, i.e., two distinct T-discharging situations (see Figure 2) must be attached. Thus one of the following occurs.







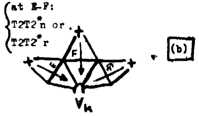
Thus $\Delta \neq \Delta^*$, q.e.d. ■

Proof of Lemma (T2,T). Assume in an arbitrary triangulation Δ, V_k is a major vertex which receives a T2-, and an arbitrary T-discharging across two consecutive 6-6 edges E, F (in that order). Then a configuration of the (2-parameter) class (a) (see drawing to the right) occurs in Δ with V_k, E, F as indicated in the drawing. But each configuration of (a) either contains a configuration of $\mathcal{U}$ or complies with one of the four cases described in the statement of the lemma (see the class check list for (a)). Thus, if $\Delta = \Delta^*$, then one of the four cases applies as stated in the lemma, q.e.d. ■


(a)

Proof of Lemma (T2,T2). Assume in an arbitrary triangulation Δ, V_k is a major vertex which receives two T2-dischargings across two consecutive 6-6 edges E, F. We claim that either $\Delta \neq \Delta^*$ (i.e., Δ contains a member of $\mathcal{U}$) or Δ contains T2T2*n or T2T2*r (see Figure 11) with V_k, E, F as indicated in the drawing. We may assume that Δ contains T2T2n or T2T2r as described in Lemma (T2,T) since otherwise the claim would follow immediately from Lemma (T2,T). Moreover, the vertex A (see the drawing to the right) must be minor (since otherwise only 10 would be discharged across one of the 6-6 edges). But if deg(A) = 5 then 1-6 occurs, if deg(A) = 6 then T2T2*n or T2T2*r occurs, q.e.d. ■

Proof of Lemma (T2,T2,T). Assume in an arbitrary triangulation Δ, V_k is a major vertex which receives a T2-, another T2-, and an arbitrary T-discharging across three consecutive 6-6 edges E, F, G. We claim that either Δ is different from Δ^* or Δ contains T2T2*rT1 as described in the lemma. We may assume that the T2-dischargings across E and F are induced by a configuration T2T2*n or T2T2*r in Δ, since otherwise the claim would follow immediately from Lemma (T2,T2). Then a configuration of the class (b) (see drawing to the right) occurs in Δ with V_k, E, F, G as indicated in the drawing. (We regard (b) as a 2-parameter class, where the first parameter ranges only over the two values T2T2*n and T2T2*r and the second parameter ranges over all T-discharging situations and their reflections.) But every configuration of (b) contains either a configuration of $\mathfrak{U}$, or T1#1, or T1#3r as described in the lemma (see the class check list for (b)). Q.e.d. ■



Lemma (T2,T2,T2) is an immediate consequence of Lemma (T2,T2,T). ■

Proof of Lemma (T2,T,T2). Assume in an arbitrary triangulation Δ, V_k is a major vertex which receives a T2-, an arbitrary T-, and another T2-discharging across three consecutive 6-6 edges E, F, G. We claim that either $\Delta \neq \Delta^*$ or the T-discharging across F is induced by T1#1. We may assume that Lemmas (T) and (T2,T) hold also for Δ (since otherwise the claim would follow immediately from those lemmas). Then one of the four cases of Lemma (T2,T) must apply to the edges E-F, and one of them (reflected) must apply to the edges G-F. But by Lemma (T), the T-discharging situations which are attached at F must be identical in both cases; this leaves the only possibility that T1#1 is attached to F in both cases, q.e.d. ■

Proof of Lemma (T,T2,T2,T). Assume in an arbitrary triangulation Δ, V_k is a major vertex which receives an arbitrary T-, a T2-, another T2-, and another arbitrary T-discharging across four consecutive 6-6 edges E, F, G, H. We claim that $\Delta \neq \Delta^*$. Let us assume that Lemma (T2,T2,T) holds also for Δ. Then one of the two cases of Lemma (T2,T2,T) applies to the edges F-G-H and one of the cases (reflected) applies to the edges G-F-E. But such a merging is impossible. Thus Lemma (T2,T2,T) does not hold for Δ, hence $\Delta \neq \Delta^*$, q.e.d. ∎

(2) Next we present a collection of 163 classes of configurations which are drawn on the following five pages (Figures I1,...,I5) and we establish that each configuration which belongs to any one of those classes contains a configuration of $\mathcal{U}$ by proving the following:

Lemma (I). If an arbitrary triangulation Δ contains a configuration out of one of the classes I1-1,...,I5-35 as defined in Figures I1,...,I5 (pp. 7,...,11 of this supplement) then Λ contains a configuration of $\mathcal{U}$.

After we have proved this lemma we may treat the classes I1-1,..., I5-35 as if they were additional configurations in $\mathcal{U}$, just containing the symbols  and  (as defined in the paper, see Figure 4).

Proof of Lemma (I). We assume that Δ contains a configuration out of one of I1-1,..., I5-35. We assume further that Lemmas (T), (T2,T), (T2,T2), (T2,T2,T), and (T2,T,T2) hold also for Δ. It remains to consider all those configurations in I1-1,..., I5-35 which may occur in Δ and to exhibit some configuration of $\mathcal{U}$ in each of them.

I1-1,..., I1-24 are 1-parameter classes and are treated in the corresponding class check lists. I1-26,...,I1-29 are 2-parameter classes; I1-30 is regarded a 1-parameter class where, by Lemma (T2,T2), the parameter ranges only over the two values T2T2*n and T2T2*r, (see class check lists).

Table J

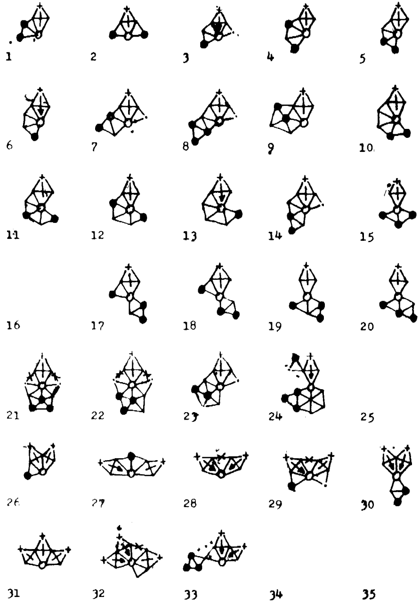


Figure I1

Table J

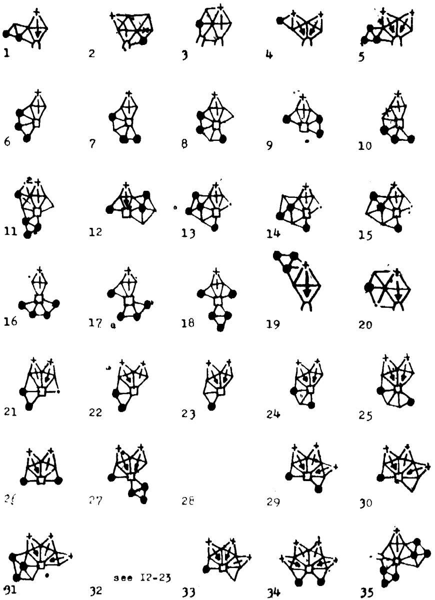


Figure I2

Table J

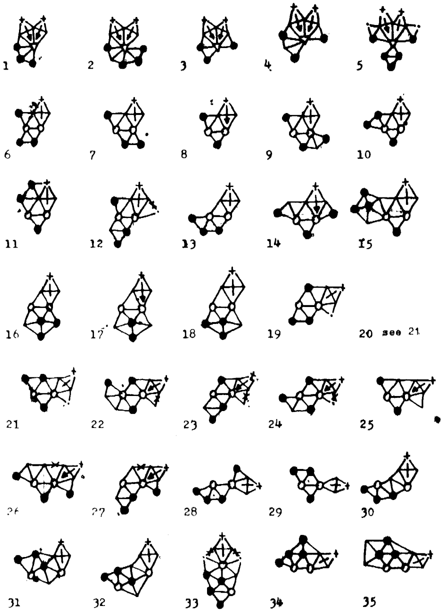


Figure 13

Table J

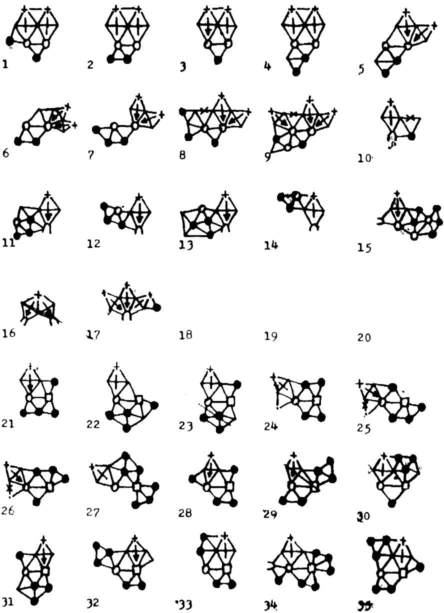


Figure I4

Table J

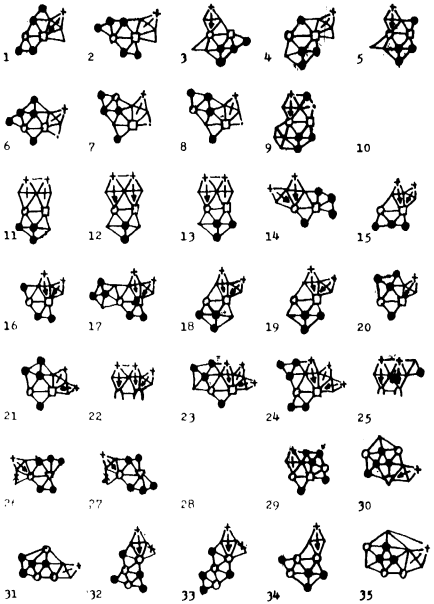


Figure I5

Proof of Lemma (I), continued

I1-31 is partitioned into the following sub-classes.
(We need to consider only those cases where the T-situation at F is not reflected, by symmetry).

(I1-31,a)

if T1#1 or
T2#5 at F

(I1-31,b)

if T1#2 or
T2#6 at F

(I1-31,c)

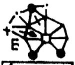
if T1#3 or
T2#7 at F

I1-32 is regarded a 2-parameter class, I1-33 a 1-parameter class. I2-1,..., I2-27 are 1- or 2-parameter classes. (See class check lists.)

I2-29,..., I2-31 may be regarded 2-parameter since by Lemma (T2,T,T2), the second T-discharging must be induced by T1#1. I2-32 and I2-33 are regarded 1-parameter since by Lemma (T2,T2,T), the two T2-dischargings must be induced by T2T2*r; moreover, the third T-discharging must be induced by one of T1#1, T1#3r. (See the class check lists.)

I2-34 is covered by the following case distinctions. Note that Lemma(T2,T) applies to the edges E-F and also to the edges G-H. If T1#1 is attached at (at least) one of F, G, then we may assume by the symmetry of I2-34, that it is attached at F; then (I2-34,a) occurs (see drawing below). Otherwise we have (I2-34,b)

I2-34



(I2-34,a)

T2T2n,
T2T2r, or
$\widehat{T2T1}$n
at E-F

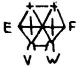
T2T2n,
T2T2r, or
$\widehat{T2T1}$r
at G-H

(I2-34,b)

Proof of Lemma (I), finished

I2-35,..., I3-35 are 1-parameter classes (see check lists).

In the remaining part of the proof of Lemma (I, we shall need a lemma on the configuration class (T‖T) (see drawing below) and its sub-classes (T2‖T) and (T2‖T2).



(T‖T)

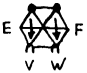
(T2‖T)

(T2‖T2)

note that T‖T/1⁺ contains T‖T/1 as a sub-configuration

T‖T/1n

T‖T/1r



T‖T/2

T‖T/1⁺

Lemma (T‖T). If Δ contains a configuration of the class (T‖T) then either Δ contains a configuration of $\mathcal{U}$ or Δ contains one of T‖T/1n, T‖T/1r, T‖T/2; in particular, if the configuration belonged to the sub-class (T2‖T) then Δ contains one of T‖T/2, T‖T/1⁺, and if the configuration belonged to (T2‖T2) then Δ contains T‖T/2 (with vertices V, W and edges E, F as indicated in the drawings above).

Proof see class check list for (T‖T). ∎

Using this lemma, we may treat I4-1,..., I4-4 as 1-parameter classes (see class check lists). All the remaining classes I4-5,..., I5-35 can be treated as 1- or 2-parameter classes (see class check lists) where for I5-11,..., I5-13, and I5-22 the above lemma is used again, and for I5-23, I5-24 Lemma (T2,T,T2) is used in order to conclude that the occurrence of I5-23 implies the occurrence of (I5-23,a) and that the occurrence of I5-24 implies the occurrence of (I5-24,a) (see drawings below).


(I5-23,a)


(I5-24,a)

(3) Next we prove Lemma S^{+}.

Proof of Lemma S^{+}. Assume that an arbitrary triangulation Δ contains an S-situation S, i.e., that there is a simplicial immersion $f: S \to \Delta$ which respects the degree specifications. Let $f^{+}: S^{+} \to \Delta$ be the extension of f over the enlarged S-situation (as drawn in Table 1 where the additional vertices - which belong to S^{+} but not to S - are indicated by "clip marks"). Now assume further that f^{+} does not respect the degree specifications (of S^{+}). Then we claim that Δ contains a configuration of $\mathcal{U}$ and thus $\Delta \neq \Delta^{\#}$. In order to prove the claim we have to consider the additional vertices, say $V^{(1)}, \ldots, V^{(p)}$, of S^{+}; each one of them has either the degree specification ≥ 6 or ≥ 7. If $V^{(i)}$ has the degree specification ≥ 6 then we have to consider the configuration, say $S_5^{(i)}$, which is obtained from S by adding vertex $V^{(i)}$ with degree specification 5 (instead of ≥ 6); if $V^{(i)}$ has the degree specification ≥ 7 then we have to consider two configurations $S_5^{(i)}$ and $S_6^{(i)}$ which are obtained from S by adding vertex $V^{(i)}$ with degree specification 5 or 6, respectively.

Each one of the configurations $S_j^{(i)}$ defined above contains some configuration of $\mathcal{U}$ (or equivalently, some configuration of Table J); see the check list for Lemma S^{+} (pp. C 23, ..., C 51).

Our above assumptions about Δ imply that Δ contains at least one of the configurations $S_j^{(i)}$ and thus $\Delta \neq \Delta^{*}$, q.e.d. ■

(4) Now we are prepared to prove the $q_{TS}(V_5)$ - Lemma.

Proof of $q_{TS}(V_5)$ - Lemma. We assume that Δ is an arbitrary triangulation that contains a 5-vertex V so that $q_{TS}(V) > 0$. We claim that Δ contains a configuration in one of the configuration classes CTS#01,...,CTS#33, as drawn in Table 3, with its central $\hat{V}_5$ identified to V, or Δ contains some configuration of $\mathcal{U}$ (or equivalently, some configuration of Table $\mathcal{J}$). In order to prove the claim we may assume that the lemmas on T-dischargings and Lemma S^+, as proved for Δ^* in Sections (1) and (3) of the supplement, hold also for Δ (since otherwise the claim would follow immediately from those lemmas).

The proof is divided into the six cases $\mu = 0, \ldots, 5$. The treatment of each case begins with an arrangement check list" (except for the case $\mu = 5$ which begins with some preliminary lemmas) in which all possible arrangements of minor and major vertices around the central 5-vertex V are listed; we use the symbol "m " for "minor" and "U " for "major"; in some instances we choose to distinguish between "5 " and "6 " (degree-specifications) for minor vertices or to distinguish between "R " for "regular discharging"(i.e., major neighbor of V such that the edge joining it to V is not an S-edge), and "S " for "small discharging" (i.e., major neighbor of V, joined to V by an S-edge). Furthermore we may choose to distinguish between "S0 ", "S1 ", and "S2 " for "S0-, S1-, or S2-discharging", respectively.

Some arrangements imply immediately the presence of a configuration of $\mathcal{U}$ or of Table 3. This is indicated by writing the number in Table $\mathcal{U}$ (e.g., 1-5) or in Table 3 (e.g., CTS#03) next to the arrangement.

For some arrangements it follows from Lemmas (5-γ-6) or (6-6-6) that T-dischargings of a total value of 20 or 30 leave V. This is indicated by writing $\tau = 20$ or $\tau = 30$ next to the arrangement.

Proof of $\varphi_{TS}(V_5)$ - Lemma
Introduction, continued

In each arrangement check list we try to avoid listing essentially the same arrangement several times, i.e., listing cyclic permutations and reflections of the same arrangement. For this purpose we may state some restrictive rules, as for instance in the case $\mu = 2$, that one of the smallest S-dischargings which leaves V must be induced by an S-situation which is attached at A (the first neighbor of V in the listing) and which is not reflected.

Some arrangements cannot be realized so as to fulfill all the requirements for the reason that no suitable S-situation exists. This is indicated by a remark next to the arrangement.

Some arrangements are regarded as 1-parameter or 2-parameter configuration classes which are treated further in the corresponding class check lists. We use the symbol " ÷ " in order to indicate that certain statements and/or drawings define a configuration class the number of which, e.g., (1b), follows " ÷ ".

In the more complicated cases we define certain configuration classes, such as (2a) or (2b), following the arrangement check list and we indicate next to each arrangement by which of those classes it is covered. For instance, " → (2a) " written next to an arrangement (without the symbol " ÷ " in between) means that every configuration which realizes the arrangement contains some configuration of the class (2a) as defined below, with central V_5 identified to the central V_5 of the arrangement. It should be noted that all classes we define are 1-parameter or 2-parameter classes and thus can be conveniently treated in the corresponding class check lists.

Some of the configuration classes, for instance (2a), contain some "critical configurations", i.e., configurations which do not contain configurations from $\mathcal{U}$ or from Table 3. In these cases we define "critical sub-classes", for instance, the critical sub-classes of (2a) are C2a#1,..., C2a#25, which contain all the critical configurations (but may contain also non-critical configurations).

Proof of $q_{TS}(V_5)$ - Lemma

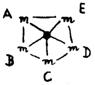
Introduction, finished

Each critical sub-class leads to one or more "derived configuration classes", e.g., C2a#5 leads to (2aa), where the configurations in the derived classes are obtained from configurations in the original class by attaching another S-situation. By this we mean the following. If some configuration of the critical sub-class occurs in Δ with its central V_5 identified to V (and if $q_{TS}(V) > 0$ and the value of μ is as prescribed in the case-hypothesis) then some configuration of one of the derived classes must occur in Δ with its central V_5 identified to V. - Some of the derived classes may again contain critical sub-classes which again lead to "second derived configuration classes"; e.g., the derived class (3ab) contains the critical sub-class C3ab which leads to the second derived class (3aba). In the corresponding class check lists it is verified that each configuration of a certain class contains a configuration of $\mathcal{U}$ or of Table 3 or belongs to one of the critical sub-classes.

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 0$

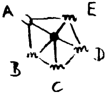
Arrangement check list



A	B	C	D	E	
5	5	5	5	5	→ 1-1
5	5	5	5	6	
5	5	5	6	6	
5	5	6	5	6	→ 1-2
5	5	6	6	6	→ 1-5
5	6	5	6	6	→ 1-2
5	6	6	6	6	→ 1-7
6	6	6	6	6	→ 1-8

Case $\mu = 1$

Arrangement check list



A	B	C	D	E	
U	5	5	5	5	→ 1-1
U	5	5	5	6	
U	5	5	6	5	→ 1-2
U	5	5	6	6	→ CTS#01
U	5	6	5	6	→ 1-2
U	5	6	6	5	→ 1-4
U	5	6	6	6	→ $v = 30$
S	5	B.	6	6	no such S

A	B	C	D	E	
U	6	5	5	6	→ CTS#02
U	6	5	6	6	→ CTS#03
U	6	6	6	6	→ $v = 30$
S	6	6	6	6	no such S

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 2$

Arrangement check list

(~~the~~ smallest S leaving the V_5 is at A and is not reflected)

A	B	C	D	E
S	m	m	m	U
S	5	5	5	$U \to 1-1$
S	5	5	6	$U \doteq (1a)$
S	5	6	5	$U \to 1-2$
S	5	6	6	$U \to \tau = 20$
S0	5	6	6	$U \doteq (1b)$
S1	5	6	6	$S \doteq (1c)$
S	6	5	5	$U \doteq (1d)$
S	6	5	6	$U \doteq (1e)$
S	6	6	5	$U \to \tau = 20$
S0	6	6	5	$U \doteq (1f)$
S1	6	6	5	$S \doteq (1g)$
S	6	6	6	$U \to \tau = 20$
S0	6	6	6	$U \doteq (1h)$
S1	6	6	6	$S \doteq (1i)$

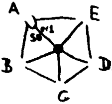
S	m	m	U	m
S	5	5	U	5 no such S
S	5	5	U	$6 \doteq (1j)$
S	5	6	U	5 no such S
S	5	6	U	$6 \doteq (1k)$
S	6	5	U	5 S reflected
S	6	5	U	6 no such S
S	6	6	U	5 S reflected
S	6	6	U	6 no such S

A	B	C	D	E	
S	m	U	m	m	
S	5	U	5	5	no such S
S	5	U	5	6	no such S
S	5	U	6	5	no such S
S	5	U	6	$6 \pm (1\ell)$	
S	6	U	5	5	S reflected
S	6	U	5	6	no such S
S	6	U	6	5	S reflected
S	6	U	6	6	no such S
S	U	m	m	m	S reflected

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 3$

Arrangement check list



S0 or S1 at A. If S1 at A then no S0 anywhere.

If S1 of Type U - U at A then S2 at E and S2 at B.

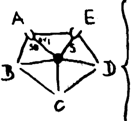
If S0 of Type U - U at A then S1 or S2 at E.

If the S at A is
- S0 of Type U - U then it may be reflected
- #248 then it is reflected (248r)
- none of the above then it is not reflected

A	B	C	D	E	
S0	m	m	S	R	→ (2b)
S0	m	m	U	S	→ (2a)
S0	m	S	m	R	→ (2c)
S0	m	U	m	S	→ (2a)
S0	m	U	U	m	no such S0
S0	U	m	m	S	→ (2a)

A	B	C	D	E	
S1	m	m	S1	R	→ (2d)
S1	m	m	U	S	→ (2a)
S1	m	S1	m	U	→ (2e)
S1	m	S2	m	S	→ (2a)
S1	m	R	m	S1	→ (2a)
S1	m	S1, U	m	→ (2e)	
S1	m	S2	S2	m	→ (2f)
S1	m	U	S1	m	→ (2e)
S1	S2	m	m	S2	→ (2a)

Configuration classes



(2a)

S0 or S1 at A

S at E

Precisely two of B, C, D are minor ([illegible], $\mu = 3$)

If S1 at A then not S0 at E

If S1 of Type U - U at A then S2 at E and S2 at B.

If the S at A is
- S0 of Type U - U then it may be reflected or no
- #248 then it is reflected (248r)
- none of the above then it is not reflected

The class (2a), as defined above, contains several critical sub-classes, denoted C2a#1,..., 25 which are described on the following [illegible] pages. Some of the C2a - classes lead to derived classes (2aa),..., (2ar) of enlarged configurations (where the enlargement consists of attaching and/or S-situation).

Proof of $q_{TS}^{\circ}(V_5)$ - Lemma

Case $\mu = 3$

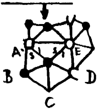
The C2a - cases

C2a # 1: { # 011 at A and one of ## 265r, 266r, 274, 325r at E
or
one of ##013,...,016 at A and #261n at E

(if one of these configurations occurs - in the case $\mu = 3$ - then deg(C) = 6 and thus some member of the configuration class below occurs)

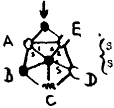
S0 at A and S2 at E as specified above → CTS #14.

C2a #2: one of ##032,...,035 at A and #248r at E



→ CTS #15

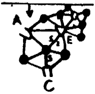
C2a #3: #021 at A and one of ## 265r, 266r, 274, 314r,...,318r, 325r at E



{ S2 at E as specified above
S1 or S2 at D

= (2aa)

C2a #4: any S1 of Type 5 - U at A and [illegible] 145r at E,



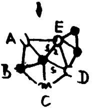
S at α → no such S

Proof of $q_S(V_5)$ - Lemma

Case [illegible] = 3

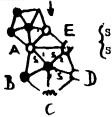
The C2a - cases, continued

C2a #5: any S? of Type 5 - U at A and #261n at E



S1 or S2 at D $\doteq$ (2ac)

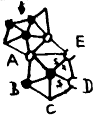
C2a #6: #025 at A and #285n or 312n at E



{S2 at E as specified above
{S1 or S2 at D

$\doteq$ (2ad)

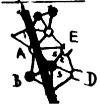
C2a #7: # 026 or 029 at A and one of ##264n, 285n, 312n at E



{S2 at E as specified above
{S1 or S2 at D

$\doteq$ (2ae)

C2a #8: # 027 or 028 at A and #264n or 285n at E



{S2 at E as specified above
{S1 or S2 at D

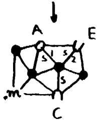
$\doteq$ (2af)

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 3$

The C2a - cases, continued 2

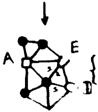
C2a #9: #030 or 031 at A and any S2 of Type 5 - U at E



S1 at A
S2 at E
S at C → #001 (the only possible)

→ 2-3

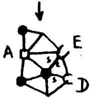
C2a #10: one of ##032,...,036 at A and one of ##263r, 267r, 268r, 270r, 271r, 274, 313, 324r, 325r at E



S2 at E as specified above
S1 or S2 at D

≐ (2ag)

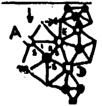
C2a #11: one of ## 037,...,041 at A and #285n or 312n at E



S2 at E as specified above,
S1 or S2 at D

≐ (2ah)

C2a #12: any S1 of Type 5 - U or 6 - U at A and #272n at E



S1 at A
S2 at E
S1 or S2 at D → no such S

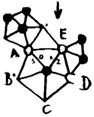
Proof of $q_{TS}(V_5^{\cdot})$ - Lemma

Case $\mu = 3$

The C2a - cases, continued[3]

C2a #13: #151 at A' and #261n at E

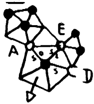
↓



→

→ CTS #18

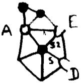
or



S at D $\doteq (2ai)$

C2a #14: one of ##161,...,164 at A and #266r or 325r at E

↓

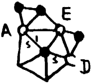


S2 at E as specified above
S1 or S2 at D

$\doteq (2aj)$

C2a #15: one of ##161,..., 164 at A and #274 at E

↓



S1 at A as specified above
S1 or S2 at D

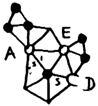
$\doteq (2ak)$

Propf of $q_{TS}(V_5)$ - Lemma

Case μ - 3

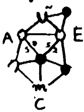
The C2a - cases, continued[4]

C2a #16: one of ##165, 166, 167 at A and #26th at E ;



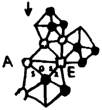
{S1 at A as specified above
{S1 or S2 at D $\doteq$ (2a ℓ)

C2a #17: one of ##231r, 234r,..., 237r at A and #052 at E



S0 at A as specified above $\longrightarrow$ {if deg(c)=5 $\rightarrow$ CTS #17
{if deg(c)=6 $\rightarrow$ CTS #16

C2a #18: one of ##231r, 234r, 235r, 237r at A and #125 at E



S0 at A as specified above $\longrightarrow$ CTS #16

C2a #19: one of ##231r, 234r,..., 237r at A and #054 or 055 at E

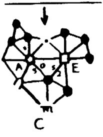
{S0 at A as specified above
{S2 at E as specified above $\longrightarrow$ CTS #17

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 3$

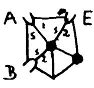
The C2a - cases, continued[5]

C2a #20: #235r or 236r at A and #118 at E



SO at A as specified above → $\begin{cases} \text{if } deg(C)=5 \rightarrow \text{CTS \#17} \\ \text{if } deg(C)=6 \rightarrow \text{CTS \#16} \end{cases}$

C2a #21: one of ##244n,..., 247n, 250n, 252n, 253n at A and any S2 of Type 5 - U at E (where "Type 5 - U" means those S - situations of Type 5 - U which do not allow another V_5 at the bottom, i.e., the S2's are ## 069,...,075, 084,...,094, 096, ..., 101, 104,..., 116, 120,..., 123, 125,..., 145, 147,..., 150.

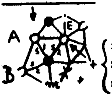


$\left\{\begin{array}{l}\text{S1 at A as specified above} \\ \text{S2 at E as specified above} \\ \text{S2 at B}\end{array}\right. \doteq (2am)$

C2a #22: one of ##244n, 246n, 247n, 250n at A and any S2 of Type 6 - U at E (i.e., one of ##172, 173, 175, ..., 179, 182,..., 190, 193,..., 197, 200,..., 230.1)

$\left\{\begin{array}{l}\text{S1 at A as specified above} \\ \text{S2 at E as specified above} \\ \text{S2 at B} \\ \text{no T-discharging leaving the central } V_5\end{array}\right. \doteq (2an)$

C2a #23: one of ##245n, 246n, 247n at A and #174 at E



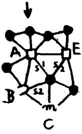
$\left\{\begin{array}{l}\text{S1 at A as specified above} \\ \text{S2 at B} \\ \text{no T-discharging leaving the } V_5\end{array}\right. \doteq (2ao)$

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 3$

The C2a - cases, finished.

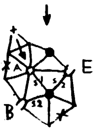
C2a #24: #253n at A and one of ##062,..., 068 at E



S2 at E as specified above
S2 at B

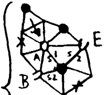
$\doteq (2ap)$

C2a #25: #245n at A and one of ##172, 175, 176, 220, 221, 224, 225 at E



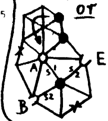
S2 at E as specified above
S2 at B
no T-discharging leaving the central V_5

$\rightarrow$

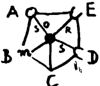


$\doteq (2aq)$

or



$\doteq (2ar)$



(2b)

S0 of Type m - U at A (whence C cannot be of degree 5)
S at D
no S at E

Class (2b) is partitioned into sub-classes C2b#1,...,6 as described on the next page. C2b#6 leads to the derived class (2ba).

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 3$

The 2b - sub-classes

C2b #1: #011 at A

C2b #2: one of ##012,..., 016 at A

→ one of ## 171, 174, 180, 181, 191, 192, 198, 199 at D → CTS #20

C2b #3: #151 at A and S0 at D

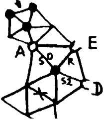
→ #151 at D → CTS #27

C2b #4: #151 at A and S1 at D

→ one of ## 161,...,167 at D → CTS #26

C2b #5: #151 at A and one of ##171,..., 173, 175, 176, 181,..., 190, 193, 194, 199,..., 204, 211,..., 221, 224,..., 229 at D → CTS #25

C2b #6: #151 at A and one of ##174, 179, 180, 191, 192, 195,..., 198, 230, 230.1 at D



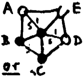
S2 at D as specified above
no T-discharging leaving the central V_5 ≐ (2ba)

(2d)

S1 at A
S1 at D

(if some S1 is of Type 6 - U then this requires that none of B, C is a V_5)

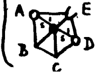
→



S1 at A
S1 at D ≐ C2d#1

= CTS #24

or

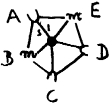


S1 at A
S1 at D ≐ C2d#2 (see class check 2)

Proof of $q_{TS}(V_5)$ - Lemma

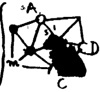
Case $\mu = 3$

Configuration classes, finished

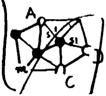


(2e)

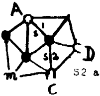
S1 at A → #002 or 003
(not reflected)
S1 at C or at D →



S1 at C ≐ C2e#1



S1 at D ≐ C2e#2



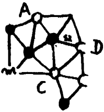
S2 at C

(2f) (see class check list)

The C2f - cases

C2f#1: #052 at C

↓



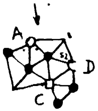
S2 at D ≐ (2fa)

Proof of $q_{T5}(V_5)$ - lemma-

Case μ = 3

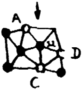
The C2f - cases, finished

C2f#2: one of ##061, 063,..., 066, 068 at C



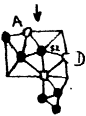
S2 at D $\doteq (2fb)$

C2f#3: #067 or 076 at C



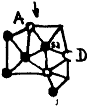
S2 at D $\doteq (2fc)$

C2f#4: one of ##080, 081, 082 at C



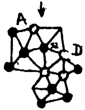
S2 at D $\doteq (2fd)$

C2f#5: #083 or 102 at C



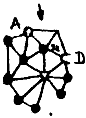
S2 at D $\doteq (2fe)$

C2f#6: #124 at C



S2 at D $\doteq (2ff)$

C2f#7: #146 at C

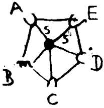


S2 at D $\doteq (2fg)$

Proof of $q_{4S}(V_5)$ - Lemma

Case $\tilde{\mu} = 4$

Arrangement check list



$$\left\{\begin{array}{l}\text{(discharging to A)} + \text{(discharging to E)} \\ \qquad \leq \text{(discharging to C)} + \text{(discharging to D)} \\ \text{minor at B}\end{array}\right.$$

$$\longrightarrow \left\{\begin{array}{l}\text{S at A (not reflected)} \\ \text{S at E}\end{array}\right.$$

A	B	C	D	E
S0	m	S	U	S0
S0	m	R	S	S0
S0	m	S0	U	S1
S0	m	S1	U	S1
S0	m	S2	S	S1
S0	m	R	S0	S1
S0	m	R	S1	S1
S0	m	S0	U	S2
S0	m	S1	S	S2
S0	m	S2	S0	S2
S0	m	S2	S1	S2
S0	m	R	S0	S2

(3a)

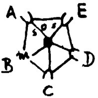
A	B	C	D	E
S1	m	S0	U	S0
S1	m	S1	U	S0
S1	m	S2	S	S0
S1	m	R	S0	S0
S1	m	R	S1	S0
S1	m	S0	U	S1
S1	m	S1	U	S1
S1	m	S2	S0	S1
S1	m	S2	S1	S1
S1	m	R	S0	S1

(3b)

A	B	C	D	E
S2	m	S0	U	S0
S2	m	S1	S	S0
S2	m	S2	S0	S0
S2	m	S2	S1	S0
S2	m	R	S0	S0

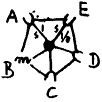
(3c)

Configuration classes



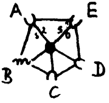
$\left\{\begin{array}{l}\text{S0 at A} \\ \text{S at E}\end{array}\right.$

$\boxed{(3a)}$



$\left\{\begin{array}{l}\text{S1 at A} \\ \text{S0 or S1 at E}\end{array}\right.$

$\boxed{(3b)}$



$\left\{\begin{array}{l}\text{S2 at A} \\ \text{S0 at E}\end{array}\right.$

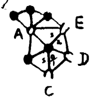
$\boxed{(3c)}$

Proof of $q_{TS}(V_5)$ - Lemma

Case μ - 4

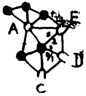
The C3a - cases and their derivatives

C3a #1: #011 at A and one of ##265r, 266r, 274, 325r at E



S2 at E as specified above
S0 or S1 at C

$\doteq (3aa)$



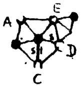
S2 at E as specified above
S0 or S1 at D

$\doteq (3ab)$

C3aa #1: S2 at E as specified above and one of ##012, 015, 016 at C

→ CTS #30

C3aa #2: S2 at E as specified above and one of ##021, 022, 023, 025,..., 029, 033, 037,..., 040 at C

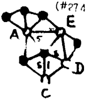


S1 at C as specified above
S at D

$\doteq (3aaa)$

or

(#274 at E)

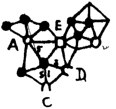


S1 at C as specified above
S at D with d< 25

$\doteq (3aab)$

or

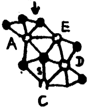
(#325r at E)



S1 at C as specified above
S at D with d< 25

$\doteq (3aac)$

C3ab: #274 at E and #248r at D



S at C
with d< 25

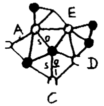
$\doteq (3aba)$

Proof of $q_{TS}(V_5)$ - Lemma.

Case $\mu = 4$

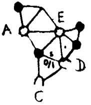
The C3a - cases and their derivatives, continued

C3a #2: one of ##013,..., 016 at A and #261n at E



SO at A as specified above
SO or S1 at C

$\doteq (3ac)$



SO or S1 at.D.

$\doteq (3ad)$

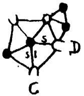
C3ac#1: #013 or 014 at A and one of ##012,..., 016 at C.

or

#015 or 016 at A and one of ##011,..., 016 at C.

→ CTS #30

C3ac #2: one of ##021,..., 023, 025,..., 029, 032,..., 041 at C



S1 at C as specified above
S at D

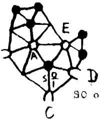
$\doteq (3ac\dot{a})$

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 4$

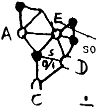
The C3a - cases and their derivatives, finished

C3a #3: #151 at A and #261n at E $\longrightarrow$



S0 or S1 at C

$\doteq (3ae)$



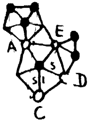
S0 or S1 at D

$\doteq (3af) = (3ad)$

C3ae #1: #151 at C $\rightarrow$ CTS#32

C3ae #2: one of ##162,..., 167 at C

$\downarrow$



S1 at C as specified above
S at D

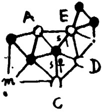
$\doteq (3aea)$

Proof of $q_{TS}(V_5)$ Lemma

Case μ = 4

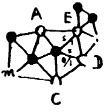
The C3b - cases and their derivatives

C3b #1: #030 or 031 at A and #249n or 251n at C



#249n or 251n at E
S0 or S1 at C

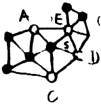
$\doteq$(3ba)



#249n or 251n at E
S0 or S1 at D

$\doteq$(3bb)

C3ba #1: #030 at A and one of ##021, 022, 030 at C

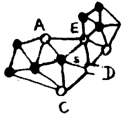


(#249 at E)

S at D

$\doteq$(3baa)

or

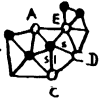


(#251 at E)

S at D

$\doteq$(3bab)

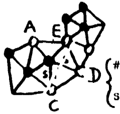
C3ba #2: #031 at A and one of ##021, 022, 023 at C



#021, 022,
or 023 at C
S at D

$\doteq$(3bac)

or



#021, 022,
or 023 at C
S at D

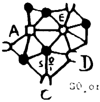
$\doteq$(3bad)

Proof of $4_{TS}(V_5)$ - Lemma

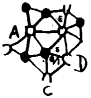
Case $\mu = 4$

The C3b'- cases and their derivatives, finished

C3b #2: one of ##032,.. , 035 at A and #248r at E



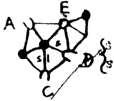
S0 or S1 at C $\doteq(3bc)$



S0 or S1 at D $\doteq(3bd)$

C3bc #1: one of ##011,. ., 016 at C → CTS #31

C3bc #2: one of ##021, 022, 023, 025,...,029, 032,..., 041 at C



S1 at C as specified above
S at D

$\doteq(3bca)$

Note that (3bca) contains (3aaa) as a sub-class and that (3bca) - (3aaa) consists of those configurations which have one of ##032, 034, 035, 036, ~~044~~ at C

Proof of $q_{PS}(V_5)$ - lemma

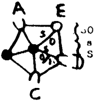
Case $\mu = 4$

The C3c - case and its derivatives

C3c: any S2 of Type 5-U at A and one of ##231n, 234n,..., 237n at E

S0 at E as specified above
S0 or 31 at C

$\doteq$(3ca)



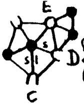
S0 at E as specified above
S0 or 31 at D

$\doteq$(3cb)

C3ca #1: one of ##011,..., 016 at C → CTS#29

C3ca #2: one of ##021, 022, 023, 025,..., 041 at C

(if #231 at E)

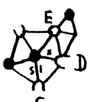


31 at C as specified above
S at D

$\doteq$(3caa)

or

(if one of ##232,..., 237 at E)



31 at C as specified above
S at D

$\doteq$(3cab)

Note that (3caa) contains (3aca) as a sub-class and that (3caa) - (3aca) consists of those configurations which have #030 or 031 at C

Note that (3cab) contains (3bca) as a sub-class and that (3cab) - (3bca) consists of those configurations which have #030 or 031 at C

Proof of $q_{TS}(V_5)$ - Lemma

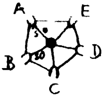
Case $\mu = 5$

This case is mainly treated by the consideration of pairs of S-situations (of Type U - U) with pivots identified to V and distinguished edges consecutive about V_5. We shall define four classes (4a),..., (4d) of such pairs and prove the following lemmas about their critical sub-classes.

Lemmas (4a),...,(4d). If Δ contains some configuration, say K, of the class (4a),..., or (4d), respectively; then K contains a configuration of $\mathcal{U}$ or K belongs to the critical sub-class (C4a) of (4a), (C4b) of (4b), or to one of the critical sub-classes (C4c#1), (C4c#2) of (C4c), or to one of the critical sub-classes (C4d#1),..., (C4d#6) of (C4d), respectively.

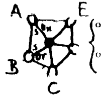
Proof see class check lists for (4a),..., (4d) (pp. C125ff). ∎

Configuration classes



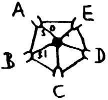
$\begin{cases}\text{S0 at A}\\ \text{S0 at B}\end{cases}$

$\doteq$ $\boxed{(4a)}$



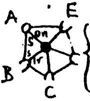
$\begin{cases}\text{one of \#\# 231n, 234n,..., 237n at A}\\ \text{one of \#\# 231r, 234r,..., 237r at B}\end{cases}$

$\doteq$ $\boxed{(C4a)}$



$\begin{cases}\text{S0 at A}\\ \text{S1 at B}\end{cases}$

$\doteq$ $\boxed{(4b)}$



$\begin{cases}\text{one of \#\# 231n, 234n,..., 237n at A}\\ \text{one of \#\# 241r,..., 244r, 246r, 247r, 249r,..., 252r at B}\end{cases}$

$\doteq$ $\boxed{(C4b)}$

Proof of $q_{TS}(V_5)$ - Lemma

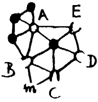
case μ = 5

Configuration classes, continued

{one of ## 231r, 234r,..., 237r at A
{S2 at B

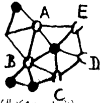
≐ $\boxed{(4c)}$

(#231r at A)



deg(B) = 7 or one of ##
323n,..., 325n
at B

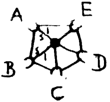
≐ $\boxed{(C4c\#1)}$



(#261r at B)

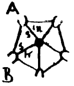
one of ##234r,..., 237r
at A

≐ $\boxed{(C4c\#2)}$



{S1 at A
{S1 at B with # not smaller than # at A

≐ $\boxed{(4d)}$



one of ## 241n,..., 244n, 246n, 247n, 249n,..., 252n at A

one of ## 241r,..., 244r, 246r, 247r, 249r,...,252r at B

≐ $\boxed{(C4d\#1)}$

#245n at A and #245r at B ≐ $\boxed{(C4d\#2)}$

#253n at A and #253r at B ≐ $\boxed{(C4d\#3)}$

#244r or 245r at A and one of ##244n,...,248n, 253r at B ≐ $\boxed{(C4d\#4)}$

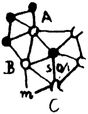
#246r or 247r at A and one of ##246n,...,248n at B ≐ $\boxed{(C4d\#5)}$

#248r at A and one of ##249n, 252n, 253n, 253r at B ≐ $\boxed{(C4d\#6)}$

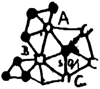
Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 5$

Configuration classes, finished



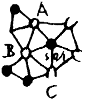
$\doteq$ (4e)



$\doteq$ (4f)

(#323n, 324n, or 325n at B)

(4e) and (4f) may be regarded to be derived from (C4c#1).



$\doteq$ (4g)

(#261r at B)

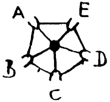
In all three classes (4e),..., (4g):

either S0 at C;
or S1n, except # 248n, at C;
or # 248r at C.

(4g) may be regarded to be derived from (C4c#2).

Arrangement check list

From now on we may assume that if Λ contains some configuration, say K, of one of the classes (4a),..., (4d) then K belongs to one of the critical subclasses (C4a),.., (C4d#6), since otherwise the claim of the $q_{TS}(V_5)$ - lemma would follow immediately from lemmas (4a),...,(4d).



We may assume that the sequence of discharging values from V to A, B, C, D, E (if read as a 5-digit number) is minimal compared with the other 9 possible (clockwise or counter-clockwise) cyclic readings of the discharging values.

A B C D E

S0 S0 S0 U U impossible, since the S0 attached at B has to be part of a configuration K of (C4a) with A(K) (the vertex A of K as denoted in the drawing on p.38) identified to A and

Proof of $i_{TS}(\Gamma_5)$ - Lemma

Case μ = 5

Arrangement check list, continued

A B C D E

B(K) identified to B; but simultaneously, the S0 has to be part of a configuration K' of (C4a) with A(K') identified to B and B(K') identified to C. Thus S0 has to be non-reflected and reflected simultaneously and one of ##231, 234,..., 237 which is impossible.

S0 S0 S1 U U impossible, since the S0 at B has to be part of some K of (C4a) with A(K), B(K) identified to A, B; and the S0 has to be part of some K' of (C4b) with A(K'), B(K') identified to B, C.

S0 S0 S2 S0 U covered by (4e), (4f), (4g). For the S0 at B is part of some K of (C4a) with A(K), B(K) identified to A, B, and thus is one of ## 231r, 234r,...,237r; thus the S2 at C is part of some K' of (4d) with A(K'), B(K') identified to B, C; but K' must belong to one of (C4c#1), (C4c#2). Thus some K" of (4e), (4f), or (4g) must occur with A(K"), B(K"), C(K") identified to B, C, D.

S0 S0 S2 S1 S2 covered by (4e), (4f), (4g).

For by symmetry of the arrangement (may read clockwise starting at B), we may assume that the S1 at D is non-reflected if it is not #248 and that it is reflected if it is #248. Then, by the same conclusions as in the previous arrangement, some K" of (4e), (4f), or (4g) must occur.

S0 S1 S0 U U impossible. For the S1 at B has to be part of some K of (C4b) with A(K), B(K) identified to A, B. The S1 has also to be part of some K' of (C4b) with A(K'), B(K') identified to C, B. Thus the S1 has to be reflected and non-reflected which is impossible since the S1's are not symmetric.

Proof of $q_{T3}(V_5)$ - Lemma

Case $\mu = 5$

Arrangement check list, continued

A B C D

S0 S1 S1 S0 S covered by CTS#33.

For the S0 at A is part of some K of (C4b) with A(K), B(K) identified to A, B, and thus is one of ##231n, 234n,...,237n. The S0 at D is part of some K' of (C4b) with A(K'), B(K') identified to D, C, and thus is one of ##231r, 234r,...,237r.

S0 S1 S1 S1 S1 impossible. For the S0 at A has to be part of some K of (C4b) with A(K), B(K) identified to A, B and has to be part of some K' of (C4b) with A(K'), B(K') identified to A, E. Thus the S0 has to be one of ##231, 234,...,237 reflected and non-reflected.

S0 S1 S1 S1 S2 covered by (4e), (4f), (4g).

For the S0 at A is part of some K of (C4b) with A(K), B(K) identified to A, B, and thus is one of ## 231n, 234n,...,237n. ~~Thus the S0~~ and the S2 are the essential part of some K' of (C4e#1) or (C4e#2) with A(K'), B(K') identified to A, E. On the other side, the S1 at B is a part of K and thus is one of ##241r,...,244r, 246r, 247r, 249r,...,252r. Consequently, the S1 at C is part of some K" of (C4d#4) or (C4d#5) with A(K"), B(K") identified to B, C, and thus is one of ##244n,...,248n, 253r. Consequently, the S1 at D is part of some K"' of (C4d#1) or (C4d#2) with A(K"'), B(K"') identified to C, D, and thus is one of ~~## 241r,...,~~ 247r, 249r,..., 252r. Thus the S1 at D and the configuration K' form the essential part of some configuration K"" of (4e), (4f), or (4g) with A(K""), B(K""), C(K"") identified to A, E, D.

S0 S1 S1 S2 S1 impossible for the same reason for which S0 S1 S1 S1 S1 is impossible.

Proof of $q_{TS}(V_5)$ - Lemma

Case $\mu = 5$

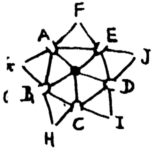
Arrangement check list, finished

A B C D E

S0 S1 S2 S0 S2 covered by (4e), (4f), (4g).

For the S0 at A is part of some K of (C4b) with A(K), B(K) identified to A, B, and thus is one of ##251n, 254n,...,257n. Thus the S0 at A and the S2 at E are the essential part of some K' of (C4c#1) or (C4c#2) with A(K'), B(K') identified to A, E. Thus the S0 at D and the configuration K' form the essential part of some configuration K'' of (4e), (4f), or (4g) with A(K''), B(K''), C(K'') identified to A, E, D.

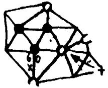
S1 S1 S1 S1 S1 impossible since by Lemma (4d), none of the S1's can be #241,..., 243, 248,..., 253. For otherwise we could assume by symmetry that the S1 at A is one of ## 241r,..., 243r, 248n, 249r,..., 253r; then the S1's at A and at B would form the essential part of a configuration K of one of (C4d#1),..., (C4d#6) with A(K), B(K) identified either to A, B, or to B, A; but none of ## 241r, ...,243r, 248n, 249n,...,253r can be contained in K at A(K), and none of ## 241n,...,243n, 248r, 249n,...,253n can be contained in K at B(K). Thus the only S1's that could possibly occur are ## 244,...,247. But then the vertices F, G, H, I, J (see drawing to the left) would have to be alternatingly of degree 5 and non-5, in cyclic order, which is impossible.



This finishes the proof of the $q_{TS}(V_5)$ - Lemma. ■

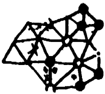
(5) Proof of the L-Lemma. CTS#01 contains the L4-situation #401 with discharging edge identified to the edge marked X in Table 3. CTS#02 contains the L6-situation #411. CTS#03 contains #421.

CTS#06
contains
#422

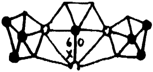


CTS#08
contains
#491

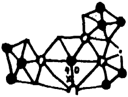
CTS#13
contains
#511



CTS#18
contains
#475
(by Lemmas(6-6),8*



CTS#27
contains
#621



CTS#32
contains
#597

The other CTS-situations are treated as 1-parameter and 2-parameter configuration classes in corresponding class check lists (see pp. C 133ff). ■

(6) In the proof of the $q_{TL}(V_k)$ - lemma we have frequently to consider all possible arrangements of 5- and non-5-vertices which form a consecutive string of p neighbors of V_k (p = 2,3,...,7; k = 7,...,10). We present therefore below and on p. 46 complete lists of all possible " 3-digit,..., 7-digit" arrangements " of this type. The arrangements are listed in lexicographic order and each one of them is given a number, e.g., 6/21 (the 21st 6-digit arrangement). We do not list however, those arrangements which would yield a larger reading (regarded as a decimal number with " " replaced by " 0 ") if read backwards; e.g., the list of 6-digit arrangements does not include . . 5 . . 5 which would be 6/23 read backwards, denoted by " 6/23r ". We shall use these numbers (with or without the symbols " n " and " r " for "non-reflected" and "reflected") later for reference.

5-digit arrangements

5/1	5 5 5 5 5
5/2	5 5 5 5 .
5/3	5 5 5 . 5
5/4	5 5 5 . .
5/5	5 5 . 5 5
5/6	5 5 . 5 .
5/7	5 5 . . 5
5/8	5 5 . . .
5/9	5 . 5 5 .
5/10	5 . 5 . 5
5/11	5 . 5 . .
5/12	5 . . 5 .
5/13	5 . [illegible] 5
5/14	5 . . . .
5/15	. 5 5 5 .
5/16	. 5 5 . .
5/17	. 5 . 5 .
5/18	. 5 . . .
5/19	. . 5 . .
5/20	. . . . .

4-digit arrangements

4/1	5 5 5 5
4/2	5 5 5 .
4/3	5 5 . 5
4/4	5 5 . .
4/5	5 . 5 .
4/6	5 . . 5
4/7	5 . . .
4/8	. 5 5 .
4/9	. 5 . .
4/10	. . . .

3-digit arrangements

3/1	5 5 5
3/2	5 5 .
3/3	5 . 5
3/4	5 . .
3/5	. 5 .
3/6	. . .

7-digit arrangements

c 7/1* 5 5 5 5 5 5 5
C 7/2* 5 5 5 5 5 5 .
7/3 5 5 5 5 5 . 5
c 7/4* 5 5 5 5 5 . .
7/5 5 5 5 5 . 5 5
c 7/6* 5 5 5 5 . 5 .
7/7 5 5 5 5 . . 5
c 7/8* 5 5 5 5 . . .
7/9 5 5 5 . 5 5 5
c 7/10* 5 5 5 . 5 5 .
7/11 5 5 5 . 5 . 5
C 7/12* 5 5 5 . 5 . .
7/13 5 5 5 . . 5 5
7/14 5 5 5 . . 5 .
7/15 5 5 5 . . . 5
c 7/16* 5 5 5 . . . .
7/17* 5 5 . 5 5 5 .
7/18 5 5 . 5 5 . 5
C 7/19* 5 5 . 5 5 . .
7/20 5 5 . 5 . 5 5
C 7/21* 5 5 . 5 . 5 .
7/22 5 5 . 5 . . 5
c 7/23* 5 5 . 5 . [illegible] .
7/24 5 5 . . 5 5 .
7/25 5 5 . . 5 . 5
c 7/26* 5 5 . . 5 . .
7/27 5 5 . . . 5 5
7/28 5 5 . . . 5 .
7/29 5 5 5
C 7/30* 5 5 . . . . .
7/31 5 . 5 5 5 5 .
7/32 5 . 5 5 5 . 5
7/33† 5 . 5 5 5 . .
7/34* 5 . 5 5 . 5 .
7/35 5 . 5 5 . . 5
7/36* 5 . 5 5 . . .

7/37 5 . 5 . 5 5 .
7/38 5 . 5 . 5 . 5
C 7/39* 5 . 5 . 5 . .
7/40* 5 . 5 . . 5 .
7/41 5 . 5 . . . 5
c 7/42* 5 . 5 [illegible] . .
7/43 5 . . [illegible] 5 .
7/44* 5 . . 5 5 . .
7/45 5 . . 5 . 5 .
7/46 5 . . 5 . . 5
c 7/47* 5 . . 5 . . .
7/48 5 . . . 5 5 .
7/49 5 . . . 5 . .
7/50 5 . . . . 5 .
7/51 5 5
C 7/52* 5 . . . . . .
7/53 . 5 5 5 5 5 .
7/54 . 5 5 5 5 . .
7/55 . 5 5 5 . 5 .
7/56 . 5 5 5 . . .
7/57† . 5 5 . 5 5 .
7/58† . 5 5 . 5 . .
7/59 [illegible] 5 . . 5 .
7/60† . 5 5 . . . .
7/61† . 5 . 5 5 . .
7/62* . 5 . 5 . 5 .
7/63* . 5 . 5 . . .
7/64* . 5 . . 5 . .
7/65† . 5 . . . 5 .
7/66* . 5 . . . . .
7/67 . . 5 5 5 . .
7/68† . . 5 5 . . .
7/69† . . 5 . 5 . .
7/70* . . 5 . . . .
7/71* . . . 5 . . .
C 7/72* . . . .

6-digit arrangements

6/1 5 5 5 5 5 5
6/2 5 5 5 5 5 .
6/3 5 5 5 5 . 5
6/4 5 5 5 5 . .
6/5 5 5 5 . 5 5
6/6 5 5 5 . 5 .
6/7 5 5 5 . . 5
6/8 5 5 5 . . .
6/9 5 5 . 5 5 .
6/10 5 5 . 5 . 5
6/11 5 5 . 5 . .
6/12 5 5 . . 5 5
6/13 5 5 . . 5 .
6/14 5 5 . . . 5
6/15 5 5 . . . .
6/16 5 . 5 5 5 .
6/17 5 . 5 5 . 5
6/18 5 . 5 5 . .
6/19 5 . 5 . 5 .
6/20 5 . 5 . . 5
6/21 5 . 5 . . .
6/22 5 . . 5 5 .
6/23 5 . . 5 . .
6/24 5 . . . 5 .
6/25 5 5
6/26 5 . . . . .
6/27 . 5 5 5 5 .
6/28 . 5 5 5 . .
6/29 . 5 5 . 5 .
6/30 . 5 5 . . .
6/31 . 5 . 5 . .
6/32 . 5 . . 5 .
6/33 . 5 . . . .
6/34 . . 5 5 . .
6/35 . . 5 . . .
6/36 . . . . . .

(7). We prove the $q_{TL}(V_k)$ - Lemma for the single cases $k = 7$, $k = 8$, $k = 9$, $k = 10$, and $k = 11$. The case $k \geq 12$ is proved in Section 3 of the paper as an immediate consequence of the Upper Bound Lemma.

Proof of the $q_{TL}(V_7)$ - Lemma. We denote by λ the number of L-discharging edges that go to V (the central vertex of degree 7 which is q_{TL} - positive by hypothesis of the lemma). We distinguish the cases $\lambda = 0$, $\lambda = 1$, and $\lambda \geq 2$.

Case $\lambda = 0$

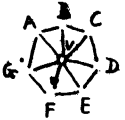
We denote the neighbor vertices of V in consecutive order, reading clockwise or counter-clockwise, by A, B, C, D, E, F, G, so that the corresponding reading of "5" or " " ("non-5") is maximal (among the 14 possible readings of this kind). Then the reading of "5" or "." corresponds to one of the 7-digit arrangements which are "cyclically maximal" and denoted by a "c" on p. 46. We consider these arrangements in the order of their numbers (see p. 46). For some of these arrangements we have to consider all possible "sub-arrangements" of 6- and major vertices at consecutive strings of non-5-vertices. For instance, for the main arrangement 7/30, 5 5 . . . . . , we have to consider 20 sub-arrangements 5 5 6 6 6 6 6,...., 5 5 U U U U U , (where "U" stands for "major vertex" and the enumeration of the sub-arrangements is in the order 5/1,...,5/20 of the list of 5-digit arrangements presented on p. 45 - with "5" replaced by "6" and "." replaced by "U").

In general, we have to consider only those sub-arrangements which contain either at least three 5's, or two 5's and at least one pair of adjacent 6's, or one 5 and at least two pairs of adjacent 6's, or no 5's but at least four pairs of adjacent 6's (since in all other cases V could not be q_{TL} - positive). This eliminates for instance 9 of the 20 sub-arrangements of 5 5 . . . . . from our list. For the sub-arrangements we have to consider all possible distributions of T-discharging values "T2", "T1", or "x" ("no T-discharging") to the 6-6-edges as far as they yield a positive q_{TL} - value for V. Here we still assume that the lemmas on T-dischargings, as proved in Section (1) of the supplement, hold for V,

Proof of $q_{TL}(V_7)$ - Lemma

Case λ = 0

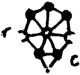
Arrangement check list



	A	B	C	D	E	F	G	
7/1,...,8	5	5	5	5				→2-1
7/10	5	5	5	.	5	5	.	
	5	5	5	6	5	5	.	→2-2
	5	5	5	U	5	5	U	→


5-1.

or


CTL#1

	A	B	C	D	E	F	G	
7/12	5	5	5	.	5	.	.	
	5	5	5	6	5	.	.	→2-2
	5	5	5	U	5	6	6	→3-26
	5	5	5	U	5	6	U	→
	5	5	5	U	5	U	6	→CTL#3
	5	5	5	U	5	U	U	→CTL#3


5-2

or


CTL#2

	A	B	C	D	E	F	G	
7/16	5	5	5	.	.	.	.	
	5	5	5	. ×	. ×	. ×	.	→
	5	5	5	6^T	6	.	.	→I1-1
	5	5	5	.	6^T	6	.	→I1-15

1-1


1-2

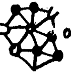

CTL#4


1-4


CTL#4


CTL#5

	A	B	C	D	E	F	G	
7/19	5	5	.	5	5	.	.	
	5	5	6	5	5	.	.	→2-2
	5	5	U	5	5	6	6	→3-26
	5	5	U	5	5	6	U	→CTL#6
	5	5	U	5	5	U	U	→


or

CTL#7 CTL#8

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list continued

	A	B	C	D	E	F	G
7/21	5	5	.	5	.	5	.
	5	5	6	5	.	5	. → 2-2
	5	5	U	5	.	5	U → CTL #9
7/23	5	5	.	5	.	.	.
	5	5	6	5	.	.	. → 2-2
	5	5	U	5	.	.	.
	5	5	U	5	6	6	6 → 3-27
	5	5	U	5	6	6	U
	5	5	U	5	6^{T2}	6	U → CTL #11
	5	5	U	5	6^{T1}	6	U → CTL #12
	5	5	U	5	$6^{\times}$	6	U → CTL #10
	5	5	U	5	6	U	→ CTL #10
	5	5	U	5	U	6	6
	5	5	U	5	U	6^{T}	6 → I1-1
	5	5	U	5	U	$6^{\times}$	6 → CTL #10
	5	5	U	5	U	6	U → CTL #10
	5	5	U	5	U	U	→ CTL #10

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = \sigma$

Arrangement check list, continued[2]

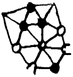
	A	B	C	D	E	F	G
7/26	5	5	.	.	5	.	.
	5	5	6	6	5	.	. → 3-26
	5	5	6	U	5	6	U → CTL #13
	5	5	6	U	5	U	6 → CTL #14
	5	5	6	U	5	U	U →
	5	5	U	6	5	6	U → CTL #19 .
	5	5	U	6	5	U	6 → CTL #13 (r)
	5	5	U	6	5	U	U →
	5	5	U	U	5	6	U → (reflected)
	5	5	U	U	5	U	6 → (reflected)
	5	5	U	U	5	U	U →

or

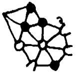
CTL #15 . CTL #16

or

CTL #20 . CTL #21



CTL #17



CTL #18

CTL #22

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list continued[3]

	A	B	C	D	E	F	G
7/30	5	5	.	.	.	.	.
5/1	5	5	6	6	6	6	6 → 4-28
5/2	5	5	6	6	6	6	U
	5	5	6	6	6	6	U → I1-1
	5	5	6	6^{T}	6	5	U → I1-5
	5	5	6	6	6^{T}	6	U → I1-10
5/3,4	5	[illegible]	[illegible]	6		U	.
	5	[illegible]	[illegible]	6	6	U	. → I1-1
	5	[illegible]	[illegible]	6^{T}	6	U	. → I1-5
5/5, ..., 8	5	5	[illegible]	6	U	.	.
	5	5	6^{T}	6	U	.	. → I1-1
5/9	5	5	6	U	6	6	U
	5	5	6	U	6^{T2}	6	U → CTL#23
	5	5	6	U	6^{T1}	6	U → CTL#24
5/15	5	5	U	6	6	6	U
	5	5	U	6^{T2}	6^{T2}	6	U → CTL#25
	5	5	U	6^{T2}	6^{T1}	6	U → CTL#26
	5	5	U	6^{T2}	$6^{\times}$	6	U → CTL#23
	5	5	U	6^{T1}	6^{T1}	6	U → CTL#27
	5	5	U	6^{T1}	$6^{\times}$	6	U → CTL#24
5/16	5	5	U	6	6	U	U
	5	5	U	6^{T2}	6	U	U → CTL#23
	5	5	U	6^{T1}	6	U	U → CTL#24

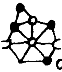
Proof of $q_{TL}(V_7)$ - Lemma

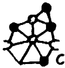
Case $\lambda = \dot{0}7$

Arrangement check list, continued[4]

7/39

A	B	C	D	E	F	G	
5	.	5	.	5	.	.	
5	6	5	6	5	.	.	→ 3-28
5	6	5	U	5	6	6	→ 4-14
5	6	5	U	5	6	U	→ CTL#28
5	6	5	U	5	U	6	→ CTL#29
5	6	5	U	5	U	U	→ CTL#29
5	U	5	U	5	6	6	
5	U	5	U	5	6 T	6	→ I1-2
5	U	5	U	5	6 ×	6	→ CTL #36
5	U	5	U	5	6	U	
5	U	5	U	5	U	6	(ref2.)
5	U	5	U	5	U	U	


CTL#30


CTL#31

CTL#32

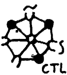

47-15


47-1

CTL#33


46-2

CTL#34



CTL#35

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list, continued[5]

	A	B	C	D	E	F	G	
7/42	5	.	5	.	.	.	.	
	5	6	5	6	6	6	6→4-15	
	5	6	5	6	6	6	U	
	5	6	5	6^{T}	6	6	U→I1-4	
	5	6	5	6	6^{T}	6	U→I1-11	
	5	6	5	6^{T}	6	U	.→I1-4	
	5	6	5	U	6	6	U	
	5	6	5	U	6^{T2}	6	U→CTL#37	
	5	6	5	U	6^{T1}	6	U →	$\frac{T1\#1 \quad T1\#2 \quad T1\#4}{CTL\#38 \quad CTL\#39 \quad 16\text{-}18}$
	5	U	5	6	6	6	6→4-15	
	5	U	5	6	6	6	U	
	5	U	5	6^{T}	6^{T}	6	U→I1-26	
	5	U	5	6^{T2}	$6^{\times}$	6	U→CTL#44	
	5	U	5	6^{T1}	$6^{\times}$	6	U→CTL#45	
	5	U	5	$6^{\times}$	6^{T2}	6	U→I1-3	
	5	U	5	$6^{\times}$	6^{T1}	6	U→CTL#46	
	5	U	5	6	6	U	.	
	5	U	5	6^{T2}	6	U	.→CTL#47	
	5	U	5	6^{T1}	6	U	.→CTL#48	
	5	U	5	U	6	6	U	
	5	U	5	U	6^{T2}	6	U→CTL#40	
	5	U	5	U	6^{T1}	6	U →	$\frac{T1\#1 \quad T1\#2 \quad T1\#4}{CTL\#41 \quad CTL\#43 \quad 16\text{-}18}$.

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list, continued[6]

	A	B	C	D	E	F	G
7/47	5	.	.	5	.	.	.
	5	6	6	5	6	6	6→3-27
	5	6	6	5	6	6	U
	5	6^{T}	6	5	6	6	U→I1-2
	5	6	6	5	6^{T}	6	U→I1-12
	5	6^{T}	6	5	6	U	.→I1-2
	5	6^{T}	6	5	U	.	U→I1-2
	5	6	U	5	6	6	6→3-27
	5	6	U	5	6	6	U
	5	6	U	5	6^{T2}	6	U→CTL#50
	5	6	U	5	6^{T1}	6	U→CTL#49
	5	6	U	5	U	6	6
	5	6	U	5	U	6^{T2}	6→CTL#51
	5	6	U	5	U	6^{T1}	6→CTL#52
	5	U	U	5	6	6	6→3-27
	5	U	U	5	6	6	U
	5	U	U	5	6^{T2}	6	U→CTL#51
	5	U	U	5	6^{T1}	6	U→CTL#52

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list, continued[7]

	A	B	C	D	E	F	G
7/52	5	.	.	.	.	.	.
6/1	5	6	6	6	6	6	6 → 4-29
6/2	5	6	6	6	6	6	U
	5	6^{T2}	6^{T}	6	6	6	U → I1-26
	5	6^{T2}	$6^{\times}$	6^{T}	6	6	U → I1-29
	5	6^{T2}	$6^{\times}$	$6^{\times}$	6^{T2}	6	U → I1-13
	5	6^{T1}	6^{T}	6	6	6	U → I1-26
	5	6^{T1}	$6^{\times}$	6^{T2}	6	6	U → I1-6
	5	6^{T1}	$6^{\times}$	6^{T1}	6^{T2}	6	U → I1-13
	5	$6^{\times}$	6^{T2}	6	6	6	U → I1-3
	5	$6^{\times}$	6	6^{T2}	6	6	U → I1-6
	5	$6^{\times}$	6	6	6^{T2}	6	U → I1-13
6/3,4	5	6	6	6	6	U	
	5	6^{T2}	6^{T}	6	6	U	. → I1-26
	5	6^{T2}	$6^{\times}$	6^{T2}	6	U	. → I1-6
	5	6	6^{T2}	6	6	U	. → I1-3
	5	6	6	6^{T2}	6	U	. → I1-6
6/5	5	6	6	6	U	6	6
	5	6^{T2}	6^{T}	6	U	6	6 → I1-26
	5	6^{T2}	$6^{\times}$	6	U	6^{T2}	6 → I1-27
	5	6^{T1}	6^{T}	6	U	6	6 → I1-26
	5	$6^{\times}$	6^{T2}	6	U	6^{T2}	6 → I1-3
6/6	5	6^{T2}	6^{T2}	6	U	6	U → I1-26
6/7,8	5	6^{T2}	6^{T2}	6	U	U	. → I1-26

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list, continued[8]

	A	B	C	D	E	F	G
7/52, finished							
6/9	5	6^{T2}	6	U	6^{T2}	6	U→CTL#53
6/12	5	6^{T2}	6	U	U	6^{T2}	6→I1-27
6/16	5	6	U	6^{T2}	6^{T2}	6	U→CTL#55
6/27	5	U	6	6	6	6	U
	5	U	6^{T}	6^{T}	6^{T}	6	U→I1-31
	5	U	6^{T2}	6^{T2}	$6^{\times}$	6	U→CTL#54
	5	U	6^{T2}	$6^{\times}$	6^{T2}	6	U→I1-28
	5	U	$6^{\times}$	6^{T2}	6^{T2}	6	U→CTL#54
6/28	5	U	6^{T2}	6^{T2}	6	U	U→CTL#55

7/72							
7/1	6	6	6	6	6	6	6
	6^{T2}	6^{T2}	6^{T}	6	6	6	6→I1-31
	6^{T2}	6^{T2}	$6^{\times}$	6^{T}	6	6	6→I1-32
	6^{T2}	6^{T2}	$6^{\times}$	$6^{\times}$	6	6	6 impossible to obtain $\tau > 60$ with a cyclically maximal reading of the T-discharging values
	6^{T2}	6^{T1}	6^{T}	6	6	6	6→I1-31
	6^{T2}	6^{T1}	$6^{\times}$	6	6	6	6 imposs.
	6^{T2}	$6^{\times}$	6^{T2}	6	6	6	6 imposs.
	6^{T2}	$6^{\times}$	6^{T1}	6	6	6	6 imposs.
	6^{T1}	6^{T1}	6^{T1}	6	6	6	6→I1-31
	6^{T1}	6^{T1}	$6^{\times}$	6	6	6	6 imposs.

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 0$

Arrangement check list, finished

7/72, finished	A	B	C	D	E	F	G
7/2	6	6	6	6	6	6	U
	6^{T2}	6^{T2}	6^{T}	6	6	6	U→I1-31
	6^{T2}	6^{T2}	$6^{\times}$	6^{T}	6	6	U→I1-32
	6^{T2}	6^{T1}	6^{T}	6	6	6	U→I1-31
	6^{T2}	6^{T1}	$6^{\times}$	6^{T2}	6^{T2}	6	U reflected reading is larger than non-reflected
	6^{T2}	$6^{\times}$	6^{T2}	6	6	6	U→I1-28
	6^{T1}	6^{T2}	6^{T}	6	6	6	U→I1-31
	6^{T1}	6^{T2}	$6^{\times}$	6^{T2}	6^{T2}	6	U reflected reading is larger than non-reflected
7/4	6	6	6	6	6	U	U
	6^{T2}	6^{T2}	6^{T1}	6^{T2}	6	U	U→I1-31

Case $\lambda = 1$

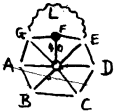
We partition this case into sub-cases according to the "width" w of the L-situation with pivot V and according to the "nominal contribution value. (counting 30 for each V_5 of the L-situation which is adjacent to V and which is not distinguished by a large discharging value in Table 2). For some sub-cases we have to consider all possible arrangements of 5- and non-5-vertices which are adjacent to V but do not belong to the L-situation. For some of the arrangements we have again to consider the sub-arrangements of 6- and major vertices at the places of the non-5's. However, we may ignore all those arrangements which cannot yield positive q_{TL} - values for V. For instance, if the v-value of the L-situation is 40 then among the additional neighbors of V we must have either at least one V_5 or several V_6 so as to provide for at least two 6 - 6 edges (in order to have $\tau > 20$).

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 1$

Sub-case $v = 3$, $v = 70$

L = 401, 402, or 403



In this sub-case an L-situation L which is one of ## 401, 402, 403 (reflected or non-reflected) is attached so as to induce an L-discharging of 40 to V. Those neighbor vertices of V that do not belong to L are denoted A, B, C, D in counter-clockwise order;

the neighbors E, F, and G of V belong to L as indicated in the drawing to the right.

We present a 4-digit arrangement check list for A, B, C, D. In this list we shall have to indicate occasionally whether or not a T-discharging goes to V across one of the edges G - A or D - E. Thus we shall mention the vertices E and G also. In or after the arrangement check list we define (using the symbol "≐") some configuration classes, (5a),..., (5d), which are treated further in the corresponding class check lists (in order to show that each configuration in these classes either contains a configuration of $\mathcal{U}$ or belongs to one of the CTL-classes of Table 4). If a configuration class is quoted after some arrangement without the symbol ≐ in between, such as 6 5 5 → (5b), then this means that every configuration of the arrangement contains some configuration of the class (but not necessarily vice versa).

Arrangement check list

	G	A	B	C	D	E	
4/1,...,7		5					$\doteq (5a)$
4/8		.	5	5	.		
		6	5	5	.		→ (5b)
		U	5	5	U		→ if L = 401 or 402: CTL #57 if L = 403: CTL #56

$\doteq \boxed{(5b)}$

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 1$

Sub-case $w = 3$, $v = 7\dot{0}$, continued

" Ln " means: L is not reflected

" Lr " means: L is reflected

G	A	B	C	D	E	
	.	5	.	.		
	6	5	.	.		→ (5b)
	U	5	6	6		≐ (5c)
	U	5	6	U		→ if Ln: CTL #59 if Lr: CTL #58
	U	5	U	6		
	U	5	U	6	T	→ I1-1
	U	5	U	6	×	→ if Ln: CTL #60 if L = 401r or 402r: CTL #63 if L = 405r: 21-8
	U	5	U	U		→ if Ln: CTL #60 if Lr: CTL #61

Proof of $q_{TL}(V_7)$ - Lemma

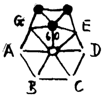
Case λ = 1

Sub-case w = 3, v = 70, finished

```
          G   A   B   C   D   E

4/10      ┌── .   .   .   .
          │ ⎧ T 6   .   .   .     → I1-1
          │ ⎪
          │ ⎪   6 T 6   .   .        → ⎧ if Ln: I1-1
          │ ⎪                          ⎩ if Lr: I1-5
          │ ⎪   6   6 T 6   .        → ⎧ if Ln: I1-5
          │ ⎪                          ⎩ if Lr: I1-10
          └→⎨ × 6 × . × . × 6 ×      → ⎧ if L = 401 or 402: CTL#64
            ⎪                          ⎩ if L = 403: 21-8
            ⎪ × 6 × . × .   U      ─┐  ⎧ if I = 401n or 402n: CTL#64
            ⎪                       └→ ⎨ if L = 403n: 21-8
            ⎪                          ⎩ if Lr: CTL#62
            ⎪   U   6 T 6   U      ─┐
            ⎪                       │  ⎧ if L = 401: I1-22
            ⎩   U   . × .   U → CTL#62 ⎨ if L = 402: I3-33
                                    └→ ⎪ if I = 403: → ⎧ if T1: CTL#70
                                       ⎩               ⎩ if T2: CTL#69
```

Sub-case w = 3, v = 60

(L = 41i)



Arrangement check list

```
              G   A   B   C   D   E

4/1,...,7         5                   → 2-11

4/8           ┌── .   5   5   .
              └→⎧ 6   5   5   .       → 2-9
                ⎩ U   5   5   U       → CTL#65
```

Proof of $q_{TL}(V_7)$ - Lemma

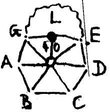
Case $\lambda = 1$

Sub-case w = 3, v = 60, finished

	G	A	B	C	D	E	
4/9		.	5	.	.		
		6	5	.	.		→ 2-9
		U	5	6	6		→ 3-27
		U	5	6	U		→ CTL #66
		U	5	U	6		
		U	5	U	6^{T}		→ I1-7
		U	5	U	6$^{\times}$		→ CTL #66
		U	5	U	U		→ CTL #66
4/10		.	.	.	.		
		^{T}6	.	.	.		→ I1-7
		$^{\times}$6	^{T}6	.	.		→ I1-8
		$^{\times}$6	$^{\times}$6	^{T}6	$^{\times}$.		→ I1-21
		.	6	^{T}6	U		→ I1-21

Sub-case w = 3, v = 40

L = 421, ..., 425,
427, or 428



Arrangement check list

	G	A	B	C	D	E	
4/1, ..., 7		5					≐ (5e)
4/8		.	5	5	.		
		6	5	5	.		→ (5f)
		U	5	5	U		→

if L = 421, ..., 425, or 427
↳ CTL #71

if L = 428 →


55-1


CTS #72

Proof of $q_{TL}(V_7)$ - Lemma

Case $\lambda = 1$

Sub-case $w = 3$, $v = 40$, finished

$\doteq \boxed{(5f)}$

	G	A	B	C	D	E	
4/9		.	5	.	.		
		6	5	.	.		→ (5f)
		U	5	6	6		≐ (5g)
		U	5	6	U		→ if Ln: CTL#75 if Lr: CTL#74
		U	5	U	6		
		U	5	U	6	T	→ (5h)
		U	5	U	6	×	→ if Ln: CTL#75 if Lr: CTL#74
		U	5	U	U		→ if Ln: CTL#75 if Lr: CTL#74

$\doteq \boxed{(5h)}$

4/10		.	.	.	.		
	T	6	.	.	.		→ (5h)
	×	6^{T2}	6	.	.	×	≐ (5i)
	×	6^{T1}	6^{T2}	6	.	×	≐ (5j)
	×	6^{T1}	6^{T1}	6^{T1}	6	×	→ I|-3|

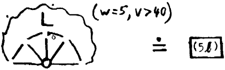
Sub-case $w = 4$

$\doteq \boxed{(5k)}$

Proof of $q_{TL}(V_7)$ - Lemma

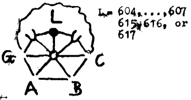
Case λ = 1

Sub-case w = 5, v > 40



The configuration class (5ℓ) contains the critical sub-class (C5ℓ) for which we have to consider the possible arrangements.

(C5ℓ): L is one of ## 604,..., 607, 615,616, 617

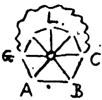


Arrangement check list for (C5ℓ)

G	A	B	C	
	5			≐(5ℓa)
1	6			→ I3-11
	6	6		→ 16-15
×	6	u		≐(5ℓb)
	u	u		→ CTL #76

Proof of $q_{TL}(V_7)$ Lemma

Case λ = 1



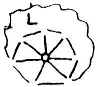
(w = 5, v = 40 or 35)

Sub-case w = 5, v = 40 or 35

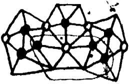
Arrangement check list

G	A	B	C	
	5			≐ (5m)
	6	6		
^{T2}	6^{T}	6		≐ (5n)
^{T2}	$6^{\times}$	6^{T}		≐ (5o)
^{T1}	6^{T2}	6		≐ (5p)
^{T1}	6^{T1}	6^{T1}		→ I1-31

Sub-case w = 6 or 7



≐ $\boxed{(5q)}$



2-9

Proof of $q_{TL}(V_7)$ - Lemma.

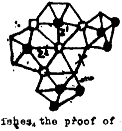
Case $\lambda \geq 2$

In this case we may assume that there are two L-situations, say $L^{(1)}$ and $L^{(2)}$, with their pivots identified to the 7-vertex V. Denote the distinguished 5-vertices of $L^{(1)}$ and $L^{(2)}$ by $V_5^{(1)}$ and $V_5^{(2)}$. Now we claim that Δ contains a configuration of $\mathcal{U}$ or contains two configurations $K^{(1)}$ and $K^{(2)}$ so that each of them belongs to one of CTL#56,..., CTL#77, their pivots are identified to V, and so that the distinguished 5-vertex of $K^{(i)}$ $(i = 1,2)$ is identified to $V_5^{(i)}$ The claim follows from our treatment of the Case $\lambda = 1$, since there we have shown the following. If Δ contains (1) an L-situation, say $L^{(1)}$, with pivot identified to V and (2) a 5-vertex $V_5^{(2)}$ which is different from the distinguished 5-vertex of $L^{(1)}$ but is adjacent to V then Δ contains a configuration of $\mathcal{U}$ or contains a CTL-situation as asserted. (Note that the presence of $L^{(1)}$ and $V_5^{(2)}$ alone guarantee that $q_{TL}(V) > 0$.)

Now we shall prove that the merging of any two CTL-situations $K^{(1)}$ and $K^{(2)}$ as described in the paragraph above yields some configuration of $\mathcal{U}$. This will finish the proof of the $q_{TL}(V_7)$ - Lemma.

The merging of $K^{(1)}$ and $K^{(2)}$ is only possible if either both $K^{(1)}$ and $K^{(2)}$ belong to the class CTL#74 or both of them belong to CTL#76 (since otherwise it would not even be possible to merge those parts of $K^{(1)}$ and $K^{(2)}$ which are actually drawn in Table 2). Thus we are left with the following two classes of configurations.

one pair of $\{421, \ldots, 425,$
$427,\ 428\}^2$
at (U^1, U^2) $\doteq \boxed{(5r)}$

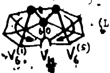


one pair of $\{604, \ldots, 607,$
$615, 616, 617\}^2$
at (Z^1, Z^2) $\doteq \boxed{(5s)}$

(See class check lists.) This finishes the proof of the $q_{TL}(V_7)$ - Lemma. ∎

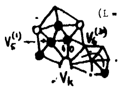
(8) In this section we prove some lemmas on L-dischargings and especially on combinations of L-dischargings and on combinations of L- and T-discharging which go to the same vertex V. These lemmas will be needed in the following Section (9) in which we shall prove the $q_{TL}(V_k)$ - Lemma for the degrees k = 8, k = 9, k = 10 and k = 11. Some of these lemmas are formulated in Section 3 of the paper where they are used for proving the $q_{TL}(V_k)$ - Lemma for $k \geq 12$. These are Lemmas (60 or 50, T), (60 or 50, T2, T2), (5, L, 50), (5, L), (5, L, T2), (5, L, . , 60 or 50), (5, L, . , L, 5), (60 or 50, . , 60 or 50), (60 or 50, [illegible] 60 or 50). In addition we shall prove the following six lemmas.

Lemma (T2, L, T2). If $V_6^{(1)}$, $V_6^{(2)}$, $V_5^{(3)}$, $V_6^{(4)}$, $V_6^{(5)}$ are five consecutive neighbors of a major vertex V_k of Δ^* so that two T2-dischargings go across the 6-6 edges $V_6^{(1)} - V_6^{(2)}$ and $V_6^{(4)} - V_6^{(5)}$ and an L-discharging goes from $V_5^{(3)}$ to V_k then the configuration drawn to the right occurs in Δ^* (with V_k as indicated in the drawing):



(L = 411)

Lemma (5, L, T1). If $V_5^{(1)}$, $V_5^{(2)}$, $V_6^{(3)}$, $V_6^{(4)}$ are four consecutive neighbors of a major vertex V_k of Δ^* so that an L-discharging goes from $V_5^{(2)}$ to V_k and a T1-discharging goes across the edge $V_6^{(3)} - V_6^{(4)}$ then the L-discharging is induced by one of ## 401, 402, 403, 431, 432, 465 and either the T1-discharging is induced by T1#1 or the configuration drawn to the right occurs in Δ^* (with V_k as indicated in the drawing)



(L = 432)

Lemma (L^2). Suppose that Δ^* contains two L-situations $L^{(1)}$ and $L^{(2)}$ with the same pivot V_k but different discharging edges. Then we have the following.

(1) If no more than five neighbors of V_k in Δ^* are occupied by (the images of), the fully degree-specified vertices of $L^{(1)}$ and $L^{(2)}$ then either one of the configurations (denoted L^2#801,...,894) drawn in Table $\mathcal{L}^2$, pages 1 and 2, occurs in Δ^* with pivot identified to V_k and with L-discharging edges (indicated in

Table $\mathcal{L}^2$ by the discharging values written to these edges) identified to the L-discharging edges of $L^{(1)}$ and $L^{(2)}$, or one of the L-situations, say $L^{(1)}$, is one of ## 530, 549, 550 (see Table 2) and contains $L^{(2)}$ (where $L^{(2)}$ is # 441).

(ii) If precisely six consecutive neighbors of V_k in Δ^* are occupied by (the images of) the fully degree-specified vertices of $L^{(1)}$ and $L^{(2)}$ then the following hold.

(ii.a) If each of $L^{(1)}$, $L^{(2)}$ has width $w = 4$ then one of the configurations (denoted L^2#901,...,977) drawn in Table $\mathcal{L}^2$, pages 3 and 4, occurs in Δ^* with pivot identified to V_k and with L-discharging edges identified to the L-discharging edges of $L^{(1)}$ and $L^{(2)}$.

(ii.b) If one of the L-situations, say $L^{(1)}$, is of width $w = 6$ then it is one of ## 701, 711,...,728 (see Table 2) and $L^{(2)}$ is #441 and is contained in $L^{(1)}$

Lemma (5,L,.,60 or 50)$^+$ If $V_5^{(1)}$, $V_5^{(2)}$, $V^{(3)}$, $V_5^{(4)}$ are four consecutive neighbors of a major vertex V_k in Δ^* so that the hypothesis of Lemma(5,L,.,60 or 50) is fulfilled (see Section 5 of the paper) then one of ## 530, 701, 716, 811, [illegible] (see Table 2 or $\mathcal{L}^2$, respectively) occurs in Δ^* (with pivot identified to V_k and L-discharging V_5's identified to $V_5^{(2)}$ and $V_5^{(4)}$).

Lemma (L, L). If $V_5^{(1)}$, $V_5^{(2)}$ are two consecutive neighbors of a major vertex V_k in Δ^* so that from each of them an L-discharging goes to V_k then one of ## 801, 802, 803, 828,...,833, 861,...,872, 905,...,910, 926,...,977 (see Table $\mathcal{L}^2$) occurs in Δ^* (with pivot identified to V_k). In particular, in Δ^* there is a third 5-vertex which is adjacent to both $V_5^{(1)}$ and $V_5^{(2)}$.

Lemma (L, 5, L). If $V_5^{(1)}$, $V_5^{(2)}$, $V_5^{(3)}$ are three consecutive degree-5-neighbors of a major vertex V_k of Δ^* so that L-dischargings go from $V_5^{(1)}$ and from $V_5^{(3)}$ to V_k then one of ## 820, 827, 902, 903, 904, 921,...,925 (see Table $\mathcal{L}^2$) occurs in Δ^* (with pivot identified to V_k) or one of the L-dischargings is induced by #531 or by #701.

Table $\mathscr{L}^2$


801


802


803

w = 4

v = 80


811


812


813


820

w = 5

v = 120 / 110


821


822


823


824


825

v = 110 or 105

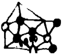

826


827


828


829


830


831


832


833


834


835


841


842


843

v = 100


851


852


853


854

v = 90 or 8

Table $\mathcal{L}^2$

v = 5

v = 80 or 75


861


862


863


864


865


866


867


868


869


870


871


872


873


874


875


876


877


878


879


880


881


882


883


884


885


886


887


888


889


891


892


893


894

Table $\mathcal{L}^2$

v = 6

v = 140


901


902


903


904


905


906


907


908


909


910

v = 110


921


922


923


924


925


926


927


928


929


931


932


933


934


935

Table $\mathcal{L}^2$

$v = 6$

$v = 80$


951


952


953


954


955


956


957


958


959


961


962


963


966


967

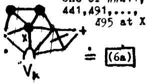

971


976


977

Proof of Lemmas (5, L) and (5, L, 5). In every L-situation (see Table 2) the distinguished 5-vertex has two neighbors which are also neighbors of the pivot; at least one of these two vertices is of degree greater than five. In particular, if the L-situation is L5 or L6 then both of these vertices are of degree greater than five. This implies the lemmas. ■

Proof of Lemma (60 or 50, T). Assume that in an arbitrary triangulation Δ, there is a major vertex V_k which receives an L6- or L5-discharging and a T-discharging as described in the lemma. Then the L-discharging is induced by one of ## 411, 441, 491,...,495 and a configuration of the configuration class (6a) (see drawing to the right) occurs in Δ (with V_k as indicated in the drawing). But every configuration of (6a) contains either a configuration of $\mathcal{U}$ or is a configuration of one of the classes defined in Figure 12(a) and (b) (see the class check list for (6a)). This implies the lemma. ■

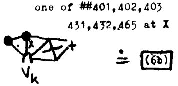


Proof of Lemma (60 or 50, T2, T2). Assume in an arbitrary triangulation Δ, there is a vertex V_k which receives an L6- or L5-discharging and a T2-discharging as described in the lemma. Then by Lemma (60 or 50, T), a configuration according to Figure 12(a) occurs in Δ (with V_k as indicated in the figure). Now assume further that V_k receives a second T2-discharging across a 6-6 edge consecutive to the 6-6 edge of the first T2-discharging. Then I2-5 occurs in Δ and thus $\Delta \neq \Delta^*$. This proves the lemma. ■

Proof of Lemma (60 or 50, T, 60 or 50) By Lemma (60 or 50, T), Δ^* contains a configuration, say $K^{(1)}$, according to Figure 12(a) or (b) so that $K^{(1)}$ contains the L-situation which induces the L-discharging from $V_5^{(1)}$ to V_k. Again by Lemma (60 or 50, T), Δ^* contains a configuration $K^{(2)}$ according to Figure 12(a) or(b) which induces the L-discharging from $V_5^{(4)}$ to V_k. Both $K^{(1)}$ and $K^{(2)}$ induce the T-discharging across $V_6^{(2)}$- $V_6^{(3)}$. Thus the T-discharging situation must be symmetric (by Lemma(T)) and thus must be T1#1 (see Figure 2). This proves the lemma. ▮

Proof of Lemma (T2, L, T2). Assume that in an arbitrary triangulation Δ, there is a vertex V_k which receives two T2-dischargings and an L-discharging as described in the lemma. Then the L-discharging is induced by one of ## 411, 421, 422 (no other L-situation can be attached in this case). But if the L-situation is #421 then I2-1 occurs; if the L-situation is #422 then I4-12 occurs; thus in these cases we have $\Delta \neq \Delta^*$. If the L-situation is #411 then by Lemma (60 or 50, T), either again $\Delta \neq \Delta^*$ or two configurations according to Figure 12(a) occur, each of which induces one of the T2-dischargings and the L-discharging so that the configuration of Lemma (T2, L, T2) occurs. This proves the lemma. ▮

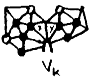
Proof of Lemmas (5, L, T2) and (5, L, T1). Assume that in an arbitrary triangulation Δ, there is a major vertex V_k which receives an L- and a T-discharging as described in the lemmas. Then the L-discharging must be induced by one of ## 401, 402, 403, 431, 432, 465 (the only L-situations in which the distinguished V_5 has a V_5- and a V_6-neighbor which are also adjacent to the pivot). Thus a configuration of the class (6b) (see drawing to the right) occurs (with V_k as indicated in the drawing). But if the T-discharging is not induced by T1#1 then a configuration of (6b) contains either a configuration of $\mathcal{U}$, or is the configuration of Lemma (5, L, T1), or is the configuration C6b drawn to the right, (see the class check list for (6b)). If in configuration C6b the degree of V_k is specified to be 7, 8, or 9 then 2-9, 7-7, or 12-10, respectively, occur; if the degree is specified ≥ 10 then it is the configuration of Figure 13. This proves the lemmas. ▮



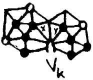
Proof of Lemma (5, L, . , L, 5). Assume that in an arbitrary triangulation Δ, there is a vertex V_k which receives two L-dischargings as stated in the Lemma. Then each one of the L-dischargings must be induced by one of ## 401,402,403,431, ...,437,465 and a configuration of the class (6c) (see drawing to the right) occurs in Δ (with V_k as indicated in the drawing). But a configuration of (6c) either contains a configuration of $\mathcal{U}$ or is one of the configurations C6c#1,..., C6c#4 drawn below, (see the class check list for (6c)).

one pair out of $\{401,402,403,431,432,465\}^2$ or $\{433,\ldots,437\}^2$ at (X,Y)

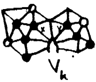
$\doteq$ (6c)



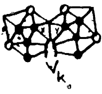
C6c#.1



C6c# 2



C6c# 3



C6c# 4

If in one of the configurations C6c#1,...,4 the degree of V_k is specified to be [illegible], [illegible], or [illegible] then a configuration of $\mathcal{U}$ occurs; moreover, if in one of C6c#2, 3,4 the degree of V_k is specified to be 10 then a configuration of $\mathcal{U}$ occurs, (see the class check list for C6c). If the degree of V_k is ≥9 in C6c#1 then we have the configuration of Figure 12(f); if the degree of V_k is >10 in C6c# 2,3, or 4, then we have a configuration according to Figure 12(g) or (h). This finishes the proof of the lemma. ∎

Proof of Lemma (60 or 50, . , 60 or 50). Assume that in an arbitrary triangulation Δ, there is a vertex V_k which receives two L-dischargings as described in the lemma. Then each one of the L-dischargings must be induced by one of ## 411, 441, 491,...,495, 551,...,553,720 and a configuration of the class (6d) (see right)

at (X,Y): one pair out of $\{411,441r,491r,\ldots,495r\} \times \{411,441n,491n,\ldots,495n\}$ or $\{441n\} \times \{441\ell,551n,\ldots,553n,720n\}$

$\doteq$ (6d)

occurs in Δ (with V_k as indicated in the drawing). But a configuration of (6d) either contains a configuration of $\mathcal{U}$ or is one of the configurations of Figure 12(c), (*), (see the class check list for (6d)). This proves the lemma. ■

Proof of Lemma ($\dot{L}^2$). In order to determine all possible mergings of two L-situations as considered in the lemma, we consider for each L-situation in Table 2 the "degree-reading" of the neighbors of the pivot. For instance, ## 401, 402, 403 have the reading 5 $\bar{5}$ 6, in their reflections have the reading 6 5 5. The following readings occur in List 2.

w = 3

5 $\bar{5}$ 6
6 $\bar{5}$ 5
6 $\bar{5}$ 6
6 $\bar{5}$ 7
6 $\bar{5}$ 7
6 $\bar{5}$ 7 U
6 $\bar{5}$ 8
7 $\bar{5}$ 6
7 $\bar{5}$ 6
U/ 5 6
8 $\bar{5}$ 6

w = 4

5 5 $\bar{5}$ 6
5 $\bar{5}$ 7 5
5 $\bar{5}$ 7 6
5 $\bar{5}$ 7 7
5 7 $\bar{5}$ 5
5 7 $\bar{5}$ 6
5 8 $\bar{5}$ 6
5 9 $\bar{5}$ 6
6 $\bar{5}$ 5 5
6 5 5 6
6 $\bar{5}$ 7 5
6 $\bar{5}$ 7 6
6 $\bar{5}$ 7 7
6 $\bar{5}$ 8 5
6 $\bar{5}$ 8 6
6 $\bar{5}$ 9 5
6 7 $\bar{5}$ 5
6 7 $\bar{5}$ 6
6 8 $\bar{5}$ 6
7 7 $\bar{5}$ 5
7 7 $\bar{5}$ 6

w = 5

5 5 $\bar{5}$ 7 6
5 $\bar{5}$ 7 5 6
5 7 $\bar{5}$ 7 5
5 7 $\bar{5}$ 7 6
5 7 $\bar{5}$ 7 7
5 7 8 $\bar{5}$ 6
5 8 $\bar{5}$ 7 6
6 5 $\bar{5}$ 7 6
6 5 7 $\bar{5}$ 5
6 5 7 5 6
6 $\bar{5}$ 8 6 6
6 $\bar{5}$ 8 7 5
6 6 8 $\bar{5}$ 6
6 7 $\bar{5}$ 5 5
6 7 $\bar{5}$ 5 6
6 7 $\bar{5}$ 7 5
6 7 $\bar{5}$ 7 6
6 7 $\bar{5}$ 7 7
6 7 $\bar{5}$ 8 5
6 7 $\bar{5}$ 8 6
6 8 $\bar{5}$ 7 6
7 7 $\bar{5}$ 7 5
7 7 $\bar{5}$ 7 6
7 7 $\bar{5}$ 7 7

w = 6

5 5 $\bar{5}$ 7 5 6
5 7 $\bar{5}$ 7 5 6
5 8 7 $\bar{5}$ 7 6
6 5 $\bar{5}$ 7 5 6
6 5 7 $\bar{5}$ 7 5
6 5 7 $\bar{5}$ 5 6
6 5 7 $\bar{5}$ 7 5
6 5 7 $\bar{5}$ 7 6
6 5 7 $\bar{5}$ 7 7
6 5 7 $\bar{5}$ 8 6
6 5 8 $\bar{5}$ 7 6
6 7 $\bar{5}$ 7 5 6
6 7 $\bar{5}$ 7 8 5
6 7 $\bar{5}$ 8 5 6
6 8 $\bar{5}$ 7 5 6
7 7 $\bar{5}$ 7 5 6

w = 7

6 5 7 $\bar{5}$ 7 5 6

In most of the readings, the distinguished V_5 can occur only in one place. In this event it is marked by an upper bar.

Proof of Lemma (L), continued

(i) In view of these readings we have to consider (in an arbitrary triangulation Δ) the following possibilities for mergings of $L^{(1)}$ and $L^{(2)}$ with resulting configuration of width ≤ 5 (and with non-identified distinguished V_5's).

(6e) ≐ $\overbrace{-\,\bar{5}\,6}^{L^{(1)}}\,\underline{5}\,-$ $L^{(2)}$ → pairs of $\{401n,\ldots,422n,421r,\ldots,428r\} \times \{401r,\ldots,422r,421n,\ldots,428n\}$

(6f) ≐ $6\,5\,\bar{5}\,6$ → $\{401n,\ldots,403n\} \times \{465n\}$ → no merging possible

(6f*) ≐ $6\,\bar{5}\,5\,6$ → $\{401r,\ldots,403r\} \times \{401n,\ldots,403n\}$

(6g) ≐ $6\,\bar{5}\,5\,\underline{5}\,6$ → 1-1

(6h) ≐ $6\,\bar{5}\,\underline{5}\,6$ reflection of (6f)

(6i) ≐ $6\,\bar{5}\,5\,-\,-$ → $\{401r,\ldots,403r\} \times \{431n,\ldots,437n,451n,\ldots,464n\}$

(6j) ≐ $6\,\bar{5}\,\underline{5}\,7\,6$ → $\{401r,\ldots,403r\} \times \{620\}$ → no merging possible

(6k) ≐ $6\,\bar{5}\,7\,\underline{5}\,6$ → (423n,423r) → 2-1

(6ℓ) ≐ $6\,\bar{5}\,7\,5\,-$ → $\{423n\} \times \{433r,\ldots,441r,466r,\ldots,478r\}$

(6m) ≐ $6\,\bar{5}\,7\,\underline{5}\,-$ → $\{42\cancel{3}n\} \times \{530r,549r,550r\}$ → no merging possible

$7\,\underline{5}$ → $\{4[illegible],\ldots,[illegible],[illegible],\ldots,[illegible]n\} \times \{433r,\ldots,441r,466r,\ldots,478r\}$

(6o) ≐ $-\,\underline{5}\,[illegible]\,\bar{5}\,6$ → $\{441r,466r,\ldots,478r\} \times \{530n,549n,550n\}$ → the only possible merging is the inclusion of 441r in 530n,549n,550n

(6p) ≐ $6\,\bar{5}\,5\,\underline{5}\,6$ → $\{431r,432r\} \times \{431n,432n\}$ → the only possible merging is 820

(6q) ≐ $6\,\bar{5}\,\underline{5}\,6$ → (465r,465n) → no merging possible

(6r) ≐ $-\,\bar{5}\,7\,\underline{5}\,-$ same as (6n)

(6s) ≐ $6\,\bar{5}\,7\,\underline{5}\,-$ reflection of (6o)

(6t) ≐ $6\,\bar{5}\,8\cdot\underline{5}\,6$ → $\{479n,480n\} \times \{479r,480r\}$ → 7-1 or 11-10 or 894

(6u) ≐ $6\,\bar{5}\,\dot{9}\,\underline{5}\,6$ → (481n,481r) → 12-1

(6v) ≐ $6\,\bar{5}\,7\,\underline{5}\,6$ → $\{549n,550n\} \times \{549r,550r\}$ → no merging possible

Proof of Lemma (L^2), finished

Each configuration of (6e),..., (6v) either contains a configuration of $\mathcal{U}$ or corresponds to one of the cases stated in Part (i) of the lemma, (see the class check lists for (6e), (6f*), (6i), (6ℓ), (6n)). This proves Part (i) of the lemma.

(ii.a) Because of the readings of the L-situations (page 75) we have to consider (in an arbitrary triangulation Δ) only the following two possibilities for mergings of $L^{(1)}$ and $L^{(2)}$, both of width 6, with resulting configuration of width 6.

(7a) $\doteq$ $\overbrace{-\ -\ 5\ 5}^{L^{(1)}}$ $\underbrace{5\ 5\ -\ -}_{L^{(2)}}$ $\longrightarrow$ pairs of $\{431r,\ldots,437r,451r,\ldots,464r\} \times \{431n,\ldots,437n,451n,\ldots,464n\}$

(7b) $\doteq$ $\overbrace{-\ 5\ 7\ 7\ 5}^{L^{(1)}}$ $\underbrace{7\ 7\ 5\ -}_{L^{(2)}}$ $\longrightarrow \{464n,494n\} \times \{464r,494r\} \longrightarrow$ no merging possible

Each configuration of (7a) either contains a configuration of $\mathcal{U}$ or is one of ## 901,...,977, (see the class check list for (7a)). This proves Part (ii.a) of the lemma.

(ii.b) We have to consider the following possibilities for mergings of $L^{(1)}$ of width 6 and $L^{(2)}$ with resulting configuration again of width 6.

(7c) $\doteq$ $\overbrace{-\ -\ 5\ 7\ 5\ 6}^{L^{(1)*}}$ $\underbrace{7\ 5\ 6}_{L^{(2)}}$ $\longrightarrow \{701n, 711r,\ldots,715r,716n,718r,\ldots,728r\} \times \{403r\}$ $\longrightarrow$ no merging possible

(7d) $\doteq$ $\overbrace{-\ -\ 5\ 7\ 5\ 6}^{L^{(1)}}$ $\underbrace{5\ 7\ 5\ 6}_{L^{(2)}}$ $\longrightarrow \{701r, 711r,\ldots,715r, 716n, 718r,\ldots,728r\}$
$\times \{441r, 466r,\ldots,478r\} \longrightarrow$ the only possible mergings are inclusions of 441r

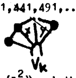
(7e) $\doteq$ $\overbrace{6\ 5\ 5\ 7\ 5\ 6}^{L^{(1)}}$ $\underbrace{6\ 5\ 5}_{L^{(2)}}$ $\longrightarrow \{716n\} \times \{401r,\ldots,403r\} \longrightarrow$ no merging possible

(7f) $\doteq$ $\overbrace{6\ 7\ 5\ 8\ 5\ 6}^{L^{(1)}}$ $\underbrace{5\ 8\ 5\ 6}_{L^{(2)}}$ $\longrightarrow \{730r\} \times \{479r, 480r\} \longrightarrow$ no merging possible

This proves Part (ii.b) of the lemma and finishes the proof of Lemma (L^2). ∎

<u>Proof of Lemma$(5,L,.,60 \text{ or } 50)^+$ (which supersedes$(5,L,60 \text{ or } 50)$).</u> Assume in an arbitrary triangulation Δ, there is a vertex V which receives two L-dischargings as stated in the lemma. Then the first L-discharging must be induced by one of ## 401, 402, 403, 431,...,437, 465, 530, 701, 716; if it is induced by 530, 701, or 716 then the second L-discharging is induced by 441 (see Lemma (L^2)) and the lemma is

$\begin{cases} 401,402,403,431,\ldots,437, \text{ or } 465 \text{ at } X \\ 411,441,491,\ldots,495,551,552, \text{ or } 553 \text{ at } Y \end{cases}$



$\doteq$ $\boxed{(7g)}$

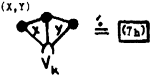
verified. If the first L-discharging is induced by 401, 402, 403, 431,...,437, or 465 then the second L-discharging must be induced by one of 411, 441, 491, ...,495, 551,...,553 and a configuration of the class (7g) see drawing on p.77) occurs in Δ (with V_k as indicated in the drawing). But a configuration of (7g) either contains a configuration of $\mathcal{U}$ or is one of 811, 812, 813 (see the class check list for (7g)). This proves the lemma. ■

Proof of Lemma (L_o, L). Assume that in an arbitrary triangulation Δ, there is a vertex V which receives two L-dischargings as stated in the lemma. Then each L-discharging is induced by one of ## 401, 402, 403, 433,...,437, 451,..., 464, 530. If none of the L-dischargings is induced by 530 then the hypothesis of Lemma (L^2) (i) or (ii.a) is fulfilled and Lemma (L, L) follows from Lemma (L^2). If one of the L-dischargings is induced by 530 then the other L-discharging is induced by one of 437, 461,...,463, 530 and 2-1 occurs. This finishes the proof of the Iemma. ■

Proof of Lemma (L, 5, L). Assume that in an arbitrary triangulation Δ, there is a vertex V_k which receives two L-dischargings as stated in the lemma. Then each L-discharging is induced by one of ## 401, 402, 403, 431,...,437, 451,...,464, 530, 531, 701. If one of the L-dischargings is induced by 531 or 701 then there is nothing to prove. If one of the L-dischargings is induced by 431 or 432 then the hypothesis of Lemma (L^2) (i) or (ii.a) is fulfilled and Lemma (L, 5, L) follows from Lemma (L^2).

If each L-discharging is induced by one of 401, 402, 403, 433,...,437, 451,...,464, 530 then a configuration of the class (7h) (see drawing to the right) occurs in Δ (with V_k as indicated in the drawing. But each configuration of (7h) contains a configuration of $\mathcal{U}$ (see the class check list for (7h)). This finishes the proof of the lemma. ■

a pair of $\{401,402,403,433,\ldots,437,451,\ldots,464,530\}^2$ at (X,Y)



(9) Now we shall prove the $q_{TL}(V_k)$-Lemma for vertex degrees k = 8, k = 9, k=10, and k=11. We assume that Δ is an arbitrary triangulation which contains a q_{TL}-positive vertex of degree k. We have to prove that Δ contains a configuration of 𝒰 or contains a configuration according to one of the CTL-situations CTL ## 81,...,152 of Table 4 (with central vertex identified to V) so that no additional L-dischargings go to V. Further we assume that the lemmas on T- and L-dischargings which have been proved for Δ* in sections (7) and (8) of this supplement hold also for Δ.

We denote by w the maximal width of L- or L^2-situations with pivots identified to V. For instance w = 5 means that some configuration of ## 510, ..., 691, 811,..., 894 (see Table 2 or $\mathcal{L}^2$, respectively) is contained in Δ with pivot identified to V, but that no configuration of greater w (i.e., none of ## 701,...,760, 901,...,977) is attached to V. If no L-situation is attached to V then we define w to be zero. For each degree k = 8, 9, or 10, our case-distinctions will be according to the value of w.

Further we denote by $\bar{v}$ the maximal contribution value of those L- or L^2-situations which are of width w (and are attached to V). For instance, "w = 5, $\bar{v}$ = 100 or 95" means that one of ## 531,...,541, 841,...,843 is attached to V, but none of ## 530, 811,...,835 is attached (and also none of larger w). We shall use the $\bar{v}$-values mainly for distinguishing sub-cases.

Proof of the $q_{TL}(V_8)$-Lemma.

Case w ≥ 6

Sub-case $\bar{v}$ > 80

(L̄ = 702) (w = 7)

51-8

or

L̄ of w = 6, v > 80
(i.e., L̄ = 701, 711,...,728,
901,...,935)

≐ $\boxed{(8a)}$

Proof of $q_{TL}(7_8)$ - Lemma

Case # $\geq_8$ 6.

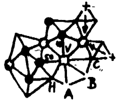
Sub-case $\bar{7} > 80$, continued

In this sub-case, 51-8 or a configuration of the class (8a) occurs (see drawings on page 79) with V as indicated in the drawings. Each configuration of (8a) contains a configuration of $\mathcal{U}$ or corresponds to one of the following critical cases C8a#1,2,3,(see the class check list for (8a)).

C8a#1: Σ is # 722, **C8a#2:** Σ is # 723, **C8a#3:** Σ is # 728.

For the configurations C8a#1,2,3, we have to consider the possible arrangements of 5- and non-5-vertices, etc., for the remaining two neighbors of V.

Arrangement check list for C8a#1



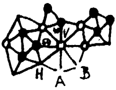
H	A	B	C	
	5			→ 22-10
	6	5		→ 22-7
	6	6		
	^{T2}6	6		→ T2#7n (by Lemma (60 or 50, T)) → 22-20
	^{T1}6	6		→ T1#1 (by Lemma (60 or 50, T)) → 1-3
	x′ 6^{T2}	6^{T2}		→ f5-22
	U	5		→ 50-20

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] ≥ 6

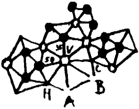
Sub-case $\hat{v} > 80$, finished

Arrangement check list for C8a#2



H'	A	B	
	5		→ 22-10
	6	5	→ 22-7
	6	6	
	$^{T2}6$	$^{T2}6$	impossible by Lemma (60 or 50, T2, T2)
	U	5	
	U	L	impossible since no L-situation can be attached at B (while A is major)

Arrangement check list for C8a#3



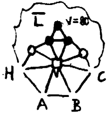
H	A	B	C	
	5			→ 22-10*
	6	5		→ 22-7
	6	6		
	6	6^{T}		→ I4'-14
	$^{T2}6$	$^{T2}6^{\times}$		impossible by Lemma (60 or 50, T2, T2)
	U	5		
	U	L		one of 423, 424, 428, 492, 493, 496, ..., 501, 503, 505, 518, 519 at B → 2-2 or 508 at B → 1-3 or one of 427, 512, 513, 514 at B → 19-3.

Proof of $q_{TL}(V_8)$ - Lemma

Case $\ell \geq 6$

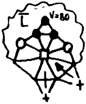
<u>Sub-case $\bar{v} = 80$</u>

($\bar{L}$ = 951,...,977)



<u>Arrangement check list</u>

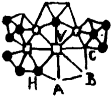
H	A	B	C	
	5			$\doteq$(8b)
	6	6		
T2	$_6$ B?	$_6$ T1		→ (8c)
T2	$_6$ T1	$_6$ T?		→ (8c)



$\doteq$ $\boxed{(8c)}$

<u>Sub-case $\bar{v} < 80$</u> (i.e., $\bar{L}$ = 729 or 730)

<u>Arrangement check list for $\bar{L}$ = 729</u>



No L5 or L6 can be attached at H

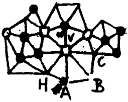
H	A	B	C	
	5	5		⟶ 23-18
	5	6		⟶ 23-19
	5	U		impossible since by Lemma (5, L), $q_{TL}(V) \leq 0$
	6	5		⟶ 23-4
	6	6		impossible since $q_{TL}(V) \leq 0$
	U	5		
L4	U	L5+		⟶ #441 at B ⟶ 1-2
R	U	L6		impossible since no L6 can be attached at B (while A is major)

(L5+ means L5 or L6)

Proof of $q_{TL}(V_8)$ - Lemma

Case # ≥ 6

Sub-case $\tilde{v} < 80$, [illegible]



Arrangement check list for L = 750

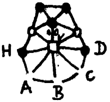
H	A	B	C	
	5			→ 43-7
	6	5		→ 23-16
	6	6		impossible since by Lemma (T?, T2, T?), $\dot{q}_{TL}(V) \le \dot{0}$
	U	5		
	U	L6		impossible since no L6 can be attached at B (while A is major)

Case # = 5

Sub-case $\tilde{v} > 80$ (i.e., L = 530,...,550, 811,...,854)
or L = 551,552,553 (v = 80) $\doteq$ $\boxed{(8d)}$

Each configuration of (8d) contains a configuration of $\mathcal{U}$ or is the critical configuration C8d, (see the class check list for (8d)).

C8d: L is # 532.



Arrangement check list for C8d

H	A	B	C	D	
	5				→ 50-1
	6	5			→ 50-4
	6	6	6		→ 50-9
	6	6	U		
L	6	6	U		→ (8da)
R	6^{T2}	6	U	L	$\doteq$ (8dd)
R	6	6	U	L5+	→ 551, 552, or 553 at D → 51-10

some L attached at H $\doteq$ $\boxed{(8da)}$

Proof of $q_{TL}(V_8)$ - Lemma

Case I = 5

Arrangement check list for C8d, finished

H	A	B	C	D	
	6	U	/		
15+	6	U	.	L	→ (8da)
	U	5	U		
L	U	5	U		$\doteq$ (8db)
R	U	L	U		$\doteq$ (8dc)
R	U	R	U	R	→ CTL#138
	U	.	U		
15+	U	.	U		→ 551, 552, or 553 at H → 51-10

" R " means "regular discharging", i.e., no L-discharging goes from the V_5 to V

Sub-case $\bar{v}$ = 80 or 75 but 551,552,553 do not occur

(i.e., I = 861,...,894)

$$\doteq \boxed{(8e)}$$

The class (8e) has the following four critical sub-classes, (see the class check list for (8e)).

C8e#1: I is # 870, C8e#2: I is # 872,

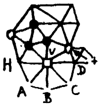
C8e#3: I is # 880 or 887, C8e#4: I is # 882 or 889.

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 5

Sub-case $\bar{v}$ = 80 or 75

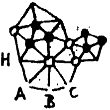
Arrangement check list for C8e#1



	H	A	B	C	D	
3/1,2,3,4		5				→ 29-13
3/2r,4r		.		5		→ 56-11
3/5		.	5	.		
		6	5	.		→ 7-21
		U	5	6		→ 30-17
		U	5	U		
		U	L5+	U		→ one of 621,…,624 attached at B → 23-3
3/6		.	.	.		
		$^{T2}6$	$^{T}6$	6		→ I2-22
		$^{T1}6$	$^{T2}6$	6		→ I2-21
		6	6	6^{T°		→ I5-13

We assume L to be non-reflected and thus we have to consider the "reflected arrangements" 3/2r, 3/4r (see page 45) also

Arrangement check list for C8e#2



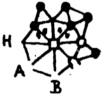
	H	A	B	C	
3/1,2,3,4		5			→ 56-29
3/2r,4r		.		5	→ 55-4
3/5		.	5	.	
		6	5	.	→ 7-21
		U	L5+	.	→ one of 491,492,493,495 attached at B → 30-1
3/6		.	.	.	
		$^{T2}6$	$^{T}6$	6	→ I2-22
		$^{T}6$	$^{T2}6$	6	→ I2-21

Proof of $q_{TL}(V_8)$ - Lemma

Case # = 5

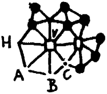
Sub-case $\bar{v}$ = 80 or 75

Arrangement check list for C8e#3



H	A	B	
	5		→ 7-27
	6	5	→ 9-16
	6	6	impossible since $q_{TL}(V) \lneq 0$
	U	5	impossible since no L5+ can be attached at B

Arrangement check list for C8e#4



H	A	B	C	
	5			→ 7-27
	6	5		→ 9-16
			6	→ 43-12
	U	5	U	impossible since no L5+ can be attached at B

Sub-case $\bar{v}$ = 70 or 65

If # 618 or 619 is attached to V then 61-4 occurs,

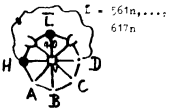
if # 620 is attached then 33-20 occurs.

Thus we may assume that one of ##.[illegible]17 is attached to V.

Proof of $4_{TL}(V_8)$ - Lemma

Case [illegible] = 5

Sub-case $\bar{v}$ = 70 or 65



Ī = 561n,...; 617n

Arrangement check list

	H	A	B	C	D	
3/1,2		5	5			$\doteq(8f)$
3/3		5	.	5		$\doteq(8g)$ critical sub-class C8g see p.88
3/4		5	.	.		by Lemma (5,L), only L4 - situations can be at H or A; by Lemma (L,L), not both H and A can L-discharge to V
	L4	5	.	.		$\doteq(8h)$ → (one of 431,..., 437 at H), C8h see p.89
	R	L4	.	.		by Lemma (5,L,T2), no T2 can go across edge B-C
	R	L4	6	6	6	$\doteq(8i)$ → (401, 402, or 403 at A)
	R	L4	U	6^{T2}	6	$\doteq(8j)$ → (one of 451,..., 463 at A)
	R	R	6^{T2}	6^{T}	6	→ I2-22
	R	R	6^{T}	6^{T2}	6	→ I2-21
3/2r		.	5	5		$\doteq(8k)$
3/5		.	5	.		
		6	5	.		$\doteq(8l)$ critical sub-classes C8l see p.89
		U	5	6		$\doteq(8m)$ critical sub-classes C8m see p.90
		U	5	U		impossible since no L5+ can be at H or B (since 551, 552,553 are ruled out by sub-case hypothesis)
3/4r		.	.	5		no L6 can be attached at H, the only possible L5 at H is 441
	L5	.	.	5		$\doteq(8n)$
		.	.	L5+		$\doteq(8o)$ → one of 411, 491,...,495 at C → deg(D) = 6 (since 494r cannot be paired with 602,3)
		6	6	5		$\doteq(8p)$

continued next page

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{w}$ = 5

Sub-case $\bar{v}$ = 70 or 65

Arrangement check list, finished

H A B C D

. . . if L5 at H then V_6 at A, since L5 = 441

L5 $6^{T} 6^{\cdot} 6$ → { if T1 then T1#1 → 22-16 / if T2 then T2#7 → 22-20 } (see Lemma(60 or 50,T))

L5 $6^{\times} 6^{T2} 6^{T2}$ → 12-23

L4 $6^{T2} 6^{T2} 6^{T1} 6$ $\doteq$ (8q) ↘ (one of 466,...,480 at H)

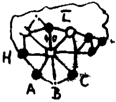
L4 $6^{T2} 6^{T1} 6^{T2} 6$ $\doteq$ (8r) ↗

L4 $6^{T1} 6^{T2} 6^{T2}$ → 12-23

Critical sub-classes

C8g: Γ is one of ## 580, 591, 592, 595, 611,...,613, 615,...,617

Arrangement check list for (C8g)



H A B C

6 → 23-4

U

L U → 433 or 434 at H → 23-26

R L U → { if one of 433,...,436 at A → 23-26 / if 437 at A → 25-23 }

R R U L → { if one of 441, 466,...,473, 479,...,481 at C → 1-2 / if 474 at C → 1-3 / if one of 476,...,478 at C → 2-2 }

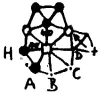
R R U R → CTL#139

Proof of $q_{TL}(V_8)$ - Lemma

Case # = 5

Sub-case $\tilde{v}$ = 70 or 65

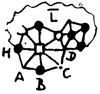
C8h: L is # 562



Arrangement check list for C8h

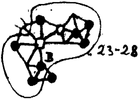
H	A	B	C	D	
		6	6		→ 50-7
		6	U		impossible since $q_{TL}(V) < 0$
		U	6^{T2}		→ by Lemma (T‖T) ⟶

C8ℓ#1: L is one of ## 589,...,597, 604,...,617

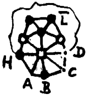


Arrangement check list for C8ℓ#1

H	A	B	C	D	
			6		→ 23-19
			U		
L5+			U		→ 441 at H → 22-10
		L5+	U		→ 491,...,495 at B ⟶



C8ℓ#2: L is #561, 562, or 602



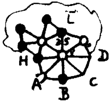
Arrangement check list for C8ℓ#2

H	A	B	C	D	
			6		→ {if L is 561 or 562 → 50-5 if L is 602 → 55-25
			U		
L5+			U		→ 441 at H → 22-10
		L5+	U		→ 491,...,495 at B ⟶

Proof of $q_{TL}(V_8)$ - Lemma

Case # =5

Sub-case $\bar{v}$ = 70 or 65



C8m# 1: $\bar{L}$ is # 563 or 564

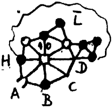
Arrangement check list for C8m# 1

H A B C D

U — no L-discharging can go from H to V (since 532,..., 541 cannot occur by sub-case hypothesis).

R U L5+ → 441 must be attached at B → 51-8

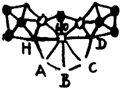
R U L4 6^{T2} 6 $\doteq$ (8ma) → the L4 at B must be of discharging value > 35 and thus must be one of 466,...,468, 471, 472, 473, 475,...,477, 479, 480



C8m#2: $\bar{L}$ is one of ## 578,...,581

→ no L-discharging can go from H to V (since 532,...,541 cannot be attached by sub-case hypothesis). No I6 can be attached at B. Thus a T-discharging must go across the edge C-D. → I4-35

Sub-case $\bar{v}$ = 60 (i.e., $\bar{L}$ = 621)



Arrangement check list
(no reflected arrangements by symmetry)

H A B C D

3/1,2 — 5 5 → 25-18

3/3 — 5 . . 5
L . 5 → (8s)

3/4 — 5 . .
L . . → (8s)
R 6^{T2} 6^{T2} → I2-4

L at A (with w ≤ 5; if w=5 then v ≤ 60) $\doteq \boxed{(8s)}$

continued next page

Proof of $q_{TL}(V_8)^*$-Lemma

Case [illegible] = 5

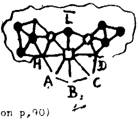
Sub-case $\bar{v}$ = 60

Arrangement check list, finished

	H	A	B	C	D	
3/5		.	5	.		
		${}^{T}6$	5	.		→ 23-19
3/6		.	.	.		
		${}^{T2}6$	${}^{T2}6$	${}^{T1}6^{T2}$		→ I2-4

Sub-case $\bar{v}$ = 50 (i.e., $\bar{L}$ is 622, 623, or 624)

Arrangement check list



	H	A	B	C	D	
3/1,2	.	5	5			→ 23-18
3/3		5	.	5		
		L	.	5		→ (8s) (as defined on p.90)
		R	.	L5+		→ 411 or 441 at C → if $\bar{L}$ = 622 → 1-3 if $\bar{L}$ = 623 or 624 → 2-2
3/4		5	.	.		
		L	.	.		→ (8s)
3/2r		.	5	5		→ 23-1
3/5		.	5	.		
		${}^{T}6$	5	.		→ 23-19
		.	5	6^{T}		→ 23-9
3/4r		.	.	5		
		${}^{T2}6$	${}^{T2}6$	L		→ I2-4
		.	.	L5+		→ 411 or 441 at C → if $\bar{L}$ = 622 → 1-3 if $\bar{L}$ = 623 or 624 → 2-2
3/6		.	.	.		impossible since $q_{TL}(V) \leq 0$, by Lemma(T2,T2,T)

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{v}$ = 5

Sub-case $\bar{v}$ = 40 or 35 (i.e., L = 631,..., 691)

Arrangement check list

L may be reflected or non-reflected; thus we have to consider only the non-reflected arrangements 3/1,..., 3/6 (see p.45).

H A B C D

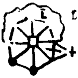
3/1 5 5 5 $\doteq$ (8t)

3/2 5 5 . (by Lemma(5,L), no L5+ can be attached)

L L $6^{T}6 \rightarrow$ (8u)

L R $6^{T2}6 \rightarrow$ (8u)

R L $6^{T}6 \rightarrow$ (8u)



L = 631,...,691

$\doteq \boxed{(8u)}$

3/3 5 . 5

L6 . 5 $\rightarrow$ (8v) (411 at A)

L5 . L $\doteq$ (8w)

L = 631,...,691

$\doteq \boxed{(8v)}$

3/4 5 . .

recall Lemma (60 or 50, T2, T2)

L6 $6^{T}6^{T}6 \rightarrow$ (8v) (411 at A; 491 cannot be attached)

3/5 . 5 .

$6^{T2}6$ L6 $6^{T}6 \doteq$ (8x) (411 at A)

$6^{T2}6$ L5 $6^{T2}6$ impossible by Lemma (T2, L, T2)

3/6 . . . impossible since $q_{TL}(V) < 0$

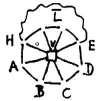
Proof of $q_{TL}(V_8)$ - Lemma

Case W = 4

Note that by case-hypothesis, no L-situation of width $w > 4$ can be attached to V; moreove , no pair of L-situations as considered in Lemma(L,L), except 801, 802, 803, can be attached to V.

Sub-case $\bar{v}$ = 100 or 95 (i.e., $\bar{L}$ = 431,...,437)

Arrangement check list



($\bar{L}$ may be reflected or non-reflected; thus we have to consider only the non-reflected arrangements 4/1,...,4/10)

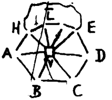
	H	A	B	C	D	E	
4/1,...,7		5					≐ (9a) .
4/8		.	5	5	.		no L-situation can be attached at E or at H, by case-hypothesis
		6	5	5	.		→ (9b)
		U	5	5	U		 $\bar{L}$ = 431, ...,437 ≐ $\boxed{(9b)}$
		U	L	5	U		→ (9c)
		U	R	R	U		if $\bar{L}$ = 431 → CTL#126 if $\bar{L}$ = 432 → CTL#127 if $\bar{L}$ = 433,...,437 → CTL#130
4/9		.	5	.	.		
		6	5	.	.		→ (9b)
		U	5	6	6		≐ (9d)  $\bar{L}$ = 431,...,437 L ($w \leq 4$) at B ≐ $\boxed{(9o)}$
		U	L	.	.		→ (9c)
		U	R	6	U		if $\bar{L}$ = 431n or 432n → CTL#129 if $\bar{L}$ = 431r or 432r → CTL#128 if $\bar{L}$ = 433n,..., 437n → CTL#132 if $\bar{L}$ = 433r,..., 437r → CTL#131
		U	R	U	.		
4/10		.	.	.	.		
	6 ^{T}	6	.	.	.		→ $\bar{L}$ = 431r or 432r → I2-6
	×	6 ^{T2}	6 ^{T}	6	.	×	≐ (9e)
	×	6 ^{T1}	6 ^{T2}	6	.	×	≐ (9f)
	×	6 ^{T}	6	6 ^{T}	6	×	→ if $\bar{L}$ = 431 or 432 → 10-13 if $\bar{L}$ = 433,...,437 → 10-5

Proof of $q_{TL}(V_8)$ - Lemma

Case # = 4

Sub-case $\hat{v}$ = 80 (i.e., L = 441, 801, 802, or 803)

Arrangement check list



no L-situations can be attached at H or at E by case-hypothesis

	H	A	B	C	D	E	
4/1,...,7							≐ (9g)
4/8		.	5	5	.		
		6	5	5	.		→ (9h)
		U	5	5	U		
		U	L	5	U		≐ (9i)
		U	R	R*	U		if L = 441 → CTL#133 if L = 801 → CTL#123 if L = 802 → CTL#125 if L = 803 → CTL#124
4/9		.	5	.	.		
		6	5	.	.		→ (9h)
		U	5	6	6		≐ (9j)
		U	5	6	U		
		U	L5+	6	U		≐ (9k)
		U	5	U	.		→ no L can be attached at B
		U	5	U	6^{T2}	6	→ L = 441r and T2#7r, by Lemmas(5,L,T2) and (60 or 50,T), → 22-20
4/10		.	.	.	.		
	6^{T2}	6	.	.	.		→ L = 441n and T2#7n → 22-20
	6^{T1}	6^{T2}	6^{T2}	6	.		→ I2-25
		6	6	6	6		if L = 441 → 10-9 if L = 801, 802, or 803 → 10-17

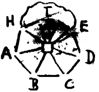
v = 80

≐ $\boxed{(9h)}$

Proof of $q_{TL}(V)$ Lemma

Case = 4.

Sub-case $\bar{v}$ = 70 or 65 (i.e., L = 451,...,481)



no L-discharging can go from H or E to V by case-hypothesis

Arrangement check list

	H	A	B	C	D	E	
4/1,...,7		5					[illegible] $\doteq$ (10a)
4/8		.	5	5	.		
		6	5	5	.		$\rightarrow$ (10d)
		U	5	5	U		
		U	L	5	U		$\doteq$ (10c) - one of 451,...,464 at B
		U	R	R	U		if L is one of 451,...,464 $\rightarrow$ CTL#134 if L is 465 $\rightarrow$ CTL#135 if L is one of 466,...,481 $\rightarrow$ CTL#136
4/9		.	5	.	.		
		6	5	.	.		$\rightarrow$ (10d)
		U	5	.	.		
		U	L5+	6	6		$\doteq$ (10e) - one of 491,...,495 at B (441 cannot occur by sub-case hypothesis)
		U	L4	6^{T2}	6		$\doteq$ (10f)
		U	L4	6	6^{T}	6	$\doteq$ (10g)
		U	R	6^{T2}	6^{T}	~~6~~	$\doteq$ (10h)
		U	R	6^{T1}	6^{T2}	6	$\doteq$ (10i)
		U	5	6	U		
		U	L6	6	U		$\doteq$ (10j) - 491 at B
		U	5	U	.		impossible since no L can be attached at B and thus $q_{TL}(V) \leq 0$

L = 451,...,481

$\doteq$ $\boxed{(10d)}$

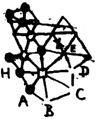
continued next page

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 4

Sub-case $\bar{v} = 70$ or 65, Arrangement check list, finished

	H	A	B	C	D	E	
4/10		.	.	.	.		
	6^{T}	6	.	.	6^{T}	6	→ L, is 465
	6^{T2}	6	.	.	6^{T2}	6	→ I4-11
	6^{T}	6	6	6	6^{T1}	6	→ 10-17
	6^{T2}	6^{T2}	6^{T1}	6^{T}	$6^{\times}$		$\doteq$ (10k)
	6^{T2}	6^{T2}	$6^{\times}$	6^{T2}	$6^{\times}$		$\doteq$ (10ℓ)
	6^{T2}	6^{T1}	6^{T2}	6^{T}	$6^{\times}$		$\doteq$ (10m)
	6^{T2}	6^{T1}	6^{T1}	6^{T2}	$6^{\times}$		$\doteq$ (10n)
	6^{T2}	$6^{\times}$	6^{T2}	6^{T2}	$6^{\times}$		$\doteq$ (10r)
	6^{T1}	6^{T2}	6^{T2}	6^{T}	$6^{\times}$		impossible by Lemma(T,T2;T2,T)
	6^{T1}	6^{T2}	6^{T1}	6^{T2}	$6^{\times}$		$\doteq$ (10p)
	6^{T1}	6^{T1}	6^{T2}	6^{T2}	$6^{\times}$		$\doteq$ (10q)



Critical sub-classes

C10a: **L** is 475r

[illegible]ement check list for C10a

	H	A	B	C	D	
3/1,...,4			5			→ 22-1
3/2r,4r					5	→ 1-9 °
3/[illegible]			.	5	.	
			6	5	.	→ 22-5
			U	5	6	→ 26-9
			U	5	U	no L can be attached at C, by case-hypothesis
		L	U	R	U	impossible since no L can be attached at A (by sub-case hypothesis)
		R	U	R	U	→ CTL # 137

continued next page

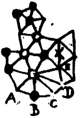
Proof of $\#_L(\mathcal{T}_8)$ - Lemma.
Case # T = 4

Sub-case # = 70 or 65

Arrangement check list for C10a, finished

H	A	B	C	D	E	
		.	.	.		no L5+ can be attached at A, (by Lemma(5,L))
		.				
	L	.	-	.		→ {if one of 401,402,403,451,...,460,464 at A[illegible]- if one of 461,...,463 at A → 2-1
	R	.	.	6^T	6	→ I4-10
	R	6^T	6^T	$6^{\times}$,		→ 10-5

<u>C10d:</u> T is 475r



Arrangement check list for C10d

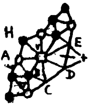
B	C	D	E	
	5			→ 22-4
	.	5		→ 1-9
	.	.		
	6	6		→ 31-29
	6	U		
L6	6	U		→ 411 attached at B → 22-6
	.	.		
	U	.		
	U	6^T		→ I4-10
L6	U	$.^{\times}$		→ 491 attached at B → 22-6

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 4

Sub-case $\bar{v}$ = 70 or 65

C1Qg: L is 476r or 477r and one of 476, 477 is attached at B.

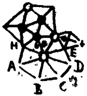


If a T2-discharging goes across one of the edges C-D, D-E, then 14-12 occurs.
If no T2-discharging goes to V then a T1-discharging must go across C-D and another T1-discharging must go across D-E (since otherwise $q_{TL}(V)$ would not be positive).

If both T1-dischargings are T1#1 then 8-14 occurs.
If a T1-discharging other than T1#1 goes to V then we may assume by symmetry that it goes across D-E Then we have:

T1#2n	T1#2r	T1#3n
6-13	2-2	6-14

Sub-case $\bar{v}$ = 60 (i.e., L = 491)



Arrangement check list

	H	A	B	C	D	E	
4/1,...,7		5					→ 14-[illegible]
4/2r,...,7r		.			5		→ 57-3
4/8		.	5	5	.		
		6	5	5	.		→ 57-4
		U	5	5	6		→ 30-5
		U	5	5	U		impossible since no l can be attached (by sub-case hypothesis) and $q_{TL} \leq 0$
4/9		.	5	.	.		
		6	5	.	.		→ 57-4
		U	5	6	6		→ 30-12
		U	5	6	U		impossible since $q_{TL}(V) \leq 0$
		U	5	U	.		impossible since $q_{TL}(V) \leq 0$

continued next page

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = .4

Sub-case $\bar{v}$ = 60

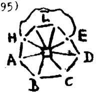
Arrángement check list finished

H A B C D E

4/9r
. . 5 .
. . 5 6 → 30-5
6 6 5 U → 8-13
6 U 5 U
U . 5 U
impossible since $q_{TL}(V) \leq 0$

4/10
. . . .
. . . 6^{T} → I5-11
$^{T2}6^{T2}6^{T1}6^{T2}6^{\times}$ → I2-33
$^{T2}6^{T1}6^{T2}6^{T2}6^{\times}$ → I2-30

Sub-case $\bar{v}$ = 50 (i.e., £ = 492,...,495)



Arrangement check list

A B C D E

4/1,..., 5 $\doteq$(10r)

4/8
. 5 5 .
6 5 5 . → (10s)
U 5 5 U impossible since $q_{TL}(V) < 0$

$\doteq \boxed{(10s)}$

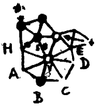
4/9
. 5 . . (critical case C10s see p.100)
6 5 . . → (10s)
U 5 . . no L6 can be attached (by sub-case hypoth.)
U L $6^{T}6^{T}6$ $\doteq$(10t)

4/10
. . . .
$6^{T2}6^{T2}6^{T1}6^{T2}6^{T}6$
$6^{T2}6^{T2}6^{T1}6^{T1}6^{T2}6$
→ if £n → I2-33
→ if £r → I5-18
$6^{T2}6^{T2}6^{\times}6^{T2}6^{T2}6$ → I5-18
$6^{T2}6^{T1}6^{T2}6^{T1}6^{T2}6$ → I2-30

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{v}$ = 4

Sub-case $\bar{v}$ = 50

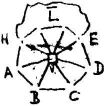


C10s: $\bar{L}$ is 493n

Arrangement check list for C10s

H	A	B	C	D	E	
			5			→ 7-7
			.	5		→ I4-23
			.	.		
			6	6		→ 30-12
			6	U		impossible since no T2 can go to V if L6 at B, by Lemma(60 or 50, T, 60 or 50)
			U	.		
6^{T}	6	L	U	6^{T}	6	→ I5-11

Sub-case $\bar{v}$ = 40 or 45 .(i.e., $\bar{L}$ = 496,...,522)



Arrangement check list

	H	A	B	C	D	E	
4/1,...,7		5					≐(10u)
			.				
4/8		.	5	5	.		at most one L at B, C, but none of width w > 3, by sub-case hypothesis
recall Lemma(5,L,T2)	6^{T2}	6	R	L4	6		→ 7-21 (since 401,402,or 403 at C)
	6^{T2}	6	R	R	6^{T}	6	→ I2-9
	6^{T1}	6	L4	R	6^{T1}	6	→ 7-21
4/9		.	5	.	.	.	the only possible L5+ at B is 411
	6^{T}	6	L6	6^{T}	6		→ if $\bar{L}$n → 8-8
	6^{T}	6	L6	$6^{\times}$	6^{T}	6	→ if $\bar{L}$r → 8-13
	[illegible]$^{\times}$	6	L6	6^{T}	6^{T}	6	→
recall Lemma(T2,L,T2)	6^{T2}	6	L4	6^{T1}	6^{T2}	6	≐(10v)
	6^{T1}	6	L4	6^{T2}	6^{T2}	6	→ if $\bar{L}$n → I2-25
	6^{T2}	6	R	6^{T2}	6^{T2}	6	→ if $\bar{L}$r → I2-26

Proof of $a_{TL}(V_8)$ - Lemma

Case $\bar{w}$ = 4

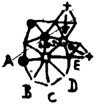
Sub-case $\bar{v}$ = 40 or 35

Arrangement check list, finished

	H	A	B	C	D	E	
4/10		.	.	.	.		
	6^{T2}	6^{T2}	6^{T1}	6^{T2}	6^{T2}	6	→ I2-33

Critical sub-classes

C10u#1 : L is 500n



Arrangement check list for C10u#1

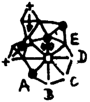
	A	B	C	D→ E	
3/1,2		5	5		→ 7-2
3/2r,3,4r				5	→ 27-17
3/4		5	.	.	at most one L can be attached at A, B
		5	6^{T}	6	→ 31-7
		5	.	6^{T2}	→ I5-12
3/5		.	5	.	the only possible L5+ at A or B is 411 (which requires deg(B) = 6)
		6	5	.	→ 7-27
		U	5	6^{T2}	→ I5-12
	L4	U	L4	6^{T}	impossible by case-hypothesis (since the L4's had to be of width w < 5, but then would form a pair as considered in Lemma (L2))
3/6		.	.	.	
	L6	6^{T}	6	.	→ I2-13
	L6	$6^{\times}$	6^{T}	6^{T}	→ 31-8
	L4	6^{T2}	6^{T2}	6^{T}	→ I2-24
	L4	6^{T}	6^{T}	6^{T2}	→ I5-12

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{v}$- 4

Sub-case $\dot{\bar{v}}$ =, 40 or ~~3~~5

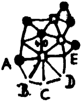
C 10 u # 2 : Z is 500 r



Arrangement check list for C 10 u # 2

	A	B	C	D	E	
3/1,...,4		5				→ 31-6
3/2r,4r			.	5		→ 27-17
3/5		.	5	.		the only possible L5+ at A or C is 411
		6	5	.		→ 31-4
		U	5	6^T		→ 27-20
3/6		.	.	.		
	L6	6^T	6^T	6		→ 9-24
	L6	6^T	$6^\times$	6^T		→ 9-24
	L6	$6^\times$	6^T	6^T		→ 9-24
	L4	6^T	6^T	6^T		→ 9-24

C 10 u # 3 : L is 503n



Arrangement check list for C 10 u # 3

	A	B	C	D	E	
3/1,...,4		5				→ 27-2
3/2r,4r				5		→ 27-31
3/5		.	5	.		the only possible L5+ at A or C is 4
		6	5	.		→ 7-27
	L4	U	5	6^T		→ the L4 at A must be 423 → 27-3
		U	L4	6^T		→ the L4 at C must be 423 → 23-8
3/6		.	.	.		
	L6	6	.	.		→ 27-3
	L4	6^T	6^T	6^T		→ if 421n at A → I2-15 if 422n at A → I5-8 if 421r or 422r at A → 27-3

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 4

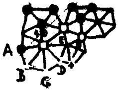
Sub-case $\dot{v}$ = 40 or 35

C10u#4: L is 503r

Arrangement check list for C10u#4

	A	B	C	D	E	
3/1,2		5	5			→ 23-2
3/2r,3,4r				5		→ 27-31
3/4		5	.	.		
		5	6^T	6		→ 8-13
		5	$6^\times$	6^T		→ 8-13
3/5		.	5	.		the only possible ~~L5+~~ at A or C is 411
		6	5	.		→ 23-4
		U	5	6^T		→ 27-33
3/6		.	.	.		
	L6	6	.	.	.	→ 23-20
	L4	6^T	6^T	6^T		→ 9-24

C10u#5: L is 519n



Arrangement check list for **C10u#5**

	A	B	C	D	E	
3/1,2		5	5			→ 7-2
3/2r,3,4r				5		→ I5-9
3/4		5	.	.		
		5	6^T	6		→ I2-8
		5	$.^\times$	6^{T2}		→ I4-16
3/5		.	5	.		the only possible L5+ at A or C is 411
		6	5	.		→ 7-27
		U	5	6^{T2}		→ I4-16 (at most one L4 can be at A,C, since L4's at A and C would form a pair of width = 5 which is ruled out by case-hypothesis)

continued next page

Proof of $q_{TL}(V_8)$ - Lemma

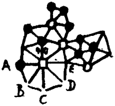
Case [illegible] =4

Sub-case $\bar{v}$ = 40 or 35

Arrangement check list for C 10 u # 5, finished

	A	B	C	D	E	
3/6		.	.	.		
	L6	.	.	.		→ 1-20
	L4	6^{T2}	6^{T2}	6^{T}		→ I2-24
	L4	6^{T}	6^{T}	6^{T2}		→ I4-16

C 10 u # 6 : L is 522 n



Arrangement check list for C 10 u # 6

	A	B	C	D	E	
3/1,...,4		.5	.			→ 45-5
3/2r,4r				5		→ 43-34
3/5		.	5	.		the only possible L5+ at A or C is 4TT
		6	5	.		→ 7-27
		U	5	6^{T}		→ I4-14
3/6		.	.	.		
	L6	6	.	.		→ 5-6
	L4	6^{T}	6^{T}	6^{T}		→ I4-14

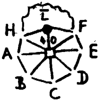
Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 3

In this case, the only L-situations which can be attached to V are 401,...., 428. The only L5+ is 411. If two L-situations are attached to V then the distance of the L-discharging V_5's (on the ring about V) must be at least three (since otherwise the two L-situations would form a pair of width w ≤ 5 which is ruled out by case-hypothesis).

Sub-case $\bar{v}_{-} = 70$ (i.e., L = 401, 402, or 403)

Arrangement check list



	H	A	B	C	D	E	F	
5/1,...,8		5	5					≐ (11a)
5/9		5	.	5	5	.		
		5	6	5	5	.		→ if L n → 7-2 if L r → 7-27
		5	U	5	5	6		→ if L n → 9-1 if L r → 7-3
		5	U	5	5	U		
		R	U	R	R	U		→ if L = 401n or 402n → CTL#97 if L = 403n → 1-2 if L r → CTL #96
5/10,13		5				5		→ 7-2
5/11		5	.	5	.	.		
		5	6	5	.	.		→ if L n → 7-2 if L r → 7-27
		5	U	5	6	6		→ if L n → 10-1 if L r → 10-3
		5	U	5	6	U		
		R	U	L	6	U		→ 423 at C → if L n → 26-1 if L r → 26-2
		R	U	R	6	U		→ if L = 401n or 402n → CTL#103 if L = 403n → 1-2 if L = 401r or 402r → CTL#98 if L = 403r → CTL#99

⋮

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{v}$ = 3

Sub-case $\bar{v}$ = 70

Arrangement check list, continued

	H	A	B	C	D	E	F	
5/11, contin.		5	U	5	U	.		
		R	U	R	U	6	$^{T}6$	→ 12-6
		R	U	R	U	. $^{\times}$		if $\mathcal{L}$ = 401n or 402n → CTL#103 if $\mathcal{L}$ = 403n → 1-2 if $\mathcal{L}$ r → CTL#100
5/12		5	.	.	5	.		
		5	.	.	5	6		if $\mathcal{L}$ n → 9-1 if $\mathcal{L}$ r → 9-14
		5	6	6	5	U		if $\mathcal{L}$ n → 9-1 if $\mathcal{L}$ r → 10-4
		5	6	U	5	U		
		R	6	U	R	U		if $\mathcal{L}$ = 401n or 402n → CTL#102 if $\mathcal{L}$ = 403n → 1-2 if $\mathcal{L}$ r → CTL#101
		5	U	.	5	U		
		R	U	6	L	U		→ (11b) (as defined on p. 108)
		R	U	.	R	U		if $\mathcal{L}$ = 401n or 402n → CTL#102 if $\mathcal{L}$ = 403n → 1-2 if $\mathcal{L}$ r → CTL#101
5/14		5	.	.	.	.		
		R	6	$^{T}6$	.	.		if $\mathcal{L}$ n → 12-6 if $\mathcal{L}$ r → 12-7
		R	. $^{\times}$	.	.	6	$^{T}6$	→ 12-6
		R	. $^{\times}$	6	$^{T}6$	$^{T}6$ $^{\times}$		if $\mathcal{L}$ n → 12-10 if $\mathcal{L}$ r → 12-8
5/15		.	5	5	5	.		
		6	5	5	5	.		if $\mathcal{L}$ n → 7 if $\mathcal{L}$ r → 9-1
		U	5	5	5	U		if $\mathcal{L}$ = 401 → 11-3 if $\mathcal{L}$ = 402 → 32-12 if $\mathcal{L}$ = 403 → 32-20

Proof of $q_{TL}(V_8)$ - Lemma

Case $\blacksquare$ = 3

Sub-case $\bar{v} \doteq 70$

Arrangement check list continued[2]

	H	A	B	C	D	E	F	
5/16		.	5	5	.	.		
		6	5	5	.	.		→ (11c) (as defined on p. 108)
		U	5	5	6	6		→ if $\mathcal{L}$ n → 9-6 if $\mathcal{L}$ r → 7-21
		U	5	5	6	U		
		U	R	L	6	U		→ if $\mathcal{L}$ n → CTL#122 if $\mathcal{L}$ r → CTL#121
		U	R	R	6	U		→ if $\mathcal{L}$ n → CTL#109 if $\mathcal{L}$ r → CTL#104
		U	5	5	U	.		
		U	R	R	U	6^{B}	6	→ T1#1 by Lemmas (5,L,T2), (5,L,T1) → CTL#107
		U	R	R	U_0^0	.$^{\times}$		→ if $\mathcal{L}$ n → CTL#108 if $\mathcal{L}$ r → CTL#104
5/17		.	5	.	5	.		
		6	5	.	5	.		→ (11c)
		[illegible]	[illegible]	6	[illegible]	U		
		U	L	6	R	U		→ (11b)
		U	R	6	R	U		→ if $\mathcal{L}$ = 401 → 11-25 if $\mathcal{L}$ = 402.or 403 → CTL#105
		U	5	U'	5	U		
		U	R	U	R	U		→ CTL#106
5/18		.	5	.	.	.		
		6	5	.	.	.		→ (11c)
		U	L	6	.	.		→ (11b)
		U	R	6^{T}	6^{T}	6		→ if $\mathcal{L}$ n → 10-5 if $\mathcal{L}$ r → 9-6
		U	R	6^{T}	6$^{\times}$	6^{T}	6	→ 10-5
recall Lemma (5,L,T2)		U	R	.$^{\times}$	6^{T2}	6^{T1}	6	→ 12-21

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 3

Sub-case $\bar{v}$ = 70

Arrangement check list, finished

5/19

H	A	B	C	D	E	F	
	.	.	5	.	.		
	6	6	5	.	.		if $\bar{L}$ n → 7-21 if $\bar{L}$ r → 9-6
	U	6	L6	6	U		if $\bar{L}$ = 401 → 10-21 if $\bar{L}$ = 402 → 32-24 if $\bar{L}$ = 403 → 32-25

if no L6 at C and none of the edges A-B, D-E is 6-6 then $q_{TL}(V) \leq 0$ since by **Lemma (5,L,T2)** the only possible T-discharging can be only T1

5/20

H	A	B	C	D	E	F	
	.	.	.	.	.		
	6	6	6	6	6		→ 10-17
	6	6	6	6	U		
6^{T1}	6^{T}	6	6	6	U		→ T1#1 across H-A by Lemma(5,L,T1) → I2-11
×	6	6	6	6	U		impossible since $q_{TL}(V) \leq 0$
	6	6	6	U	.		impossible since at most two T2-dischargings and one T1-discharging can go to V, by Lemma (5,L,T2), and thus $q_{TL}(V) \leq 0$
	6	6	U	.	.		

$\bar{L}$ = 401,402, or 403
one of 423,...,428 at B

$\doteq$ $\boxed{(11b)}$

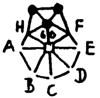
$\bar{L}$ = 401,402, or 403

$\doteq$ $\boxed{(11c)}$

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 3

Sub-case $\bar{\bar{v}}$ = 60 (i.e., L = 411)



Arrangement check list

	H	A	B	C	D	E	F	
5/1,...,8		5	5					→ 7-10
5/9		5	.	5	5	.		
		5	6	5	5	.		→ 7-27
		5	U	5	5	6		→ 9-17
		5	U	5	5	U		
		R	U	R	R	U		→ CTL # 111
5/10,13		5				5		→ 7-27
5/11		5	.	5	.	.		
		5	6	5	.	.		→ 7-27
		5	U	5	6	6		→ 8-13
		5	U	[illegible]	6	U		
		R	U	[illegible]	6	U		→ 423, at C → 26-6
		5	[illegible]	[illegible]	[illegible]	6		
		R	U	[illegible]	U	6	T	→ 12-13
		5	U	[illegible]	U	U		impossible since $q_{TL}(V) = 0$
5/12		5	.	.	5	.		
		5	.	.	5	6		→ 8-8
		5	6	6	[illegible]	U		→ 9-16
		5	6	U	5	U		impossible since $q_{TL}(V) = 0$
		5	U	6	5	U		
		R	U	6	L	U		→ if 423,424,425, or 427 at D → 26-22 if 428 at D → 43-23
		5	U	U	5	U		impossible since $q_{TL}(V) = 0$

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{W} = 3$

Sub-case $\bar{v} = 60$

Arrangement check list, continued

	H	A	B	C	D	E	F	
5/14		5	.	.	.	.		
		R	.	.	.	6^{T}		→ I2-13
		R	6^{T2}	6^{T2}	6	$.^{\times}$		→ I2-24
		R	6^{T}	6	6^{T}	$6^{\times}$		→ 10-9
		R	$.^{\times}$	6^{T2}	6^{T2}	$6^{\times}$		→ I2-23
5/15		.	5	5	5	.		→ 11-4
5/16		.	5	5	.	.		
		6	5	5	.	.		→ 8-8
		U	5	5	6	6		→ 8-13
		U	5	5	6	U		impossible since $q_{TL}(V) = 0$
		U	5	5	U	6		
recall Lemma(60 or 50,T)		U	R	R	U	6	F	→ if T2#7r across E-F → CTL#112 if T1#1 across E-F → CTL#113
		U	5	5	6	U		impossible since $q_{TL}(V) = 0$
5/17		.	5	.	5	.		
		6	5	.	5	[illegible]		→ 8-8
		U	5	6	5	U		
		U	L	6	R	U		→ if 423,424,425, or 427 at B → 26-22 if 428 at B → 4[illegible]
		U	5	U	5	U		impossible since $q_{TL}(V) = 0$.
5/18		.	5	.	.	.		
		6	5					→ 8-8
		U	5	6	6	6		→ 9-24
		U	5	6	6	U		— impossible since $q_{TL}(V) \leq 0$
		U	5	6	U	.		
recall Lemma (60 or 50,T2,T2)		U	5	U	.	.		

Proof of $q_{TL}(V_8)$ - Lemma

Case $\bar{W} = 3$

Sub-case $\bar{v} = 60$

Arrangement check list, finished

	H	A	B	C	D	E	F
5/19		.	.	5	.	.	
		6	6	5	.	.	→ 8-13
		6	U	5	6	U	impossible since $q_{TL}(V) \leq 0$
		6	U	5	U	6	
recall Lemma (T2,L,T2)	^{T2}	6	U	R	U	6^{T2}	→ 8-9
		6	U	5	U	U	impossible since $q_{TL}(V) \leq 0$
		U	.	5	.	U	
5/20		.	.	.	.	.	
		6	6	6	6	6	→ 10-19
recall Lemma (60 or 50,T2,T2)		6	6	6	6	U	
	^{T2}	6^{T1}	6^{T2}	6^{T2}	6	U	→ 12-30
		.	.	6	U	.	
	^{T2}	6^{T1}	6^{T2}	6	U	6^{T2}	→ 8-9
		6	6	U	.	.	impossible since $q_{TL}(V) \leq 0$

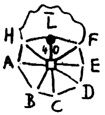
Sub-case $\bar{v} = 40$ (i.e., $L = 421,\ldots,428$)

In this sub-case, no L5- or L6-discharging can go to V, and L7-dischargings can go only from 5-vertices which do not have degree-5-neighbors on the ring about V and are at a distance of at least three (on the ring about V) from the nearest other L-discharging V_5.

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 3

Sub-case $\bar{v}$ = 40



Arrangement check list

	H	A	B	C	D	E	F	
5/1,...,8		5	5					≐(11d) (C11d see p.114)
5/9		5	.	5	5	.		
		5	6	[illegible]	5	.		→ if $\bar{L}$n or $\bar{L}$ = 421r or 422r → 7-27 if $\bar{L}$ = 423r → 27-24
		5	U	5	5	6		→ (11e) (as defined below)
		5	U	5	5	U		
		R	U	R	R	U		→ if $\bar{L}$n → CTL#117 if $\bar{L}$ = 421r, 422r, or 423r → CTL#116
5/10,15		5				5		→ if $\bar{L}$ = 421 or 422 → 7-27 if $\bar{L}$ = 423 → 22-10
5/11		5	.	5	.	.		
		5	6	5	.	.		→ if $\bar{L}$n or $\bar{L}$ = 421r or 422r → 7-27 if $\bar{L}$ = 423r → 27-24
		5	U	I	6	.		impossible since 42[illegible]r at C implies 441 at B, violating the case-hypothesis
		R	U	R	6^{T2}	6^{T}	6	→ 12-26
		R	U	R	6^{T1}	6^{T2}	6	→ 12-26
5/12		5	.	.	5	.		
		5	.	.	5	6		→ (11e)
		5	6	6	5	U		
		R	6^{T2}	6	L	U		≐(11f)
5/14		5	.	.	.	.		
		5	.	.	.	6^{T}	6	≐(11g)
		5	.	.	.	.	×	impossible since $q_{TL}(V) \leq 0$
5/15			5	5	5	.		
		6	5	5	5	.		→ (11e)
		[illegible]	R	R	R	U		≐(11h)

$\bar{L}$ = 421,..., 428

≐ $\boxed{(11e)}$

(critical sub-classes see pp. 114, 115)

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 3

Sub-case [illegible] = 40

Arrangement check list, finished

	H	A	B	C	D	E	F	
5/16		.	5	5	.	.		
		6	5	5	.	.		→ (11e)
		U	R	R	6^{T2}	6^{T}	6	→ I2-26
		U	R	R	6^{T1}	6^{T2}	6	→ I2-26
5/17		.	5	.	5	.		
		6	5	.	5	.		→ (11e)
		U	5	.	5	U		impossible since $q_{TL}(V) < 0$
5/18		.	5	.	.	.		
		6	5	.	.	.		→ (11e)
		U	5	.	.	.		
		U	L	6^{T2}	6^{T2}	6^{T}	6	→ I2-23
		U	L	6^{T2}	6^{T1}	6^{T2}	6	→ I2-29
		U	L	6^{T1}	6^{T2}	6^{T2}	6	→ I2-23
5/19		.	.	5	.	.		
	6^{T2}	6^{T}	6	5	.	.		→ I2-26
	6^{T1}	6^{T2}	6	5	.	.		→ I2-26
								we may assume by symmetry, that the T-dischargings across D-E and E-F add up to no greater value than the T-dischargings across H-A and A-B. Thus the T-dischargings across H-A and A-B must add up to more than 20.
5/20		.	.	.	.	.		by symmetry, the T-dischargings across H-A, A-B, and B-C must add up to more than 40. But by Lemma (T2,T2,T2), they cannot amount to more than 50. Thus the T-dischargings across C-D, D-E, and E-F must add up to more than 30.
	6^{T2}	6^{T2}	6^{T1}	6	6	6		→ I2-33
	6^{T2}	6^{T1}	6^{T2}	6	6	6		→ I2-30
	6^{T1}	6^{T2}	6^{T2}	6	6	6		→ I2-23

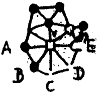
Proof of q_{TL}[illegible] Lemma

Case [illegible] 3

Sub-case [illegible] 40

Critical sub-classes

C11d: [illegible] is 426n



no L-dischargings can go from A or B to V (by sub-case hypothesis)

Arrangement check list for C11d

B	C	D	E	
	5			→ 7-2
	.	5		
	6	5		→ 9-14.
	U	5		
	U	R		→ CTL#118

impossible since $q_{TL}(V) \leq 0$

C11e#1: E is 421n

Arrangement check list for C11e#1

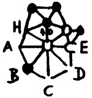
	H	A	B	C	D	E	F	
3/1,....,				5				→ 7-7
3/2r,4r						5		→ 9-17
3/5				.	5	.		
				6	5	.		→ 9-18
				U	5	6		→ 10-6
				U	5	U		
	T2		L	U	R	U		→ I2-12
3/6				.	.	.		
				6	6	6		→ 10-20
				6	6	U		impossible since $q_{TL}(V) \leq 0$
				6	U	.		
	T2			U	6^{T2}	6^{T}		→ I2-12
	T1		L	U	6^{T2}	6^{T2}		→ I2-1

Proof of $q_{TL}(V_8)$ - Lemma

Case # = 3

Sub-case $\bar{v}$ = 40

C11e# 2: L is 426n



Arrangement check list for C11e# 2

H	A	B	C	D	E	
			5			→ 7-7
			.	5		
			6	5		→ 9-18
			U	5		
T2		L	U	R		→ 423r at B → 29-2

impossible since $q_{TL}(V) \leq 0$

Case # = 0

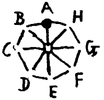
In this case, no L-dischargings go to V. We have to consider all possible arrangements of $5^{\pm}$ and non-5-vertices about V. We agree to denote the neighbors of V in some cyclic order by A,B,C,D,E,F,G,H so that the reading (as a decimal number with "0" for ".") is maximal (as compared with the 15 other possible clockwise or counter-clockwise cyclic readings). We need not consider the arrangement " ...•...." in which V has only non-5-neighbors since by Lemmas (T2,T2,T2) and (T,T2,T2,T), the only possibilities for T-dischargings amounting to more than 60 across four consecutive edges are $6^{T2}6^{T2}6^{T1}6^{T2}6$ and $6^{T2}6^{T1}6^{T2}6^{T2}6$, and the only further possibilities for more than 50 are $6^{T2}6^{T2}6^{T1}6^{T1}6$, $6^{T2}6^{T2}6 \times 6^{T2}6$, $6^{T2}6^{T1}6^{T2}6^{T1}6$, $6^{T2}6^{T1}6^{T1}6^{T2}6$ $6^{T2}6 \times 6^{T2}6^{T2}6$, $6^{T1}6^{T2}6^{T1}6^{T2}6$, $6^{T1}6^{T1}6^{T2}6^{T2}6$; thus, again by Lemmas (T2,T2,T2) and (T,T2,T2,T), no combination of two quadruplets of T-dischargings is possible which amounts to more than 120.

It remains to consider all possible arrangements in which the degree of A is 5. For this purpose, we use the list of 7-digit arrangements of p.46 for

Case [illegible] - 0

the vertices B,C,D,E,F,G,H. We have to consider only those 7-digit arrangements which yield cyclically maximal readings A,B,...,H with "5" for A. These 29 arrangements are marked by an asterisk on p. 46.

We have to consider only those sub-arrangements (specifying "6" or "U" for ".") for which 30(number of 5's, including A) + 20(number of 6-6-edges) > 120

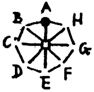


Arrangement check list

	B	C	D	E	F	G	H	
7/1,2,4,6,8	5	5	5	5				→ 7-1
7/10	5	5	5	.	5	5	.	
	5	5	5	6	5	5	.	→ 7-2
	5	5	5	U	5	5	U	→ CTL # 81
7/12	5	5	5	.	5	.	.	
	5	5	5	6	5	.	.	→ 7-2
	5	5	5	U	5	6	6	→ 8-1
	5	5	5	U	5	6	U	→ CTL # 82
	5	5	5	U	5	U	.	→ CTL # 83
7/16	5	5	5	.	.	.	.	
	5	5	5	6	6	6	6	→ 10-5
	5	5	5	6	6	6	U	
	5	5	5	6^{T}	6	6	U	→ I2-6
	5	5	5	$6^{\times}$	6^{T}	6	U	→ I2-10
	5	5	5	6	6	U	.	
	5	5	5	6^{T}	6	U	.	→ I2-4
	5	5	5	6	U	6	6	see above (reading from D to the left
	5	5	5	U	6	6	U	
	5	5	5	U	6^{T}	6	U	→ I2-16

Proof of $q_{TL}(V_8)$ - Lemma

Case # = 0



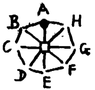
Arrangement check list, continued

	B	C	D	E	F	G	H	
7/17	5	5	.	5	5	5	.	→ 11-1
7/19	5	5	.	5	5	.	.	
	[illegible]	5	6	5	5	.	.	→ 7-3
	5	5	U	5	5	6	6	→ 9-1
	5	5	U	5	5	6	U	→ CTL # 84
	5	5	U	5	5	U	.	→ CTL # 85
7/21	5	5	.	5	.	5	.	
	5	5	6	5	.	5	.	→ 7-2
	5	5	U	5	6	5	.	→ 11-2
	5	5	U	5	U	5	6	→ 7-2
	5	5	U	5	U	5	U	→ CTL # 86
7/23	5	5	.	5	.	.	.	
	5	5	6	5	.	.	.	→ 7-2
	5	5	U	5	6	6	6	→ 10-1
	5	5	U	5	6	6	U	
	5	5	U	5	6	$^{T\rightarrow}6$	U	→ I2-17
	5	5	U	5	6	U	.	impossible since $q_{TL}(V) = 0$
	5	5	U	5	U	6	6	
	5	5	U	5	U	6	$^{T}\dot{6}$	→ I2[illegible]
7/26	5	5	.	.	5	.	.	
	5	5	6	6	5	.	.	→ 9-1
	5	5	.	.	5	6	6	→ 9-1

Proof of $q_{\mathrm{TL}}(V_8)$ - Lemma

Case ▮ = 0

Arrangement check list, continued²



	B	C	D	E	F	G	H	
7/30	5	5	.	.	.	.	.	by symmetry, we need to consider only the non-reflected 5-digit sub-arrangements
5/1	5	5	6	6	6	6	6	→ 10-13
5/2	5	5	6	6	6	6	U	
	5	5	6^{T}	6	6	[illegible]	U	→ I2-6
	5	5	6 ×	6^{T2}	6^{T2}	6	U	→ I2-10
5/3,4	5	5	6	6	6	U	.	
	5	5	6^{T2}	6^{T2}	6	U	.	→ I2-6
5/5	5	5	6	6	U	6	6	
	5	5	6^{T2}	6	U	6^{T2}	6	→ I2-6
5/15	5	5	U	6	6	6	U	
	5	5	U	6^{T2}	6^{T2}	6	U	→ T2T2 across E-F-G → 11-9
7/34	5	.	5	5	.	5	.	
	5	6	5	5	.	5	.	→ 7-3
	5	U	5	5	6	5	6	→ 7-7
	5	U	5	5	6	5	U	→ CTL# 87
	5	U	5	5	U	5	6	→ CTL # 87
	5	U	5	5	U	5	U	→
7/36	5	.	5	5	.	.	.	
	5	6	5	5	.	.	.	→ 7-3
	5	U	5	5	6	6	6	→ 10-2
	5	U	5	5	6^{T}	6	U	→ if T2 → CTL # 91 if T1 → CTL # 92
	5	U	5	5	U	6^{T}	6	see above


→ CTL#90


→ CTL#8


→ CTL#8

Proof of $q_{TL}(V_8)$ - Lemma

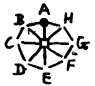
Case # = 0

Arrangement check list, continued[3]

	B	C	D	E	F	G	H	
7/39	5	.	5	.	5	.	.	
	5	.	5	.	5	6^{T}	6	→ I2-9
7/40	5	.	5	.	.	5	.	
	5	.	5	6^{T}	6	5	.	
	5	6	5	6^{T}	6	5	.	→ 10-4
	5	U	5	6^{T}	6	5	U	→ CTL#95
7/42	5	.	5	.	.	.	.	
	5	.	5	6	6	6	6	→ 10-5
	5	.	5	6	6	6	U	
	5	6	5	6^{T2}	6^{T2}	6	U	→ I2-7
	5	U	5	6^{T2}	6^{T2}	6	U	→ CTL #94
	5	.	5	U	6^{T2}	6^{T2}	6	→ I2-22
7/44	5	.	.	5	5	.	.	
	5	6^{T}	6	5	5	.	.	→ I2-9
	5	$.^{\times}$	.	5	5	6^{T}	6	→ I2-9
7/47	5	.	.	5	.	.	.	
	5	6^{T}	6	5	.	.	.	→ I2-9
	5	$.^{\times}$	.	5	6^{T2}	6^{T2}	6	→ I2-26
7/52	5	.	.	.	.	.	.	
	5	6	6	6	6	6	6	→ 10-17
	5	6	6	6	6	6	U	
	5	6^{T2}	6^{T2}	6^{T1}	6^{T2}	6	U	→ I2-33
	5	6^{T2}	6^{T1}	6^{T2}	6^{T2}	6	U	→ I2-30
	5	U	6	6	6	6	6	see 3 lines above (with direction of Tending reversed)

Proof of $q_{TL}(V_8)$ - Lemma

Case [illegible] = 0



Arrangement check list, finished

	B	C	D	E	F	G	H	
7/62	.	5	.	5	.	5	.	impossible since $q_{\dot{TL}}(V) = 0$
7/63	.	5	.	5	.	.	.	
	.	5	.	5	6^{T2}	6^{T2}	6	→ I2-26
7/64	.	5	.	.	5	.	.	
	.	5	6^{T2}	6	5	6^{T2}	6	→ 10-6
7/66	.	5	.	.	.	.	.	
	.	5	6	6	6	6	6	
	.	5	6^{T2}	6^{T2}	6^{T1}	6^{T2}	6	→ I2-34
	.	5	6^{T2}	6^{T1}	6^{T2}	6^{T2}	6	→ I2-34
7/70	.	.	5	.	.	.	.	
	6	6	5	6	6	6	6	
	6^{T2}	6	5	6^{T2}	6^{T2}	6^{T1}	6	→ I2-23
	6^{T2}	6	5	6^{T2}	6^{T1}	6^{T2}	6	→ I2-29
	6^{T2}	6	5	6^{T1}	6^{T2}	6^{T2}	6	→ I2-[illegible]
7/71	.	.	.	5	.	.	.	by symmetry we may assume that the T-disch[illegible] across B-C and C-[illegible] amo[illegible] more than 30
	6^{T2}	6^{T2}	[illegible]	5	.	.	.	→ I[illegible]
7/72	.	.	.	.	.	.	.	by symmetry we may assume that the T-dischargings across B-C, C-D, and D-E amount to more than 40; since they cannot amount to more than 50 it follow that the T-dischargings across E-F, F-G, and G-H must also amount to more than 40
	6^{T2}	6^{T2}	6^{T1}	6	6	6	6	→ I2-36
	6^{T2}	6^{T1}	6^{T2}	6	6	6	6	→ I2-30
	6^{T1}	6^{T2}	6^{T2}	6	6	6	6	→ I2-2[illegible]

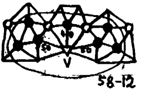
This finishes the proof of the $q_{TL}(V_8)$ - Lemma. ■

Proof of the $q_{TL}(V_9)$ - Lemma.

Case $\bar{v} \geq 6$

Sub-case $\bar{v} \dot{>} 110$ (i.e., L = 701,...,718, 901,...,910)

(L = 702) (v = 7)

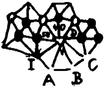


or

$\doteq$ $\boxed{(12a)}$

Critical sub-classes

C12a#1: L is 712 or 713



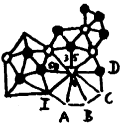
Arrangement check list for C12a#1

	I	A	B	C	D
3/1,...,4		5			→ 58-10
3/2r,4r				5	→ 1-1
3/5		.	5	.	
		.	5	6	→ 58-12
		.	5	U	no L5+ can be attached at D
recall Lemma (6Q or 5Q T)	T	6	5	U	if T2#7n across I-A → 34-22 if T1#1 across I-A → 1-3
		.	5	U	impossible since no L6 can be attached at B
3/6		.	.	.	
		.	.	6	→ 58-12
		.	.	U	no L5+ can be attached at D; thus impossible ($q_{TL} < 0$)

Proof of q_{TL}([illegible]a

Case [illegible]

Sub-case $\bar{v} > 110$



C12a#2: L is 715

Arrangement check list for C12a#2

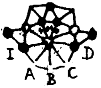
	I	A	B	C	D	
3/1,2		5	5			→ 36-6
3/2r			5	5		→ 36-13
3/3		5	.	5		
		L	.	5		→ if V_5 adjacent to I and A → 1-3 if V_6 adjacent to I and A → 1-6 if 422n,476r,...,478r at A → 2-6
		R	.	L		→ if V_5 adjacent to C and D → 36-12 if 437 at C → 35-6 if 530 at C → 1-3
		R	.	R	L	→ 433r or 434r at D → 36-12
3/4		5	.	.		
		5	6	6		→ 13-4
		L5+	. ×	..		→ 411,441r,491r,...,495r at A → 1-3
		L4	. ×	.	L6	impossible since no L6 can be attached at D
3/4r		.	.	5		
		6	6	5		→ 36-10
	T2	6	U	[illegible]		→ 926n,...,929n at C,D by Lemma(L,L) → 18-2
3/5		.	5	.		
	T2	6	5	.		→ T2#7n across I-A by Lemma(60 or 50,T[illegible] → 34-22
		.	5	.	L5+	→ 441r at D → 58-12
		.	L5+	.		→ if 441r at B → 34-14 if 491,...,493, or 495 at B → 36-[illegible]
3/6		.	.	.		
		.	.	.	L5+	→ 441r at D → 58-12

Proof of $q_{TL}(V_9)$ - Lemma

Case # ≥ 6

Sub-case $\bar{v} > 110$

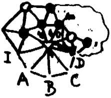
C12a # 3: L is 908.



Arrangement check list for C12a # 3

	I	A	B	C	D	
3/1,...,4		5				→ 58-33
3/5		.	5	.		
		6	5	.		→ 38-2
		U	5	U		
	L	U	5	U		→ 532 at I → 58-1
	R	U	L5+	U	R	impossible since no L5+ can be attached at B
3/6		.	.	.		
	L5+	.	.	.		→ 441 at I → 58-31
	L4	6^{T2}	6^{T}	6		→ if L5+ at D → 441 at D → 58-31 if no L5+ at D or I → 13-5[illegible]

Sub-case $\bar{v} = 110$ and L one of 921,...,925



Arrangement check list

	I	A	B	C	D	
3/1,...,4		5				≐(12b)
3/2r,4r				5		≐(12c)
3/5		.	5	.		
		$^{T2}6$	L	6^{T2}		→ 411 at B by Lemma(T2,L,T2) → 12-10
		$^{T2}6$	L6	.		→ 411 or 491 at B → 12-10
		.	L6	6^{T2}	6	→ if 411 at B → 12-10 if 491 at B → I2-20
		$^{T2}6$	L5	6^{T1}	6	impossible since no L5 can be attached at B
		$^{T1}6$	L5	6^{T2}	6	
3/6		.	.	.		impossible since $q_{TL}(V) \leq 0$

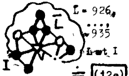
Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] $\geq$ 6

Sub-case $\hat{v}$ = 110 and $\bar{L}$ is one of 926,...,935.

Arrangement check list

	I	A	B	C	D	
3/1,2		5	5			→ if $\bar{L}$ = 926,...,930 → 36-22 if $\bar{L}$ = 931,...,935 → 35-1
3/2r		.	5	5		$\doteq$ (12d)
3[illegible]		5	.	5		
	L	5	.	5		→ (12e)
	R	5	6	5		→ if $\bar{L}$ = 926,...,930 → 38-1 if $\bar{L}$ = 931,...,935 → 35-4
	R	5	U	5		
	R	5	U	5		$\doteq$ (12f)
	R	R	U	L5+		→ 441 at C → 35-28
3/4		5	.	.		
	L	5	.	.		→ (12e)
	R	5	.	.		impossible since $q_{TI}(V) \leq 0$ by Lemma(5,L,T2)
3/4r		.	.	5		no L6 can be attached at I
	L5	6	6	5		→ 441 at I → 35-22
	L5	6	U	L6		→ 441 at I and 491 at C → 1-3
	L5	U	.	L6		impossible since no L5 can be attached at I
	L4	6^{T}	6	L5+		→ 411 attached at C → 14-25
	R	6^{T2}	6	L6		
3/5		.	5	.		
	L5+	.	5	.		→ 441n at I → 35-18
		.	L6	.		→ 411 or 491r at B → if $\bar{L}$ = 926,...,930 → 38-7 if $\bar{L}$ = 931,...,935 → 35-6
	L4	.	L5	6^{T2}	6	→ 441 at B → if $\bar{L}$ = 926,...,930 → 58-14 if $\bar{L}$ = 931,...,935 → 58-21
3/6	.	.	.	.		impossible since no L6 can be at I

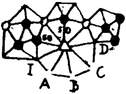


$\doteq$ $\boxed{(12e)}$

Proof of $q_{TL}(V)$-Lemma

Case [illegible] ≥ 6

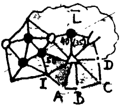
Sub-case $\bar{v}$ = [illegible] (i.e. L = 720)



Arrangement check list

	I	A	B	C	D	
3/1,2		5	5			→ 4-7
3/2r		.	5	5		no L5+ at B or C
	T	6	5	5		→ 1-15
3/3		5	.	5		
		L5+	.	5		→ 411 or 141 at A → 1-[illegible]
		5	.	L5+		→ 411 or 141 at C → 1-[illegible]
3/4		5	.	.		
		L5+	6^T	6^T		→ 411 at A → 1-[illegible]
3/4r		.	.	5		
	T	6^T	6	L5+		→ 411 at C → 1-[illegible]
3/5		.	5	.		
recall Lemma (60 or 50,T)	T2	6	L5+	6^T		impossible by Lemma(60 or 50, T, 60 or 50)
	T1	6	L6	6^{T2}		→ 411 at B and T2#7n across C-D → 1-11
3/6		.	.	.		- impossible since $q_{TL}(V) \le 0$

Sub-case $\bar{v}$ = 90 or 85 (i.e., L = 721,...,728)



In this sub-case, at least two of the vertices A, B, C (see drawing to the right) must be of degree 5 since otherwise $q_{TL}(V)$ would be not positive. (For a positive $q_{TL}(V)$ with only one V_5 at A, B, or C we would have to attach an L6 and two T2 which is ruled out by Lemmas (60 or 50,T2,T2) and (60 or 50,T,60 or 50).)

Proof of $q_{TL}(V)$ - Lemma

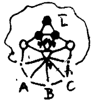
Case [illegible] ≥ 6

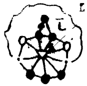
Sub-case $\bar{v}$ = 40 or 45

Arrangement check list

	I	A	B	C	D	
3/1,2		5	5			→ 4-7
3/2?				5		no L5+ can be attached at B or C
	T	6	5	5		→ 34-45
?/3		5	.	[illegible]		no L5+ can be attached at A, by Lemma (60 or 50,..,60 or 50)
		L4	.	L6		→ 411 at C → 1?-?

Sub-case $\bar{v}$ = 80 (i.e., L = 951,...,977)





In this Sub-case, all three vertices A, B, C (see drawing to the right) must be of degree 5 since otherwise $q_{TL}(V)$ would be not positive. For a positive $q_{TL}(V)$ with only two 5-vertices at A, B, or C we would have to attach an L6 and an L5+ (ruled out by Lemma(60or50, . ,60 or 50)) or an L5+, and a T (impossible by Lemma(5,L)). Thus the configuration class (12g) (as defined to the right) must occur. (See class check list.)

Sub-case $\bar{v}$ = 70 (i.e., L = 729 or 730)

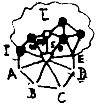
This sub-case cannot occur since $q_{TL}(V)$ would be not positive. If three 5-vertices are attached to V (besides those belonging to $\bar{L}$), then at most two L-dischargings (besides the one along the discharging edge of $\bar{L}$) could go to V but by Lemma(5,L), none of them could be L5+. If two 5-vertices are attached then at most one L5+ but no T can go to V, or one T but no L5+. The maximal $q_{TL}(V)$ possible with only one V_5 attached would be -10 (if an L6 and two T2 are attached).

Proof of $i_{TL}(T_L)$ = Lemma

Case [illegible] = 5.

Sub-case $\hat{v}$ = 120 or 115 (i.e., L = 530, 811, 812, or 813).

Arrangement check list



	I	A	B	C	D	E	
4/1,...,7		5					→ 36-[illegible]
4/2r,...,5r,7r					5		→ 35-18
4/8		.	5	5	.		
		6	5	5	.		→ 38-[illegible]
		U	5	5	6		→ 5-32
		U	5	5	U		
	L	U	5	5	U		→ if 433,434, or 435 at I → 58-11 if 436 at I → 58-24 if 437 or 530 at I → L = 530 or 813 → 2-
	R	U	L	5	U		→ if 433, ... , 436 at B → 35-28 if 437 at B → 58-24
	R	U	R	L	U		→ if 451,...,460 at C → 35-28 if 461 at C → 1-3 if 462 at C → 2-2 if 463 at C → 1-2
4/9		.	5	.	.		
		6	5	.	.		→ 38-17
		U	5	6	6		→ 35-25
		U	5	.	U		impossible since no L6 can be attached at B and no L5+ at I
recall Lemma (60 or 50,T)		U	5	U	6	T	→ if T2#7r across D-E → 35-27 if T1#1 across D-E → 35-20
4/9r		.	.	5	.		
		. .	.	5	6		→ 35-22
		6	6 ·	5	U		→ 38-26
		. ×	.	L6	U		→ if 491 at C → 59-4 if 621 at C → 1-2

continued next page

Proof of $q_{TL}(V)$ - Lemma

Case [illegible] = 5

Sub-case $\bar{v}$ = 120 or 115

Arrangement check list, finished ,

	I	A	B	C	D	E	
4/10		.	.	.	.		
		.	.	.	6	T	→ if T2 # 7r across D-E → 35-27 if T1 # 1 across D-E → 35-20
		.	.	.	.	×	impossible since $q_{TL}(V) \leq 0$

Sub-case $\bar{v}$ = 110 or 105 (i.e., $\mathfrak{L}$ = 820,...,835)

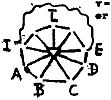
In this sub-case, we consider first the possibility that an additional L5+ is attached to V so that one of the 5-vertices which belong to $\mathfrak{L}$ becomes L5- or L6-discharging to V. Then a configuration of the class (13a) occurs (see drawing to the right and the corresponding class check list).

$\mathfrak{L}$ = 828n, ..., 833.
L5+ at E
≐ $\boxed{(13a)}$
↳ 441 at E

Second we consider the possibility that a T2-discharging goes to V across a 6-6 edge one of whose vertices belongs to $\mathfrak{L}$. In this case a configuration of the class (13b) occurs (see drawing to the right).

$\mathfrak{L}$ = 827, ..., 826, 834, 835
≐ $\boxed{(13b)}$
(820, 822, ..., 833 are ruled out by Lemma (5,L,T2))

In the remainder of this sub-case, we assume that no configuration of one of the classes (13a), (13b) occurs.



v = 110 or 105

no L5+ at I or E
no T2 across I-A or D-E

<u>Arrangement check list</u>

	I	A	B	C	D	E	
4/1,...,7		5					≐ (13c)
4/8		.	5	5	.		
		6	5	5	.		→ (13d) (see p.129)
		U	5	5	U		
		U	L	5	U		≐ (13e)

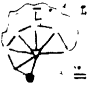
continued next page

Proof of $q_{TL}(V)$-Lemma

Case V = 5

Sub-case $\bar{V}$ = 510 or 1.5

Arrangement check list, finished



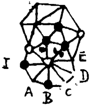
L = 820, ..., 835

≐ $\boxed{(13a)}$

	I	A	B	C	D	E	
4/9		.	5	.	.		
		6	5	.	.		→ (13d) critical sub-classes see below
		U	5	6	6		≐ (13f)
		U	5	.	U		impossible since $q_{TL}(V) \leq 0$
		U	5	U	6		
	L4	U	L6	U	6^{T1}	6	impossible since no L5 can be attached at B
4/10		.	.	.	.		impossible since $q_T(V) \leq 0$

Critical sub-classes

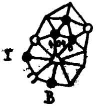
C13d: L is 832r



Arrangement check list for C13d

I	A	B	C	D	E	
			5			→ 58-5
			.	5		→ 60-8
			.	.		
			6	6		→ 14-3
			.	U		impossible since $q_{TL}(V) \leq 0$
			U	6		
L4		L6	U	6^{T1}		→ 491 at B → 58-7

C13f#1: L is 831r, 832r, or 833r



→ if L at I → if 532 at I → 58-1

→ if L at I → if 533n at I → 58-4

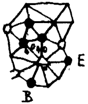
if L5t at B → 441 at B → 5[illegible]14

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 5 .

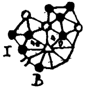
Sub-case $\bar{v}$ = 110 or 105

C10f # 2: $\bar{L}$ is 832n



→ L5+ at B → 491,...,493, or 495 at B → 38-19

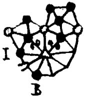
C13f # 3 : $\bar{L}$ is 834n



no L can be attached at I since if one of 433,...,436 were attached at I then a pair of width w = 6 (as considered in Lemma(ℓ^2)) would be attached, contradicting the case-hypothesis

→ L5+ at B → 441 at B → 35-25

C13f # 4: $\bar{L}$ is 835n

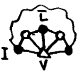


→ if L at I → 530r at I → 2-1

→ if L5+ at B → ~~441 at~~ B → 35-25

Sub-case $\bar{v}$ = 100 or 95 (i.e., $\bar{L}$ = 531,...,~~541~~, 841,...,843)

First we consider the possibility that an additional L5+ is attached to V so that one of the 5-vertices which belong to $\bar{L}$ becomes L5- or L6-discharging to V. Then a configuration of the class (13g) occurs (see drawing to the right). The class (13g) has two critical subclasses which are treated next.

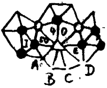


$\bar{L}$ = 532,
...,
541
L5+ at I
≐ $\boxed{(13g)}$

Proof of $q_{TL}(7)$ - Lemma

Case $\bar{V}$ = 5

Sub-case $\tilde{v}$ = 100 or 95



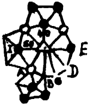
C13g#1: is 538 or 559n and 441 is attached at I

Arrangement check list for C13g#1

	I	A	B	C	D	E	
3/1,...,4			5				→ 58-10
3/2r,4r					5		→ 1-1
3/5			.	5	.		
			.	5	6		→ 58-12
			6	5	U		
recall Lemma (60 or 50,T)		T2	6	5	U		→ Ť2#7n across A-B → 1-5
		T1	6	L5+	U	L4	→ 441 at C → 34-14
			U	5	U		impossible since $q_{TL}(V) \leq 0$
3/6			.	.	.		
			.	.	6		→ 58-1
			.	.	U		impossible since $q_{TL}(V) < 0$

C13g#2: L is 532 and one of 551, 552, 553 is attached at I

Arrangement check list for C13g#2



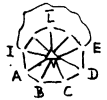
I	A	B	C	D	E	
				5		→ 58-1
			5	.		
			L5+	.	L	→ 411 or 441 at C → 1-2
			.	.		impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 5

Sub-case $\bar{v}$ = 10 or 25

For the remainder of this sub-case we assume that no configuration of class (1 g) occurs (i.e., that no L5+ can be attached at I or E).



Arrangement check list

I A B C D E

4/1,...,7 5 $\doteq$ (13h)

4/8 . 5 5 .

- 6 ^{T} 6 5 5 . → (13i)
- L4 . L4 L4 . → the L4 at B is of width 3 or 4 (no others can be attached); **thus the**
- L4 . L4 5 . L4 → L4 at I must be of width 5 (otherwise a contradiction against the case-hypothesis would arise); thus the L4 at I and the V_5 at C form a configuration of the class (13h) . (as defined above);

4/9 . 5 . . .

$\bar{L}$ = 531,841,...,843

- 6 ^{T} 6 5 . . → (13i) + $\doteq$ $\boxed{(13i)}$
- L4 . L5+ •6 ^{T} 6 → the L5+ at B is of width 3 or 4; thus, by case-hypothesis, the L4 at I must be of width 5; but then a configuration of the class (13g) would occur.
- L4 . L6 . × 6 ^{T2} 6 → $\bar{L}$ = 531 → I4-14

recall Lemma (60 or 50,T2,T2)

- × . L6 6 ^{T} 6 ^{T} 6 $\doteq$(13j)
- × . L6 6 ^{T2} 6 L4 → { if $\bar{L}$ = 531r → 411 or 491 at B → 12-10
  - if $\bar{L}$ = 532,...,541 → 411 at B → 13-2

4/10 •• . . . .

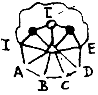
- 6 ^{T2} 6 ^{T2} 6 ^{T1} 6 ^{T2} 6 ^{T2} 6 — **impossible by Lemma(60 or 50,T2,T2) (since $\bar{L}$ would have to be 841,...,843)**
- L4 6 ^{T} 6 ^{T} 6 ^{T} 6 ^{T} 6 **impossible since $q_{TL}(V) \leq 0$**
- L4 6 ^{T} 6 ^{T} 6 ^{T} 6 L4

Proof of $q_{TL}(V_0)$-Lemma

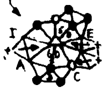
Case $\ell = 5$

Sub-case $\bar{v}$ = 90 or 85 (i.e., ℓ = 544,550,851,...,854)

Arrangement check list



	I	A	B	C	D	E	
4/1,...,7		5					≐(13k)
4/8		.	5	5	.	.	
		6	5	5	6	,	→ 34-15
		.	5	5	U		impossible since $q_{TL}(V) \leq 0$, by Lemmas (5,L),(5,L,T2)
4/9		.	5	.	.		
		6	L5+	6	6		→ 411 at B → 43-5
	T2	6	L4	6^{T2}	6^{T2}		Impossible by Lemma (T2,L,T2)
	T2	6	L6	U	6^{T2}		→ L = 851r,...,854r, 549n, or 550n (by Lemma(60 or 50,T,60 or 50)) and 491 at B



→ |-2|

(by Lemma(T‖T), see p.13)

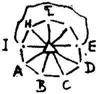
4/10 impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 5

Sub-case $\bar{v}$ = 80 or 75 and $\mathfrak{L}$ = 861,...,894

Arrangement check list



[illegible] A. B C D E

4/4,..., 5 5 ≙ (13ℓ) (critical sub-classes see p.135)

4/5 5 . 5 .

L5+ . L5+ . impossible since by Lemma(60 or 50, . ,60 or 50) and sub-case hypothesis, 551 or 552 had to be at C and 441 at A, but then $q_{TL}(V) \leq 0$.

L6 . 5. 6 ^{T} 6 → 411 at A →
- if $\mathfrak{L}$ = 861n,...,871n → 12-16
- if $\mathfrak{L}$ = 861r,...,871r, or 873,...,894 → 13-4

L5 . L4 6 ^{T2} 6 impossible since the only L5 which can be attached at A is 441; but 441r together with the L4 at C would form a pair of width 5 and v > 80, violating the sub-case hypothesis; similarly, 441n and $\mathfrak{L}$ = 878r or 885r would violate the sub-case hypothesis (by containing 843).

L4 . L5+ 6 ^{T} 6 impossible since the L5+ at C had to be 411 or 441n and the L4 at A had to be of width 3 or 4 (since width 5 and v > 80 is ruled out by sub-case hyp.); but then the L4 at A would be part of a pair of width 5 and v > 80 violating the sub-case hypothesis (the pair being completed either by the L5+ at C or by a 441r at E if $\mathfrak{L}$ = 878r or 885r).

R . L6 6 ^{T2} 6 → 411 at C →
- if $\mathfrak{L}$ = 861n,...,871n → 13-1
- if $\mathfrak{L}$ = 861r,...,872r, or 873,...,894 → 13-2

4/6 5 . . 5

5 6 6 5

→ L5+ 6 6 5 → 411 at A (since 441n at A and $\mathfrak{L}$ = 878r or 885r would yield a contradiction against the sub-case hypothesis) →
- if $\mathfrak{L}$ = 861n,...,872n or 873,...,894 → 13-2
- if $\mathfrak{L}$ = 861r,...,874r → 13-18

5 6 U 5

→ L5+ 6 U L5+ → 411 at A →
- if 49[illegible],...,493, or 495 at D → 38-29
- if 551 at D → 1-2

Proof of $q_{TL}(V_5) =$ [illegible]

Case $\bar{W} = 5$

Sub-case $\bar{V} = 8$[illegible] or 75 and [illegible] = 861,...,

Arrangement check list, finished

I A B C D E

4/7 $\overset{}{L6}$ $\overset{5}{6}\ {}^{T}\ \overset{.}{6}\ {}^{T}\ \overset{.}{6}\ {}^{T2}\ \overset{.}{6}$ $\doteq$ **(13m)** (411 at A → L = 861r,...,884r, or 891,...,894 since otherwise violation of sub-case hypothesis)

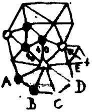
4/8 . 5 5 . impossible since $q_{TL}(V) \leq 0$ by Lemmas(5,L) and (5,L,T2)

4/9 $\overset{.}{6}\ {}^{T2}\ \overset{}{6}\ \ L6\ \ \overset{.}{6}\ {}^{T}\ \overset{.}{6}\ {}^{T}\ 6$ → 411 at A → { if L = 861n,...,871n → 12-22; if L = 861r,...,871r, or 873,...,894 → 13-5

4/10 . . . . impossible since $q_{TL}(V) < 0$

Critical sub-classes

C15ℓ#1: L is 870n



no L at A by Lemma (5,L,.,L,5)

Arrangement check list for C1ℓ#1

B C D E

5 → 12-7

. 5 → 37-7

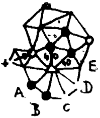
L4 $\overset{.}{6}\ {}^{T2}\ \overset{.}{6}\ {}^{T2}\ 6$ impossible by Lemma(5,L,T2)

Proof of $q_{TL}(V)$ - Lemma

Case W = 5

Sub-case $\bar{v}$ = 80 or 75

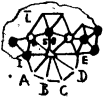
C13ℓ # 2: $\bar{L}$ is 870r



Arrangement check list for C13ℓ # 2

A	B	C	D	E	
		5			→ 38-5
		.	5		→ 47-26
		.	.	.	
L4	L4	6^{T1}	6^{T2}		→ I4-13

Sub-case $\bar{v}$ = 80 and $\bar{L}$ = 551,...,553



Arrangement check list

	I	A	B	C	D	E	
4/1,		.	5	5			→ 54-1
4/2r		.	5	5	5		→ 54-26
4/3		5	5	.	5		no L can be attached at I or A
		5	L4	.	L		impossible since the L's at B and D would form a pair of width 5 and v > 80 violating the sub-case hypothesis.
		5	R	.	L5+		→ 411 or 441 at D → 1-2
4/3r		5	.	5	5		no L can be attached at I
		L4	.	5	5		→ if 401,...,405,433,...,436 at A → 1-1 if 437 at A → 2-1
		R	.	L4	L4		→ if 401 at D → 1-3 if 402 or 403 at D → 2-2
4/4		5	5	.	.		impossible since $q_{TL}(V) \leq 0$ (no L can be attached at I or A)
4/4r			.	5	5		
	L5	6^{T1}	6	L4	L4		→ if 401 at D → 1-3 if 402 or 403 at D → 2-2
4/5		5	.	5	.		no L can be attached at I
		L4	.	L	6^{T}		impossible since violating the sub-case hypothesis
		R	.	L6	6^{T2}		→ 411 at C and T2#7n across D-E by Lemma(60 or 50,T) → 1-11

Proof of $q_{TL}(V_7)$ - Lemma

Case # = 5

Sub-case $\bar{v}$ = 60 and Σ = 51,...,[illegible]

Arrangement check list, finished

	I	A	B	C	D	E
4/5r		.	5	.	5	
	L5	.	L6	.	5	→ 441 at I and 111 at B → 1-20
	L5	.	L5	.	L	→ 441n at I and 441 r at B → 1-20
		.	5	.	L5+	→ 111 or 111 at D → 1-2
4/6		5	.	.	5	no can be attached at I
		5	6^{T}	6	L6	→ 411 at D → 1-8
4/7		5	.	.	.	impossible since $q_{TL}(V) < 0$
4/7r		.	.	.	5	
	L5	6^{T}	6^{T}	6	L6	→ 411 at [illegible] 1-2
4/8		.	5	5	.	impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,T2), (5,L,L,60 or 50)$^{+}$ and sub-case hypothesis
4/9		.	5	.	.	impossible since $q_{TL}(V) \leq 0$ by Lemmas (60 or 50, . ,60 or 50), (60 or 50,T2,T2)
4/9r		.	.	.	.	
	L	6^{T}	6	L6	6^{T2}	→ 411 at C and T2#7n across D-E → 1-11
4/10	.	.	.	.	.	impossible since $q_{TL}(V) < 0$

Sub-case $\bar{v}$ = 70 or 65 (i.e., Σ = 501,...,620)

In this sub-case, we have to consider only those arrangements of 5- and non-5-neighbors of V in which at least two 5-neighbors are attached to V besides the two V_5's contained in Σ. For otherwise $q_{TL}(V)$ would be not positive. If only one V_5 were attached then it could discharge at most 60 to V and in addition at most two T2-dischargings could go to V. Furthermore an L4-discharging could go from the boundary-V_5 of Σ to V or, if Σ = 620, another T1-discharging could go to V. This could result in $q_{TL}(V) = 0$ but in no positive charge. No L6 can be attached to the boundary-V_5 of any Σ and if an L5 is attached then only

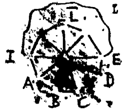
Proof of $q_{TL}(V_9)$ - Lemma

Case $\bar{W} = 5$

Sub-case $\bar{v} = 70$ or 65

one T2-[illegible] discharging could go to V, by Lemma(60 or 50,T,60 or 50)

(or the V not belonging to [illegible] could not be L6-discharging).



$L \to 561,$ 620

(no L6 can be attached at I or E)

Arrangement check list

	I	A	B	C	D	E	
4/1		5	5	5	5	$\doteq(13n)$	
4/2		5	5	5	.		
		5	5	5	6		
recall Lemma (5,L,T2)	L4	5	5	5	6	$\overset{T}{.}$ 6	if 433,...,435 at I → 34-8 if 436 or 437 at I → 36-1
	.	L4	5	L4	6		impossible since the L4's at A and C, by Lemma (L,5,L), would form a pair violating the sub-case hypothesis
	.	L4	5	R	.	L5 → 441 at E → 34-7	
	.	L4	5	R	6	^{T2} 6 → L = 620 →	if L = 620n → 12-7 if L = 620r → 13-9
		5	.	.	U		
recall Lemma (5,L,.,60 or 50)†	.	L4	R	L4	U	L4	impossible since the L4's at A and C would form a pair violating the sub-case Hypothesis, by Lemma(L,5,L).
		L4	R	.	U	L.	impossible since no L6 can be attached at E
4/3		5	5	.	5		
		5	5	6	5		
	L4	5	L4	6	5		impossible since the L4's at I and B would form a pair violating the sub-case hyp.
	L4	5	R	6	L5+ → 411 at D → 12-8		
	.	L4	L4	6	R	L4	if 433,...,435 at E → 34-12 if 436 or 437 at E → 36-29
	.	5	5	6	5	.	if L = 620n → 13-12 if L = 620r → 13-11

⋮

Proof of $q_{TL}(V)$ - Lemma

Case V = 5

Sub-case $\bar{v}$ = 70 or 65

Arrangement check list, finished

	I	A	B	C	D	E	
		5	5	U	5		
	L4	R	L4	U	5		impossible since the L4's at I and B form a pair violating the sub-case hypothesis
	L4	R	R	U	L5		→ 141 at D → 6-35
		L4	L4	U	5		impossible since by Lemma (L,L) and sub-case hypothesis, no pair of L4's can be attached at A,B
		L4	R	U	L5		→ I and E are non-5, 441 at D → {if Ī = 620n → 38-17 if Ī = 620r → 35-18
		R	L4	U	L5		impossible by Lemma (5,L, , ,60 or 50) and sub-case hypothesis
		5	5	U	L6		impossible since no L6 can be attached at D
		5	5	U	L4	L4	impossible by Lemma (L,L) and sub-case hypothesis
4/4		5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2)
4/5		5	.	5	.		
	L4	L4	.	5	.		impossible since the L4's at I and A would form a pair violating the sub-case hyp.
	L4	R	.	L6	6^{T2}	6	→ 411 at C and T2#7n → 12-27
	R	L4	.	L6	6^{T2}	6	impossible by Lemma (5,L, . ,60 or 50)$^{+}$
	.	.	.	L	6^{T}	6	impossible since Ī had to be 620 and the two L's at A and C would form a pair violating the sub-case hypothesis
recall Lemma (60 or 50,., 60 or 50)	.	L6	.	L4	.	L5	impossible since the two L's at A and C would form a pair violating the sub-case hypothesis
	.	L5	.	L5	.	L5	
4/6		5	.	.	5		
	L4	5	6^{T2}	6	L6		→ 411 at D → 12-22
	.	L5+	6^{T1}	6	L5+		→ Ī - 620 and 411 at A and at D → 13-6''
	.	L6	$.^{\times}$	.	L6		
4/8		.	5	5	.		impossible since $q_{TL}(V) \leq 0$, by Lemma(5,L,T2)

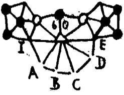
Proof of "$q_{TL}(V_9)$ - Lemma

Case $\bar{v}$ = 5

Sub-case $\bar{v}$ = 60 (i.e., L = 621)

In this sub-case we have to consider only those arrangements in which at least two V_5's are attached to V in addition to L since otherwise $q_{TL}(V) < 0$.

Arrangement check list



	L	A	B	C	D	E	
4/1		5	5	5			→ 34-26
4/3		5	5	.	5		
recall Lemma (5, L, . , 60 or 50)+		L4	R	.	L6		→ 411 at D → 1-2
4/4		5	5	.	.		impossible since $q_{TL}(V) < 0$
4/5		5	.	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (60 or 50, . , 60 or 50
4/6		5	.	.	5		
recall Lemma (60 or 50, T, 60 or 50)		L6	6^{T1}	6	L6		→ 411 at A and at E → 13-6
4/8		.	5	5	.		impossible since $q_{TL}(V) \leq 0$

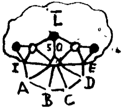
Sub-case $\bar{v}$ = 50 (i.e., L = 622,...,624)

In this sub-case, we have to consider only those arrangements in which at least three V_5's are attached to V in addition to L since otherwise $q_{TL}^{*}(V)$ would be not positive. If only two V_5's were attached then their combined L-dischargings could amount to at most 120, but the value of 120 could be achieved only if the V_5's were at distance three (on the ring around V) by Lemma(60 or 50, . , 60 or 50), and in this case no T2-discharging could go to V by Lemma(60 or 50, T , 60 or 50).

Proof of $q_{TL}(V_4)$ - Lemma

Case $\bar{L}$ = 5

Sub-case $\bar{v}$ = 50



Arrangement check list

	I	A	B	C	D	E
4/1		5	5	5	5	→ A-5
4/2		5	5	5	.	impossible since $q_{TL}(V) \leq 0$
4/3		5	5	.	5	impossible since $q_{TL}(V) \leq 0$ by Lemma (5, L, .., 60 or 50)$^+$.

Sub-case $\tilde{v}$ = 40 or 35 (i.e., $\bar{L}$ = 651,...,691) .

This sub-case is impossible since $q_{TL}(V)$ would not be positive. If four V_5's are attached to V in addition to $\bar{L}$ then at most two of them can be L-discharging but no L5+ is possible. If three consecutive V_5's are attached then their combined discharging to V can be at most 110 and one T-discharging can go to V. If three non-consecutive V_5's are attached then the combined discharging cannot exceed 140 and no T-discharging is possible. If only two V_5's are attached then either their combined dischargings can amount to 120 and one T-discharging can go to V, or, if the V_5's are consecutive, their dischargings can amount to 80 and two T-dischargings are possible.

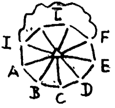
Proof of $q_{\Gamma_L}(V_9)$ - Lemma

Case [illegible] = 4

In this case, no boundary-V_5 of the L-situation $\mathcal{L}$ is L-discharging to V since otherwise $\mathcal{L}$ and the L-situation inducing the L-discharging from the boundary-V_5 would form a pair of width 5 or 6 violating the case-hypothesis.

Sub-case $\bar{v}$ = 100 or 95 (i.e., $\mathcal{L}$ = 431,...,437)

Arrangement check list



	I'	A	B	C	D	E	F	
5/1,...,8		5	5					≐(14a)
5/9		5	.	5	5	.		
		5	6	5	5	.		→ (14h) (see next page)
		5	U	5	5	6		→ (14k) (see page 144)
		5	U	5	5	U		
		L	U	5	5	U		≐(14d)
		R	U	L	[illegible]	U		≐(14e) (one of 433,...,437 at C)
		R	U	R	1	[illegible]		≐(14f)
		R	U	R	R	U		

if $\mathcal{L}$ = 431n or 432n → CTL # 147
if $\mathcal{L}$ = 431r or 432r → CTL # 146
if $\mathcal{L}$ = 433n,...,437n → CTL # 149
if $\mathcal{L}$ = 433r,...,436r → CTL # 148
if $\mathcal{L}$ = 437r

see next page

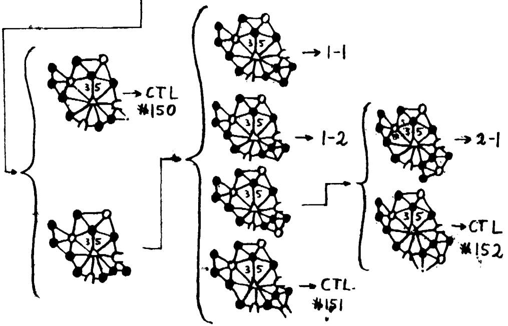
Proof of $q_{TL}(V_4)$ - L [illegible]

Case [illegible] = 4

Sub-case $\bar{v}$ = 100 or [illegible]

Arrangement R U R R U

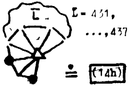
if $\bar{L}$ = 437r



Arrangement check list, continued

	I	A	B	C	D	E	F	
5/10,13		5	.			5		≐(14g)
5/11		5	.	5	.	.		
		5	6	5	.	.		→(14h)
		5	U	[illegible]	6	6		≐(14i)
		5	U	5	6	U		
		L5+	U	5	6	U		→ 441 at A → $\bar{L}$ = 431r or 432r → 34-7
		5	U	L5+	6	~~μc.~~		≐(14j) (441 at C)
		5	U	5	U	.		no L at C by case-hypothesis
		[illegible]+	U	5	U	6	^{T}6	impossible (since V_6 at F implies V_5 at I)

continued next page



Proof of $q_{TL}(V_9)$ - Lemma,

Case [illegible] 4

Sub-case $\bar{v}'$ = 100 or 95

Arrangement check-list, continued[2]

	I	A	B	C	D	E	F	
5/11, cont.		[illegible]	U	5	U	6^{T2}	6	impossible by Lemma(5,L,T2)
5/12		5	.	.	.	.		
		5	.	.	5	6		→ (14k)
		5	6	6	5	U		≐ (14m)
		5	6	U	5	U		no L at D by case-hypothesis
		L6	6	U	5	U		→ 411 at A → $\bar{L}$ = 431r or 432r → 12-[illegible]
		5	U	.	5	U		
		L5+	U	.	5	U		→ 491,...,495 at A → {if L = 431r → 6-[illegible] if L = 432r → 21-[illegible]
		5	U	.	L5+	U		→ (14s) (defined on p. 145)
5/14		5	.	.	.	.		
		L5+	.	.	.	.		→ 411,491,...,495 at A → {if L = 431r → 6-19 if L = 432r → 21-25
		L4	6	6	6	6		≐ (14n)
		L4	U	6^T	6	6^T	[illegible]	≐ (14o) (L = 431n or 432n)
		R	6^T	6	6	6^T	6	→ L = 431n or 432n → 14-5
5/15		.	5	5	5	.		≐ (14p)
5/16		.	5	5	.	.		→ S
		6	5	5	.	.		≐ (14q)
		U	5	5	6	6		≐ (14r)
		U	5	5	6	U		impossible since $q_{TL}(V) \leq 0$
		U	5	5	U	.		at most one L at B,C by case-hypothesis
		U	5	5	U	6^{T2}	6	impossible by Lemma(5,L,T2)

$\bar{L}$ = 431,...,437 ≐ $\boxed{(14k)}$

Proof of $q_{TL}(V_9)$ - Lemma

Case 4

Sub-case $\bar{v}$ = 100 or 95

~~Arrangement check list~~ finished

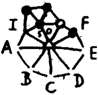
```
        I   A   B    C   D    E

5/17  ───── .   5    .   5    .
        ─── 6   5    6   5    .
        {6  6   5    6   5    .  → L = 431r or 432r → 13-27
        {5  6   L5+  6   5    .  → (14s)
        {5  6   5    6   L5+     → (14s)

        ─── 6   5    U   5    .   at most one L at B,D since a pair would
                                    violate the case-hypothesis; no L6.
        {6 T2 6 5    U   5    .  impossible by Lemma(5,L,T2)
        {6 T1 6 L5   U   5    .  → (14s)
        {6 T1 6 R    U   L5   .  → (14s)
         ┌  U   5    .   5    U
         └→ U   L5   6   5    U  → (14s)

            .   .    .   .    .
        {   e   L5+  .   .    .  → (14s)
        {6 T 6  5    6 T 6 T  6  → L = 431r or 432r → 13-10
        {   .   L4   6 T 6 T  6 T2 6 impossible by Lemma(5,L,T2)
        {   .   L4   6 T2 6 T2 6 T1 6 ≐ (14t) (L = 431n or 432n → T1#1 or
                                          T1#3n across E-F, by Lemma
                                          (5,L,T1))
5/19     ┌  .   .    5   .    .
        {6  6   6    5   .    .    → L = 431r or 432r → 13-10
        {6 T 6  U    L5+ 6 T  6  → 491,492,493,495 at C, L=431r,432r → 12-10
        {5  6 T 6    L5+ 6 T  6   5 → 411 at C → 13-24

5/20        .   .    .   .    .
         └→6 T2 6 T2 6 T1 6 T2 6 T2 6  impossible by Lemma(5,L,T2)
```

L = 431,
...,437
L5+ of V≤4
at B
≐ $\boxed{(14s)}$

Proof of $\dot{q}_{TL}(V_9)$ - Lemma

Case I = 4

Sub-case $\bar{V}$ = 80 and $\bar{L}$ = 441



no L5+ at.A by Lemma
(60 or 50, . ,60 or 50)

Arrangement check list

	I	A	B	C	D	E	F	
5/1,...,8		5	5					→ 34-7
5/2r,...,8r					5	5		→ 36-23
5/9		5	.	5	5	.		
		5	6	5	5	.		→ 34-10
		5	U	5	5	6		→ 38-21
		5	U	5	5	U		impossible since $q_{TL}(V) \leq 0$ (since no L can be at C or D by sub-case hypoth.)
5/9r		.	5	5	.	5		
		6	5	5	.	5		→ 34-15
		U	5	5	6	5		→ 36-4
		U	5	5	U	5		impossible since $q_{TL}(V) \leq 0$ (since no L can be at C or E by sub-case hyp.)
5/10,1		5						→ 35-16
5/11		5	.	5	.	.		
		5	6	5	.	.		→ 34-10
		5	U	5	.	.		impossible since $q_{TL}(V) \leq 0$ (since no pair of L's can be at A,C by case-hyp.)
5/11r		.	.	5	.	5		
		.	.	5	6	5		→ 38-17
		.	.	5	U	5		no L at E by sub-case hypothesis
		^{T}6	^{T}6	L5	U	R		→ 441 at C → 58-24
5/12		5	.	.	5	.		
		5	.	.	5	6		→ 39-4
		5	.	.	5	U		
		L4	6	^{T2}6	L5	U		→ 441 at D → 58-12

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 4

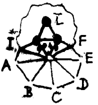
Sub-case L = 441, Arrangement check list finished

	I	A	B	C	D	E	F
5/12r			5		.	5	
		6	5		.	5	→ 45-72
		U	5	6	6	5	→ 38-26
5/14		5					impossible since $q_{TL}(V) < 0$
5/14r						5	impossible since $q_{TL}(V) \leq 0$, by Lemma (5,L,T2)
5/15			5	5	5	.	
			5	5	5	.	→ 34-15
		U	5	5	5	.	impossible since $q_{TL}(V) \leq 0$ since only one L can be [illegible] B D by Lemma(L,5,L) and case-hypothesis
5/16			5	5	.	.	impossible since $q_{TL}(V) \leq 0$, by Lemma (5,L,T2)
5/16r			. .	5	5	.	no two T2-dischargings can go across E-A and A-B by Lemma(60 or 50,T2,T2)
		.	.	5	5	6	→ 38-21
		.	.	5	5	U	impossible since $q_{TL}(V) \leq 0$ since no L can be at D by sub-case hypothesis
5/17		.	5	.	5	.	
		6	5	6	5	.	→ 34-16
		6	5	U	5	.	impossible since $q_{TL}(V) \leq 0$ since at most one L can be at B, D (a pair would violate the case-hypothesis) and no L6 can be attached
		U	5	.	5	.	impossible since $q_{TL}(V) \leq 0$, by Lemma (60 or 50, . ,60 or 50)
5/18		.	5	.	.	.	
		$^{T}6$	L5+	6	$^{T}6$	$^{T}6$	→ 411 at B → 34-14
5/18r		.	.	.	5 .	.	
		$^{T}6$	$^{T}6$	$^{T}6$	L5+	.	→ 411 at D → 38-25
5/19		.	. .	5	.	.	
		$^{T}6$	$^{T}6$	L5+	6	$^{T}6$	→ 411 at C → 13-5
5/20		.	.	.	.	.	impossible since $q_{TL}(V) < 0$

Propf of $q_{TL}(V_9)$ - Lemma

Case V = 4

Sub-case $\vec{v}$ = 80 and $\bar{L}$ = 801, 802, or 803



No L5+ at A or E, by Lemma (5,L,.,60 or 50) and case-hypothesis. No 441 by sub-case hypothesis.

No T2 across I-A or E-F, by Lemma(5,L,T2).

Arrangement check list

	I	A	B	C	D	E	F	
5/1,...,8		5	5					→ if $\bar{L}$ = 801 or 802r → 12-21 if $\bar{L}$ = 802n → 40-14 if $\bar{L}$ = 803 → 40-2
5/9		5	.	5	5			
		5	6	5	5			→ 13-11
		5	U	5	5	6		→ 13-15
		5	U	5	5	U		no L at C by sub-case hypothesis
		L4	U	R	L4	U		≙ (14u)
5/10,13		5				5		→ if $\bar{L}$ = 801 or 802 → 12-30 if $\bar{L}$ = 803 → 40-
5/11		5 .	.	5				
		5	6	5	.			→ if $\bar{L}$ = 801 or 802n → 12-19 if $\bar{L}$ = 802r → 40-13 if $\bar{L}$ = 803 → 40-4
		5	U	5	.	.		impossible since $q_{TL}(V) \leq 0$ since at most one L at A, C_4 (a pair would violate the case-hypothesis) and no L5+
5/12		5	.	.	5	.		
		5	.	.	5	6		→ 13-15
		5	6	6	5	U		→ 13-13
5/14		5	.	.	.	.		impossible since $q_{TL}(V) \leq 0$
5/15		.	5	5	5	.		
		6	5	5	5	.		→ 13-9
		U	5	5	5	· U		impossible since $q_{TL}(V) \leq 0$ since no pair of L s at B, D by case-hyp.

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] - 4

Sub-case L = 801,...,803

Arrangement check list, finished

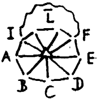
	I	A	B	C	D	E	F	
5/16		.	5	5	.	.		
		6	5	5	.	.		→ if L = 801 or 802n → Z23 if L = 802r → 12-32 if L = 803 → 40-7
		U	5	5				impossible since $q_{TL}(V) \leq 0$ since no pair of L's at B, C by **Lemma**(L,L) and case-hypothesis
5/17			5	.	5	.		
		6	5	6	5	.		→ if L = 801 or 802r → 12-32 if L = 802n → 40-17 if L = 803 → 40-8
		6	5	U	5			impossible since $q_{TL}(V) \leq 0$ since no L5+ at B or D
		U	5		5	U		impossible since $q_{TL}(V) < 0$ by **Lemma** (60 or 50,.,60 or 50)
5/18		.	5	.	.	.		
		^{T}6	L5+	6	^{T}6	^{T}6		411 at B → 12-22
5/19		.	.	5	.	.		
		6	^{T}6	L5+	6	^{T}6		→ 4[illegible] at C → 13-5
5/20		.	.	.	.	.		impossible since $q_{TL}(V) < 0$

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 4

Sub-case $\bar{v}$ = 70 or 65 (i.e., $\bar{L}$ = 451,...,481)

Arrangement check list



	I	A	B	C	D	E	F	
5/1,...,8		5	5					≐ (15a) (critical sub-classes see pp.153 154)
5/9		5	.	5	5			
		5	6	5	5	6		
	5	5	6	5	–5	6		→ if $\bar{L}$ = 451n,...,464n → 12-7 if $\bar{L}$ = 466r,...,481r → 13-15
		5	6	5	5	6	5	no pair of L's at C, D (by s-c hyp.)
		L5+	6	5	5	6	5	→ 411 at A (441 cannot occur by sub-case hypothesis) → 12-29
	6	5	6	5	5	6	6	→ $\bar{L}$ = 465 → 13-11
		5	6	5	5	U		no pair of L's at C, D
		L5+	6	5	5	U		→ 411 at A → if $\bar{L}$ = 451r,...,463r → (15b) (see p.151 if $\bar{L}$ = 465, 466n,...,481n → 12-16
		5	U	5	5			impossible since $q_{TL}(V) \leq 0$ since no L at A or C by sub-case hypothesis (423 at A would imply 441 at A) (recall Lemma(5,L,T2))
5/10,13		5	•			5		≐ (15c) (critical sub-class p.154)
5/11		5		5	.	.		no pair of L5+'s at A, C by Lemma (60 or 50,.,60 or 50) and case-hyp.
		L5+	.	L4	6	.		impossible since 411 at A and the pair at A, C violates the case-hypothesis
		L5+	.	L4	U	6^{T2}	6	→ 411 at A → $\bar{L}$ = 465 → 12-16
		L5+	.	R	6^{T}	6^{T}	6	
		L4	.	L	6	.		impossible since the pair at A, C violates the case-hypothesis
		L4	.	L6	U	6^{T2}	6	→ 491 at C and deg(I) > 5 (by sub-case hypothesis) → $\bar{L}$ = 465 → 37-29
		R	.	[illegible]	6^{T}	6^{T}	6	→ 411 at C → if $\bar{L}$ = 451,...,463 → 12-8 if $\bar{L}$ = 465,...,481 → 13-5

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 4

Sub-case v = 70 or 65

Arrangement check list, continued

L = 451r, ..., 463r $\doteq$ $\boxed{(15b)}$

I A B C D E ° F

5/12

5 5

5 6 6 5 6

L5+ 6 6 5 6 → 411 at A → {if L = 451r,...,463r → (15b); if L = 465, 466n, ...,481n → 13-2}

5 6 6 L5+ 6 → 411 at D → {if deg(I) = 5 → 12-22; if deg(F) = 5 → 13-2; if L = 465 → 13-24}

L4 6^{T2} 6 L4 6^{T2} 6 impossible by Lemma(T2,L,T2)

5 6 6 5 U

L5+ 6 6 L5+ U → 411 at A and 491r,...,495r at D → L = 465 → 13-13

L4 6^{T2} 6 L6 U → deg(I) > 5 by Lemma(5,L,T2); 491r at D → L = 465 → 13-13

L6 6^{T2} 6 L4 U → 411 at A → {if L = 451r,...,463r → (15b); if L = 465,466n,...,481n → 13-2}

5 6 U 5 .

L5+ 6 U L5+ . → 411 at A, 491n,...,493n,495n at D → 38-29

L6 6 U L4 6^{T2} 6 → 411 at A → {if L = 465n → 14-11; if L = 465r → 6-19}

L4 6 U L6 6^{T2} 6 491n at D

5 L4 6 U L6 6^{T2} 6 → 401,402, or 403 at A → 38-19

6 L4 6 U L6 6^{T2} 6 → L = 465 → 13-15

5 U . 5 6

L5+ U . L5+ 6 → 491r,492r,493r, or 495r at A, 411 at D → 38-29

L6 U . L4 6^{T2} 6 → 491r at A and L = 465 → 13-15

L4 U . L6 6^{T2} 6 411 at D

5 L4 U . L6 6^{T2} 6 → 451r,...,463r at A → (15b)

⋮ ⋮ ⋮

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 4

Sub-case $\bar{v}$ = 70 or 65

Arrangement check list, continued[2]

I A B C D E F

5/12, cont.
6 L4 U . L6 $6 \overset{T2}{} 6$ → L = 465 → 13-15
5 U . 5 U
L6 U . L6 U → 491r at A and at D → { if L = 465n → 1-3 / if L = 465r → 1-6 }

5/14
5 . . . .
L5+ $6 \,^{T}\, 6 \,^{T}\, 6 \,^{T}\, 6 \,^{T}\, 6$ → 411 at A and L = 465 → 14-12

5/15
. 5 5 5 . no pair of L's at B, D (would violate case-hypothesis)
6 5 5 5 6 → { if L = 451,...,464 → 12-7 / if L = 465 → 14-9 / if L = 466,...,481 → 13-34 }
6 5 5 5 U
recall Lemma (5, L, T2) $6 \,^{T2}\, 6$ 5 5 L4 U → deg(F) > 5 by sub-case hypothesis → L = 465 → 13-9

5/16
. . 5 5 . . no pair of L's at B, C by sub-case hyp.
$6 \,^{T}\, 6$ 5 5 $6 \,^{T}\, 6 \,^{T}\, 6$ → L = 465 → 14-10

5/17
. . 5 . 5 . no pair of L5+'s at B, D by case-hyp.
6 5 6 5 6 no pair of L's
6 L5+ 6 5 6 → 411 at B → { if deg(F) = 5 → 17-1 / if L = 451n,...,464n → 12-1 / if L = 465, 466r,..., 481r → 13-4 }
6 5 . 5 U
$6 \,^{T2}\, 6$ L6 . L4 U → 411 at B → { if L = 451r,...,454r → 38-15 / if L = 455r,...,463r → 35-30 / if L = 465, 466n,..., 481n → 13-2 }

. 5 . . .
$6 \,^{T}\, 6$ L5+ $6 \,^{T}\, 6 \,^{T}\, 6 \,^{T}\, 6$ → 411 at B, L = 465 → 12-22
$6 \,^{T2}\, 6$ L6 . $6 \,^{T2}\, 6 \,^{T2}\, 6$ → L = 465 → 14-11

Proof of $q_{TL}(V_9)$ - Lemma

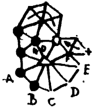
Case # = 4

Sub-case $\bar{v}$ = 70 or 65

Arrangement check list, finished

	T	A	B	C	D	E	F	
5/19		.	.	5	.	.		
recall Lemmas (T2,L,T2), (60 or 50,T2,T2)	6^{T2}	6	.	5	.	6^{T2}	6	→ Ł = 465 → I4-11
	6^{T}	6^{T}	6	L5+	6^{T}	6^{T}	6	→ Ł = 465 and 411 at C → 13-5
5/20		.	.	.	.	.		impossible since $q_{TL}(V) < 0$

Critical sub-classes

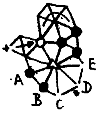


C15a#1: Ł = 453n

Arrangement check list for C15a#1

	B	C	D	E	F	
3/1,...,		5				→ 34-32
3/2r,4r				5		→ 36-20
3/5			5	.		
		6	5	.		→ 12-2
		U	5	6		→ 37-28
		U	5	U		impossible since $q_{TL}(V) < 0$
3/6 recall Lemma (5,L,T2)	L4	6^{T1}	6^{T2}	6^{T2}		→ I5-22

C15a#2: Ł = 453r



no pair of L's at A, B by sub-case hypothesis

Arrangement check list for C15a#2

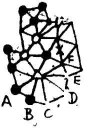
	B	C	D'	E	
3/1,...,4		5			→ 38-3
3/2r,4r				5	→ 35-33
3/5			5	.	→ L5+ at D → deg(Ç) = 6 → 38-13
3/6			.	.	impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 4

Sub-case $\check{v}$ = 70 or 65

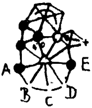
Critical sub-classes, finished



C15a#3: Σ = 475r

Arrangement check list for C15a#3

	B	C	$\dot{D}$	E	F	
3/1,...,4		5				→ 34-1
3/2r,4r				5		→ 1-9
3/5			5			
		6	5			→ 13-13
		6	5	U		→ L5+ at D → 491r,492r, or 495r at D → 57-6
		U	5			no L at B by sub-case hypothesis
		U	L4	6^{T2}		→ 14-10
3/6		.	.	.		
		6^T	6^T	6^T		→ 14-10



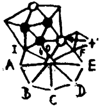
C15c Σ

Arrangement check list for C15c

	A	B	C	D	E	
3/1,...,4		5				→ 6-20
3/2r,4r				5		→ 35-3
3/5		.	5	.		
		6	5	.		→ 37-9
		U	5	6		→ 79-6
		U	5	U		impossible since $q_{T_1}(V) \leq 0$ since no L5+ can be attached at E by sub-case hypothesis
3/6		.	.	.		
	L	.	.	.		→ if 401,...,407,451,...,460,464 at A → 1-1 if 461,...,463 at A → 2 1
	R	6^T	6^T	6	L5+	→ 411 at E → 13-5

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible]-4



Sub-case $\bar{v}$ = 60 (i.e., Σ = 491)

Arrangement check list

I	A	B	C	D	E	F
5/1,...,	5	5	5			→ 60-20
5/2r,3r,4r			5	5	5	→ 38-22-
5/5,7,7r,10,13	5				5	→ 47-6
5/6	5	5	.	5	.	
	5	5	6	5	.	→ 12-29
	5	5	U	5	.	impossible since $q_{TL}(V) \leq 0$ since no L at B or D by sub-case hypothesis (423 at D would imply 441)
5/6r	.	5	.	5	5	
	6	5	6	5	5	→ 1 -1
	6	5	U	5	5	impossible since $q_{TL}(V) \leq 0$ since no L at B or D by sub-case hypothesis
Lemma (5,L,.,60 or 50)^r	U	L6	.	R	L	→ 401,402, or 403 at E → 58-19
5/8	5	5	.	.	.	impossible since $q_{TL}(V) \leq 0$ since no pair of L's at A,B by sub-case hypothesis
5/8r	.	.	.	5	5	impossible since $q_{TL}(V) \leq 0$ (no pair of L's)
5/9	5	.	5	5	.	no pair of L's at C,D by sub-case hypothesis
	L	.	5	5	.	if 411, 421r, or 422r at A → 1-3 if 421n at A → 1-6 (423r would imply 441) if 422n at A → 2-6
5/9r	.	5	5		5	
	6	5	5	6	5	→ 1 -15
	.	5	5	U	5	impossible since $q_{TL}(V) \leq 0$ since no L at C or E by sub-case hypothesis
	U	5	5	6	5	impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,.,60 or 50)$^+$ and since no L at B by sub-case hypothesis

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 4

Sub-case $\bar{v}$ = 60

Arrangement check list, continued

	I	A	B	C	D	E	F	
5/11 recall Lemma (60 or 50,T2,T2)		5	.	5	.	.		
		L	. . .	5	.	.		→ if 411, 421r, or 422r at A → 1-3 if 421n at A → 1-6 if 422n at A → 2-6
5/11r		.	.	5	.	5		no pair of L5+'s at C, E
		$^{T}6$	$^{T}6$	L	.	L		impossible since pair of L's violates the case hypothesis
		$^{T2}6$	$^{T2}6$	R	.	L6		→ 12-5
5/12		5	.	.	5			
		L5+	.	.	5	.		→ 411,491r,...,495r at A → 1-7
		5	6	$^{T}6$	L5+	6^{T}		→ 411, at D → 13-18
5/12r		.	5	.	.	5		
		6	L5+	.	.	5		→ 411, 491r,...,495r at B → 2-10
		$^{T}6$	5	6	$^{T}6$	L5+		→ 411 at E → 12-20
		6	L5	6	$^{T}6$	L5+		↗
5/14		5	.	.				
		L6	6	$^{T2}6$	$^{T1}6$	$^{T2}6^{T2}$		→ 411 at A → 1-7
5/14r		.	.	.		5		
		$^{T2}6$	$^{T2}6$	$^{T1}6$	$^{T2}6$	L6		→ 12-5
5/15		.	5	5	5	.		at most one L at B or D by Lemma(L,5, and sub-case hypothesis
		$^{T}6$	5	5	5	6^{T}		→ 12-10
5/16		.	5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,T2)
5/16r		.	.	5	5	.		
5/17		.	5	.	5	.		no pair of L5+'s at B, D
		$^{T}6$	L5+	.	5	6^{T}		→ 411 at B → 13-2
		$^{T}6$	5	.	L5+	6^{T}		→ 411 at D → 13-4

Proof of $q_{TL}(V_9)$ - Lemma

Case ▮ = 4

Sub-case $\bar{v}$ = 60

Arrangement check list, finished

I A B C D E F

5/18 $\Big[$. 5 . . . $\hookrightarrow$ T2 6 L6 · 6 T2 6 T1 6 T2 impossible by Lemma(60 or 50,T,60 or 50)

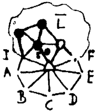
5/18r $\Big[$ . . . 5 . $\hookrightarrow$ T2 6 T1 6 T2 6 L6 6 T2 impossible since the T2-discharging across E-F would have to be T2#5r by Lemma,T#T (p.13) and would have to be T2#7n by Lemma(T2,L,T2) .

5/19 . . 5 . . impossible since $q_{TL}(V) \leq 0$ by Lemma (60 or 50,T2,T2)

5/20 . . . . impossible since $q_{TL}(V) < 0$

Sub-case v = 50 (i.e., L = 492,...,495)

In this sub-case, we have to consider only those [illegible] vertices are attached. For if only one of A,...,E is a V_5 then $q_{TL}(V) \leq 0$ by Lemma(60 or 50,T2,T2).



no L5+ at A by Lemma (60 or 50, . ,60 or 50) and case hypothesis

Arrangement check list

I A B C D E F

5/1, 5 5 5 5 → 12-2

5/2r,3r,1r 5 5 5 $\hookrightarrow$ { if L = 492,493,or 495 → 38-22 ; if L = 494 → 60-27 }

5/4,5,7,7r,10,13 5 5 $\hookrightarrow$ { if L = 492,493, or 495 → 37-6 ; if L = 495 → 60-24 }

5/4 $\Big[$ 5 5 5 . . no pair of L's at A,C $\hookrightarrow$ L4 R R . 6 T2 6 T2 6 → I5-1

5/6 $\Big[$ 5 5 . 5 . no pair of L's at A,B $\hookrightarrow$ 5 5 . L5+ 6 T 6 → 411 at D → 12-29

5/6r $\Big[$. 5 . 5 5 no pair of L's at D,E $\hookrightarrow$ T 6 L5+ . 5 5 → 411 at B → 13-2

Proof of $q_{TL}(V_9)^*$- Lemma

Case $\bar{v}$ = 4

Sub-case $\bar{v}_1$ = 50

Arrangement check list, finished

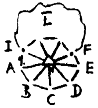
	I^1	A	B	C	D	E	F	
5/8		5	5	.		.		← impossible since $q_{TL}(V) < 0$
5/8r		.	.	.	5	5		
5/9		5	.	5	5			impossible since $q_{TL}(V) \leq 0$ (no L5+ at A)
5/9r		.	5	5	.	5		
	T	6	5	5	.	L5+		→ 411 at E → 13-15
5/11		5	.	5	.	.		
		L4	.	L6	6^{T2}	6^{T2}	6	impossible by Lemma(60 or 50,T2,T2)
5/11r		.	.	5	.	5		no pair of L5+'s at C,E by Lemma(60 or 50, ., 60 or 50) and case-hyp.
	T2	6^{T2}	6	L4	.	L6		→ 12-5
5/12		5	.	.	5	.		
		L4	6^{T2}	6.	L6	6^{T2}	6	→ 411 at D → 14-18
5/12r		.	5	.	.	5		
		.	L6	.	.	L6		→ 411 at B and at E → 14-6
	T	6	L5+	6^{T}	.	L5+		
	T2	6	L6	6^{T2}	6	L4		→ 411 at B → 14-24
	T2	6	L4	6^{T2}	6	L6		impossible by Lemma(T2,L,T2)
5/15		.	5	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,T2)
5/16		.	5	5	.	.		← impossible since $q_{TL}(V) < 0$
5/16r		.	.	5	5	.		
5/17		.	5	.	5	.		no pair of L5+'s at B,D
	T2	6	L6	.	L4	[redacted]		411 at B → 14-2
	T2	6	L4	.	L6	6^{T2}	6	→ 411 at D → 13-4

Proof of $q_{TL}(V_9)$ - Lemma

Case I = 4

Sub-case $\bar{v}$ = 40 or 35 (i.e., L = 496,...,522)

We need to consider only those arrangements in which at least two additional 5-vertices are attached to V since otherwise $q_{TL}(V)$ would not be positive.



Arrangement check list

	I	A	B	C	D	E	F	
5/1,2		5	5	5	5			
	↳	5	5	5	5	6 ^{T}	6	→ {if $\bar{L}$n → 12-2; if $\bar{L}$r → 13-7}
5/3		5	5	5	.	5		no pair of L's at A,C by Lemma(L,5,L) and case-hypothesis
	↳	5	5	5	.	L5+		→ 411 at E → 12-8
5/4		5	5	5	.	.		impossible since $q_{TL}(V) \leq 0$
5/5		5	5	.	5	5		impossible since $q_{TL}(V) \leq 0$ since no pair of L's at A,B or at D,E by sub-case hyp.
5/6		5	5	.	5	.		no pair of L's at A,B
	↳	5	5	.	L5+	6 ^{T}	6	→ 411 at D → {if $\bar{L}$n → 12-29; if $\bar{L}$r → 13-2}
5/7		5	5	.	.	5		
	↳	5	5	6 ^{T}	6	L6		→ 411 at E → 12-22
5/8		5	5	.	.	.		impossible since $q_{TL}(V) < 0$
5/9		5	.	5	5	.		
	↳	L6	.	5	5	6 ^{T}	6	→ 411 at A → {if $\bar{L}$n → 12-16; if $\bar{L}$r → 13-15}
5/10		5	.	5	.	5		no pair of L5+'s at A,C or at C,E
	↳	L5+	.	5	.	L5+		→ 411 at A and at E → 13-1
5/11		5	.	5	.	.		impossible since $q_{TL}(V) \leq 0$ (no pair of L5+'s
5/12		5	.	.	5	.		
	↳	L5+	6 ^{T}	6	L5+	6 ^{T}	6	→ 411 at A and at D → 13-6

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 4

Sub-case $\bar{v}$ = 40 or 35

Arrangement ~~check~~ list, finished

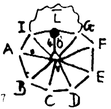
	I	A	B	C	D	E	F	
5/13		5	.	.	.	5		
		L6	6^{T2}	6^{T1}	6	L6		→ 411 at A and at E → 13-5
5/15		.	5	5	5	.		impossible since $q_{TL}(V) \leq 0$
5/16		.	5	5	.	.	.	impossible since $q_{TL}(V) < 0$.
5/17		.	5		5	.		impossible since $q_{TL}(V) \leq 0$ since no pair of L5+ s at B D

Case # = 5

In this case, any two L-situations attached to V must be at distance at least three (of their L-discharging V_5's on the ring about V) since otherwise they would form a pair violating the case-hypothesis.

Sub-case $\bar{v}$ = 70 ([illegible] 4 1, 40, or [illegible]

Arrangement check list



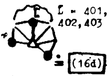
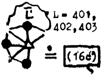
	I	A	B	C	D	E	F	G	
6/1,...,8		5	5	-5					≐ (16a)
6/9		5	5	.	5	5	.		
		5	5	6	5	5	.		→ if L n → 12-7 if L r → 12-12
		5	5	U	5	5	6		→ (16d) (as defined on p.161)
		5	5	U	5	5			no L can be attached at A, B, D, or
		R	R	U	R	R	U		if L = 401n or 402n → CTL # 14 if L = 403n → 1-2 if L = 401r or 402r → CTL # 14 if L = 403r → CTL #145
6/10,12,14		5	5				5		if L n → 12-2 if L r → 12-7

Proof of $q_{TL}(V_9)$ - Lemma

Case # = 3

Sub-case v = 70

Arrangement check list, continued

	I	A	B	C	D	E	F	G	
6/11		5	5	.	5	.	.		
		5	5	6	5	.	.		→ if $\bar{L}$n → 12-2 if $\bar{L}$r → 13-12
		5	5	U	5	.	.		no L at A,B, or D (since 423n at D would imply 441 at D, violating the case-hypothesis)
		R	R	U	R	6 ^{T}	6 ^{T}	6	→ $\bar{L}$n → 13-10
6/14		5	5	.	.	5	.		
		5	5	.	.	5	6		→ (16d)
		5	5	.	.	5	U		no L5+ can be attached
		5	5	6 ^{T}	6	5	U		→ if $\bar{L}$n → 13-7 if $\bar{L}$r → 13-15
6/15		5	5		.	.	.		at least 4 T-dischargings must go to V
		5	5	6	6	6	6		→ if $\bar{L}$n → 14-5 if $\bar{L}$r → 14-8
		5	5	U	6	6	6		impossible since $q_{TL}(V) \leq 0$ since no L can be at A or B
6/16		5		5	5	5	.		≐(16b)
6/17		5		5	5	.	5		
		5	6	5	5	.	5		→ (16c)
		5	U	5	5	U	5		no L can be attached at A,C,D, or F
		R	U	R	R	U	R		→ if $\bar{L}$ = 401 or 402 → CTL # 143 if $\bar{L}$ = 403 → 1-2
6/18		5	.	5	5	.	.		
		5	6	5	5	.	.		→ (16c)
		5	U	5	5	6	6		→ if $\bar{L}$n → 13-10 if $\bar{L}$r → 13-15
		5	U	5	5	6	U		impossible since $q_{TL}(V) \leq 0$
		5	-U	5	5	U			

Proof of $q_{TL}(V_9)$ - Lemma

Case ▮ = 3

Sub-case $\hat{v}$ = 70

Arrangement check list continued

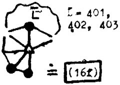
	I	A	B	C	D	E	F	G
6/19		5		5	.	5	.	
		5	6	5	.	5	.	→ (16 c)
		5	U	5	.	5	.	no L at A or C
		5	U	5	.	5	6	→ (16 d)
6/20		5	.	5	.	.	5	
		5	6	5	.	.	5	→ (16c)
		5	U	5	.	.	5	impossible since $q_{TL}(V) \leq 0$
6/21		5	.	5	.	.		
		5	6	5	.	.	.	→ (16e)
		5	U	5	.	.	.	impossible since $q_{TL}(V) \leq 0$
6/22		5	.	.	5	5	.	
		5	.	.	5	5	6	→ (16[illegible])
		5	.	.	5	5	U	impossible since $q_{TL}(V) \leq 0$ by Lemma ([illegible],L,T[illegible])
6/23			.	.	5	.	.	
		5	6	6	5	6	6	→ if Ln → 1[illegible]-10 if Lr → 1[illegible]-15
		5	6 ^{T}	6	L6	6	U	impossible since $q_{TL}(\bar{V}) \leq 0$
		5	U	6	L6	6 ^{T}	6 ^{T}	6 → 411 at D and Ln → 1[illegible]-5
6/24		5	.	.	.	5		
		5	.	.	.	5	6	→ (16d)
		5	.	.	.	5	U	impossible since $q_{TL}(V) \leq 0$
6/25		5	.	.	.	.	5	impossible since $q_{TL}(\bar{V}) \leq 0$
6/26		5	.	.	.	.	.	.
		5	6 ^{T2}	6 ^{T2}	6 ^{T1}	6 ^{T2}	6 ^{T2}	6 → Ln → [illegible]-1
6/27			5	5	5	5	.	≐ (16e)

Proof of $q_{TL}(V_3)$ - Lemma

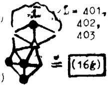
Case $\bar{W}$ = 3

Sub-case $\bar{v}$ = 70

Arrangement check list, finished



	I	A	B	C	D	E	F	G	
6/28			5	5	5	.	.		
		6	5	5	5	.	.		→ (16f)
		U	5.	5	5	6	6 ^{T}	6	→ L n → 13-10
6/29		.	5	5	.	5	.		
		6	5	5	.	5	.		→ (16f)
		U	5	5	6	5	6		→ {if L n → 13-13 if L r → 13-11
6/30					.	.	.		
		6	5	5	.	.	.		→ (16f)
		U	5	L4	6 ^{T1}	6 ^{T2}	6 ^{T3}	6	→ 401,402,403 at C → 13-33
6/31		.	5	.	5	.	.		
		.	L6	.	.	.	.		→ 411 at B → (16g)
		.	5	.	L6	.	6		→ 411 at D → {if L n → 13-5 if L r → 13-2
		.	5	.	.	5	.		
recall lemma ([illegible],L,[illegible])		.	L6	.	.	5	.		→ 411 at B → (16g)
6/33		.	5	.	.	. .	.		
		.	L6	:	.	.	.		→ 411 at B → (16g)
6/34		.	.	5	5	.	.		
		6 ^{T}	6	5	5	6 ^{T}	6		→ 14-10
6/35		. 5	.	5	.	.	.		
		6	.	L6	.	.	.		→ 411 at C → {if L n → 12-22 if L r → 13-5
		U	.	L6	.	.	.		impossible since $q_{TL}(V) \leq 0$.
6/36		.	.	.	.	.	.		impossible since $q_{TL}(V) < 0$



Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 3

Sub-case $\bar{v}$ = 60 (i.e., $\mathcal{I}$ = 411)

No L-discharging can go to V from a 5-vertex which has a V_5 - neighbor on the ring about V by sub-case hyp.

Arrangement check list

	I	A	B	C	D	E	F	G	
6/1,...,8		5	5	5					→ 12-8
6/9		5	5	.	5	5			
		R	R	.	R	R	6^T		→ 12-22
6/10,12,14 -		5	5				5		→ 12-16
6/11		5	5	.	5	.	.		
		5	5	6	5	.	.		→ 12-29
		5	5	U	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(60 or 50,T2,T2) and since no L can be attached A, B, or D
6/13		5	5	.	.	5			
				.	.	L6	.		→ E
		5	5	6^T	6	5	6^T		→ 13-1
6/15		5	5	.	.	.	.		
		5	5	6^{T2}	6^{T2}	6^{T1}	6^{T2}		→ 14-12
6/16		5	.	5	5	5	.		
		5	.	5	5	5	6^T		→ 12-22
6/17		5	.	5	5	.	5		impossible since $q_{TL}(V) = 0$
6/18		5	.	5	5	.			impossible since $q_{TL}(V) \leq 0$ by Lemma (60 or 50,T2,T2)
6/19		5	.		.	5			
		5	.	6	.	5	.		→ 411 at C → 13-1
		5	.	5	.	L6	.		→ 411 at D → 13-6
6/20		5	.	5	.	.	5		
		5	.	L6	.	.	5		→ 411 at C → 13-1

Proof of $q_{TL}(V_9)$ - Lemma

Case ■ = 3

Sub-case $\bar{v}$ = 60

Arrangement check list, continued

	I	A	B	C	D	E	F	G
6/21		5	.	5	.	.	.	
		5	.	L6	.	.	.	→ 411 at C → 13-1
6/22		5	.		5	5	.	
		5	6^{T2}	6	5	5	6^{T2}	→ 12-22
6/23		5	.	.	5	.	.	
		5	6^{T}	6	L6	6	6	→ 13-5
6/24			.	.	.		.	
recall Lemma (T2,L,T2)		5	.	.	.	L6	.	→ 411 at E → 13-6
6/25		5	.	.	.	.	5	impossible since $q_{TL}(V) < 0$
6/26		5	.	.	.	.	.	impossible since $q_{TL}(V) < 0$
6/27		.	5	5	5	5	.	
	T	6	5	5	5	5	.	→ 12-22
6/28		.	5	5	5	.	.	
	T	6	5	5	5	.	.	→ 12-22
6/29		.	5	5	.	5	.	
		.	5	5	.	L6	.	→ 411 at E → 13-6
	T	6	5	5	.	5	6^{T}	→ 12-22
6/30		.	5	5	.	.	.	
	T	6	5	5	6^{T}	6^{T}	6^{T}	→ 12-22
6/31		.	5	.	5	.	.	
		.	L6	.	5	.	.	→ 411 at B → 13-18
	T	6	5	.	L6	.	.	→ 411 at D → 13-18
6/32		.	5	.	.	5	.	
		.	L6	.	.	5	.	→ 411 at B → 13-6
	T	6	5	6^{T}	6	5	6^{T}	→ 13-24

Proof of $q_{TL}(V_9)$ - Lemma

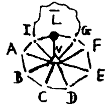
Case [illegible] - 3

Sub-case $\bar{v}$ - 60

Arrangement check list, finished

	I	A	B	C	D	E	F	G	
6/33		.	5	.	.	.	.		
recall Lemma (T2,L,T2)		.	L·6	.	.	.	.		→ 411 at B → 13-6
6/34		.	.	5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(60 or 50,T2,T2)
6/35		.	.	5	.	.	.		
		6	6	L6	6	6	6		→ 411 at C → 13-5
6/36		.	.	.	.	.	.		impossible since $q_{TL}(V) \leq 0$

Sub-case $\hat{v}$ = 40 (i.e., $\bar{L}$ - 421,...,$\hat{4}28$)



In this sub-case, no L5+ can be attached to V. An L4 requires two non-5 neighbors of the L-discharging V_5 on the ring about V. Any pair of L4's must have its L-discharging V's at distance at least three on the ring about V. Consequently, we have to consider only those arrangements in which at least three additional V_5's are attached to V. For at most 7 T-dischargings can go to V; if precisely one V_5 is attached (in addition to the one in $\bar{L}$) then at most 5 T-dischargings can go; if two adjacent V_5's are attached, then at most 4 T-dischargings can go; if two non-adjacent V_5's are attached, then at most 3 T-dischargings can go to V,

Furthermore, if deg(A) = 5 then $\bar{L}$ is non-reflected or is 421r or 422r, i.e., in every case deg(I) = 6; ($\bar{L}$ = 425r would imply that 441 occurs, violating the case-hypothesis) Likewise, if deg(A) = deg(F) = 5 then deg(I) = deg(G) = 6.

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] = 3

Sub-case $\bar{v}$ = 40

Arrangement check list

	I	A	B	C	D	E	F	G
6/1,...,4		5	5		5			→ 12-2
6/5,7,10,12,14		5	5				5	→ if L n → 12-16 if L r → 12[illegible]
6/6	┌	5	5	5	.	5		
	└→	5	5	5	.	L4	6^{T2}	→ 421 or 422 at E → 13-13
6/8		5	5	5	.	.	.	impossible since $q_{TL}(V) \leq 0$
6/9		5	5	.	5	5	.	impossible since $q_{TL}(V) \leq 0$
6/11		5	5	.	5	.	.	impossible since $q_{TL}(V) \leq 0$
6/13		5	5	.	.	5	.	impossible since $q_{TL}(V) \leq 0$
6/16		5	.	5	5	5	.	impossible since $q_{TL}(V) \leq 0$
6/17		5	.	5	5	.	5	impossible since $q_{TL}(V) < 0$
6/18		5	.	5	5	.	.	impossible since $q_{TL}(V) < 0$
6/19		5	.	5	.	5	.	impossible since $q_{TL}(V) < 0$
6/20		5	.	5	.	.	5	impossible since $q_{TL}(V) < 0$
		5						impossible since $q_{TL}(V) < 0$
6/27	┌	.	5	5	5	5	.	
	└→ T	6	5	5	5	5	6^{T}	→ L = 421 or 422 → 13-7
6/28			5	5	5	.	.	impossible since $q_{TL}(V) \leq 0$ by Lemma(T2,L,T2)
6/29			5	[illegible]	.	5	.	impossible since $q_{TL}(V) \leq$ [illegible]

Case W = ¯0

In this case, if v is the number of V_5-neighbors of V and if $v > 0$, then at most $8 - v$ T-dischargings can go to V; moreover, if the v degree-5 neighbors of V are not consecutive (on the ring about V) then at most $7 - v$ T-dischargings can go to V. From this it follows that v must be at least 3, and if $v = 3$ then the 3 V_5-neighbors of V must be consecutive; but in the latter case, the (at most 5) T-

Proof of $q_{TL}(V_9)$ - Lemma

Case [illegible] - 0

dischargings can amount to at most 90 so that $q_{TL}(V) \leq 0$. Thus we have $\nu' \geq 4$. If $\nu = 4$ then the 4 V_5-neighbors of V must be consecutive and three T2's and one T1-discharging must go to V. Then 14-5 occurs.

From now on we may assume that $\nu \geq 5$. At least two of the V_5-neighbors of V are adjacent to each other and thus the configuration drawn at the right occurs. We may assume that the reading A, B, ..., I is maximal (as compared with all possible cyclic permutations and reflections). Thus we have to consider for the degrees of C, D, ..., I only those 7-digit arrangements which are marked in the List on p.46 with an asterisk or with a dagger (and which have at least three entries "5")

Arrangement check list

	C	D	E	F	G	H	I	
7/1,2,4,6,8	5	5	5	5				→ 12-1
7/10	5	5	5		5	5		
	5	5	5	6	5	5		→ 12-2
	5	5	5	U	5	5	U	→ CTL # 1A1
7/12	5	5	5		5			
	5	5	5		5	6^{T}	6	→ 13-7
7/16	5	5	5					
	5	5	5	6^{T2}	6^{T2}	6	.	→ I3-1
	5	5	5	6^{T}	6	6^{T}	6	→ 14-4
7/17	5	5	.	5	5	5	.	→ 14-17
7/19	5	5	.	5	5	.	.	
	5	5	.	5	5	6^{T}	6	→ 13-7
7/21	5	5	.	5	.	5	.	impossible since $q_{TL}(V) = 0$
7/23	5	5	.	5	.	.	.	
	5	5	.	5	6^{T2}	6^{T2}	6	→ I3-1

Proof of $q_{TL}(V_9)$ - Lemma

Case - 0

Arrangement check list, finished

	C	D	E	F	G	H	I	
7/26	5	5	.		5	.		
	5	5	6^{T2}	6	5	6^{T2}	6	→ 13-7
7/33	5	.	5	5	5			
	5	.	5	5	5	6^{T}	6	→ 13-9
7/34	5′	.	5	5	.	5	.	impossible since $q_{TL}(V) = 0$
7/36	5	.	5	5	.	.	.	
	5	.	5	5	6^{T2}	6^{T2}	6	→ I3-1
7/39	5	.	5	.	5	.	.	impossible since $q_{TL}(V) < 0$
7/40	5	.	5	.	.	5	.	impossible since $q_{TL}(V) < 0$.
7/44	5	.	.	5	5	.		
	5	6^{T2}	6	5	5	6^{T2}	6	→ 13-9
7/57	.	5	5	.	5	5	.	impossible since $q_{TL}(V) = 0$
7/58	.	5	5	.	5	.	.	impossible since $q_{TL}(V) < 0$
7/61	.	5	.	5	5	.	.	impossible since $q_{TL}(V) < 0$
7/62	.	5	.	5	.	5	.	impossible since $q_{TL}(V) < 0$

This finishes the proof of the $q_{TL}(V_9)$ - Lemma. ∎

Proof of the $q_{TL}(V_{10})$ - Lemma.

Now we shall use the Upper Bound Lemma for $d(V_k)$ and its Corollary, as proved in Section 3 of the paper. Since by hypothesis, (see p.79) our triangulation Δ satisfies all the Lemmas on L- and T-dischargings which have been proved for Δ^*, the Upper Bound Lemma and its corollaries hold also for Δ, with our vertex V in place of V_k. Recall that by hypothesis, V is q_{TL}-positive, $q_{TL}(V) > 0$. For the case of $k = 10$, we need another corollary of the Upper Bound Lemma.

Corollary 2 of the Upper Bound Lemma. (Notation as in the proof of the Upper Bound Lemma). If the configuration of Figure 13 does not occur and if

there are two indices ℓ and m ($1 \leq \ell < m \leq k$) such that $c^{(\ell)} + c^{(m)} < 30$

or if

there are three indices ℓ, m, n ($1 \leq \ell < m < n \leq k$) such that $c^{(\ell)} + c^{(m)} + c^{(n)} \leq 50$

then

$$(3.5^*) \qquad d(V_k) \leq 40k - 10(k-v) - 20.$$

Proof. This follows from (3.3) and (3.4) (see Section 3 of the paper) since $\sum_{i=1}^{k} c^{(i)}$ is an integral multiple of 10.

We apply the Upper Bound Lemma and its corollaries to our q_{TL}-positive vertex V of degree $k = 10$. We denote the neighbors of V in counter-clockwise order by A, B,...,J and their contribution values by $c(A),\ldots,c(J)$ where we use the definition of the contribution values as given in the proof of the Upper Bound Lemma with $A = V^{(10)}$, $B = V^{(9)}$, ..., $J = V^{(1)}$, $c(A) = c^{(10)}$, $c(B) = c^{(9)}$, ..., $c(J) = c^{(1)}$. For the "partial contribution values" $c_*^{(j)}$ and $c_{**}^{(j)}$ (as defined in the proof of the Upper Bound Lemma) we use the corresponding notation $c_*(A),\ldots,c_*(J)$ and $c_{**}(A)$, ..., $c_{**}(J)$. Since $q_O(V) = -240$ and $q_{TL}(V) = q_O(V) +$ [illegible] > 0, we have $d(V) > 240$. Thus the inequalities (3.1), (3.4), (3.5), and (3.5*) give the following.

Proof of $q_{TL}(V_{10})$ - Lemma

Introduction, finished

Restrictions for the number v of degree-5 neighbors of V

(i) $v \geq 3$

(ii) If the configuration of Figure 13 does not occur then $v \geq 5$.

(iii) If the configuration of Figure 13 does not occur and if at least one of the non-5 neighbors of V has [illegible] < 20 then $v \geq 6$.

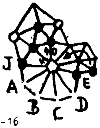
(iv) If the configuration of Figure 13 does not occur and if there are two non-5 neighbors of V the sum of whose c-values is < 30 or three non-5 neighbors of V the sum of whose c-values is < 50 then $v \geq 7$.

First we shall treat the special case that the configuration of Figure 13 occurs (with pivot identified to V). After that, we may assume that the configuration of Figure 13 does not occur and hence $v \geq 5$. Next we shall treat the special case that the configuration of Figure 12(f) occurs (with pivot identified to V). For the following cases we may then assume that the configuration of Figure 12(f) does not occur and hence that the hypothesis of Lemma(5,L,.,L,5) cannot be fulfilled. Third we treat the case $v \geq 8$, fourth the case $v = 7$. Then we are left with the case $v = 6$ or 5. This last case is again partitioned into cases [illegible] ≥ 6, [illegible] $= 5$, [illegible] $= 4$, [illegible] $= 3$, and [illegible] $= 0$ (where it is easy to see that the case [illegible] $= 0$ actually cannot occur). A further partition into sub-cases is made according to the value v (as in the previous proofs for $k = 8$ and $k = 9$). In many of the sub-cases it is easy to show that the restriction (iv) holds for v which then contradicts the case-hypothesis (and thus proves [illegible] the sub-case actually cannot occur).

Proof of $q^{\to}_{TL}(V_{10})$ - Lemma

Case 5,L,T2 (i.e., the configuration of Figure 13 occurs with pivot identified to V)

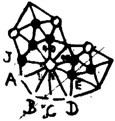
Arrangement check list



	J	'A	B	C	D	E	
4/1,...,7		5					→ 15-6
4/2r,..7,7r					5		→ 15-14
4/8		.	5	5			&
		.	5	5	.	L6	→ 411 at E → 15-16
		.	5	5	.	L5	→ 441 at E → 41-6
	L4	.	L4	L4	.	L4	→ if 431r at J → 15-29 if 437r at J → 2-1
4/9		.	. 5		.		
	L4	.	L6	.	.		impossible by Lemma(5,L,.,60 or 50)$^+$
			L5+	6^{T2}	6	L5+	impossible by Lemma(60or 50, T ,60or 50)
		.	5	6^{T}	6	L6	→ 411 at E → 15-16
4/9r		.	.	5			
		.	.	5		L5+	→ if 411 at E → 15-16 if 441 at E → 41-6
	L4	6^{T2}	6	L6		L4	→ 431 at J and 411 at C → 15-16
4/10							impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_{10})$ - Lemma

Case 5,L, . ,L,5 (i.e., the configuration of Figure 12(f) occurs with pivot identified to V)

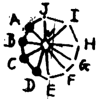


Arrangement check list

	J	A	B	C	D	E	
4/1,...,7		5					→ 15-5
4/2r,...,7r					5		→ 15-11
4/8		.	5	5	.		
		6	5	5	.		→ 15-20
		U	5	5	6		→ 15-17
		U	5	5	U		
	L4	U	5	5	U		→ 437 at J → 2-1
	R	U	L4	5	U		→ if 433 or 434 at B → 15-33 if 435,...,437 at B → 2-1.
	R	U	R	L4	U	L4	→ 433 or 434 at C → 15-33
4/9		.	5	.	.		
		6	5	.	.		→ 15-20
		U	5	.	6		→ 15-17
		U	5	.	U		
	L4	U	L6	.	U	L4	impossible by Lemma(5,L,.,60 or 50)$^{+}$
4/9r		.	.	5	.		
		.	.	5	6		→ 15-17
		.	.	5	U		
	L4	6^{T2}	6	5	U		→ 431 at J → I4-11
	L4	6^{T1}	6	L6	U		impossible since no L6 can be attached at C
	L4	6^{T1}	6	L5	U	L4	→ 811n at C,E by Lemma(5,L,.,60 or 50)* → 41-[illegible]
	L4	$.^{\times}$	.	L6	U	L4	impossible by Lemma(5,L,.,60 or 50)
	R	6^{T}	6	L6	U		impossible since no L6 can be attached at C
	R	6^{T2}	6	L5	U	L4	→ 441 at C → 41-6
4/10							impossible since $q_{TL}(V) \leq 0$

Case v ≥ 8

Arrangement check list



(reading A,. .,J cyclically maximal)

A	B	C	D	E	F	G	H	I	J	
5	5	5	5	5	5	5				→ 15-1
5	5	5	5	5	5		5	5		
5	5	5	5	5	5	6	5	5		→ 15-3
5	5	5	5	5	5	U	5	5	U	
5	5	5	5	5	L4	U	5	5	U	→ if 433 or 434 at F → 41-11 if 435,...,437 at F → 41-19
R	R	R	R	R	R	U	L4	5	U	→ if 433 or 434 at H → 42-9 if 435,...,437 at H → 41-1
5	5	5	5	5	.	5	5	5		
5	5	5	5	5	6	5	5	5		→ 15-3
5	5	5	5	5	U	5	5	5	U	
				L4	U				U	→ if 433 or 434 at E → 41-11 if 435,...,437 at E → 41-19
R	R	R	R	R	U	L4	5	5	U	→ if 433 or 434 at G → 42-9 if 435,...,437 at G → 41-1
5	5	5	5		5	5	5	5		
5	5	5	5	6	5	5	5	5		→ 15-5
5	5	5	5	U	5	5	5	5	U	
5	5	5	L4	U	5	5	5	5	U	→ if 433,...,435 at D → 41-11 if 436 or 437 at D → 41-22

Proof of $q_{TL}(V_{10})$-Lemma

Case v = 7

Arrangement check list

(reading A,...,J cyclically maximal)

A	B	C	D	E	F	G	H	I	J	
5	5	5	5	5	5	5	.	.	.	→ 15-1
5	5	5	5	5	5	.	5	.	.	
5	5	5	5	5	5	6	5	.	.	→ 15-3
5	5	5	5	5	5	U	.	6	6	→ 15-4
5	5	5	5	5	5	U	5	.	× .	no ≥6 can be attached at H
L4	R	R	R	R	L4	U	L5	.	× .	→ 441n at H by Lemma (5,..,..,60 or 50)⁺ → 41-19
5	5	5	5	5	.	5	5	.	.	
5	5	5	5	5	.	5	5	6	6	→ 15-4
5	5	5	5	5	.	5	5	.	× .	
L4	R	R	R	L4	.	L4	L4	.	× .	impossible by Lemma(5,L,.,L,5) and case-hypothesis
5	5	5	5	5	6	5	.	5		→ 15-3
5	5	5	5	5	U	5	.	5	U	
5	5	5	5	5	U	L5+	.	5	U	→ 441n at G → 41-19
L4	R	R	R	L4	U	L4	.	L4	U	
L4	R	R	R	L4	U	L4	6	L4	U	→ the L4 s at E and G form a pair of V = 5 → 834 or 835 at E,G → 41-19
L4	R	R	R	L4	U	L4	U	L4	U	→ if 532 at G → 41-19 if 533 or 535,...,541 at G → 41-1
5	5	5	5	.	5	5	5	.	.	
5	5	5	5	.	5	5	5	6	6	→ 15-13
5	5	5	5	.	5	5	5	.	× .	impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,.,L,5) and case-hypothesis

continued next page

Proof of $q_{TL}(V_{10})$ - Lemma

Case ψ = 7

Arrangement check list, finished

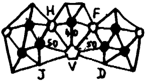
A	B	C	D	E	F	G	H	I	J	
5	5	5	5	.	5	5	.	5	.	
5	5	5	5	6	5	5	.	5	.	→ 15-5
5	5	5	5	U	5	5	.	5	.	
5	5	5	L4	U	5	5	.	5		→ if 433,...,435 at D → 41-11 if 436 or 437 at D → 42-25
5	5	5	R	U	L4	5	.	5		→ if 433,...,435 at F → 42-9 if 436 or 437 at F → 41-9
5	5	5	R	U	R	5	.	L5+		→ if 411 at I → 15-9 if 441n at I → 41-3 if 441r at I → 42-1
5	5	5	.	5	5	5	.	5	.	
5	5	5	6	5	5	5	.	5	.	→ 15-11
5	5	5	U	5	5	5	.	5	.	
5	5	L4	U	5	5	5	.	5	.	→ if 433,...,435 at C → 41-26 if 436 or 437 at C → 41-22
5	5	R	U	R	5	L4	.	L6	.	impossible by Lemma(5,L,.,60 or 50)+
L4	R	R	U	R	R	L4	.	L5	.	→ 441 at I → 41-3
5	5	5		5	5	.	5	5	.	impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,.,L,5) and case-hypothesis

Proof of $q_{TL}(7_{16}$[illegible]Lemma

Case v = 6 or 5

Case v = 6 or 5, $\bar{v}$ = 7 (i.e., L = 702)

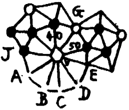
We have $c_{**}(F) = c_*'(H) = 5$ (see Figure 14) and $c_*(F) = c_{**}(H) = 5$, thus $c(F) + c(H) = 10$. Thus, by Restriction (iv) (see page 171), $v \geq 7$. Hence this case is impossible.



Case v = 6 or 5, $\bar{v}$ = 6

Sub-case $\bar{v}$ = 150 (i.e., L = 701)

We have $c_{**}(G) = 5$, $c_*(G) = 10$, and thus $c(G) = 15$, and, by Restriction (iii), $v \geq 6$. Thus we have to consider only those arrangements in which precisely two V_5's occur.

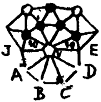


Arrangement check list

	J	A	B	C	D	E	
4/4,5,6		5					→ 42-1
4/4r,5r					5		→ 41-25.
4/8			5	5			
	L4		L4,	5			impossible by Lemma(5,L,.,.,L,5) and case-hypothesis
			5,.	L4	6 ^{T2}	6	impossible by Lemma(5,L,T2) and case-hypothesis

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, 6



Sub-case v = 140 and L = 901

We have to consider only those arrangements in which precisely one or two V_5's occur.

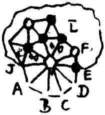
Arrangement check list

	J	A	B	C	D	E	
4/4		5	5				→ 15-5
4/5		5	. .	5	.		
		5	6	5	.		→ 15-9
		5	U	5	.		
		5	U	5	6^{T2}		Impossible by Lemma(5,L,T2) and case-hyp.
		L5	U	L5	6^{T1}		→ 441 at A → 6-19
4/6		5	.	.	5		→ 15-26
4/7		5	.	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,..,60 or 50
4/8		.	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2) and case-hyp.
4/9							impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2) and case-hyp.

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, $\bar{7}$ - 6

Sub-case $\bar{v}$ = 140 and $\bar{L}$ = 902, 0-, or 904



We have $c_*(F)$ = 10 and $c_{**}(F) \leq 5$ by Lemma (5,L,.,L,5), and thus by Restriction (iii), $v \geq 6$. Thus we have to consider only those arrangements in which precisely two V_5's occur.

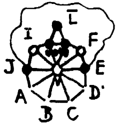
Arrangement check list

	J	A	B	C	D	E	
4/4		5	5	.	.		— impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,.,L,5), (5,L,T2) and case-hypothesis
4/4r		.	.	5	5		
4/5		5	.	5	.		
		L5+	.	5-	.		→ 11 or 411 at A → 6-19
		5	.	5	.	L5+	→ 411 at E → 40-2,
		L4	.	L6	.	L4	→ 11 at C → 1-30
4/5r		.	5	.	5		— impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,T2),(5,L,.,60 or 50)$^+$, (5,L,.,.,L,5)
4/6		5	.	.	5		
4/8		.	5	5			
	T1	6	L4	L4,	.	L5	→ 441 at E → 41-10

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, [illegible] = 6

Sub-case $\bar{v}$ = 140 and $\mathcal{L}$ = 9.5,...,910

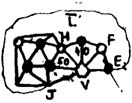


We have $c_*(F) = 10$ and $c_{**}(F) \leq 5$ by Lemma(5,L,..,L,5), and thus, by Restriction (iii), $v \geq 6$. Thus we have to consider only those Arrangements which provide precisely two V_5s.

Arrangement check list

	J	A	B	C	D	E	
4/4		5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,..,L,5) and (5,L,T?)
4/5		5	.	5	.	.	
		5	.	L5+	.	L5+	→ 441r at E → 411 or 441r at C → 1-20
4/6		5	.	.	5		impossible since $q_{TL}(V) \leq 0$ by Lemma ([illegible],..,L,5)
4/8		.	5	5	.		
	L5+	.	5	5	.		→ 441 at J → 41-10

Sub-case v = 120 or 115 and $\mathcal{L}$ = 711,...,715, or 718

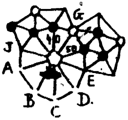


We have $c_*(H) = c_{**}(H) = c_*(F) = 5$ and $c_{**}(F) \leq 10$, and thus $c(H) + c(F) \leq 25$. Thus, by Restriction (iv), $v \geq 7$ which contradicts the case-hypothesis. Thus this sub-case is impossible.

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, $\bar{N}$ = 6

Sub-case $\bar{v}$ = 120 and $\bar{L}$ = [illegible]16

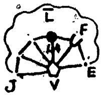


We have c(G) = 15 and thus, by Restriction (iii), v $\geq$ 6. Thus we have to consider only those arrangements which provide precisely three L_5's.

Arrangement check list

	J	A	B	C	D	E	
4/2		5	5	5	·		→ impossible since $q_{TL}(V) \leq 0$ by Lemma
4/2r		·	5	5	5		
4/3		5	5	·	5		
		5	5	·	L6+		→ 411 or 411n at D → 1-4
4/3r		5	·	5	5		
		L6	·	L4	5		impossible by Lemma(...,60 or 50)$^{+}$
		5	·	5	L4		→ if 401 at D → 1-6 if 402 or 403 at D → 7-6

Sub-case $\bar{v} \leq$ 115 and $\bar{L}$ - [illegible]



We have $c_*(F) \leq 10$, $c_{**}(F) = 0$, $c_*(E) = 0$, $c_{**}(E) \leq 15$, and thus $c(E) + c(F) \leq 25$. Thus this sub-case is impossible since by Restriction (iv), $v \geq 7$.

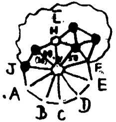
Sub-case $\bar{v}$ = 100 and $\bar{L}$ = 720

We have $c_*(H) = 5$, $c_{**}(H) = 0$ (see Figure 14), and $c(F) = 20$, thus $c(H) + c(F) = 25$. Thus this sub-case is impossible by Restriction (iv).

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, W = 5

Sub-case $\bar{v}$ = 120 or 115 (i.e., L_3 = 811,812,813, or 530)



We have c(W) = 15 and thus, by Restriction (iii), v ≥ 6. Thus we have to consider only those arrangements which provide precisely three V_5's.

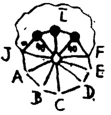
Arrangement check list

	J	A	B	C	D	E	F	
5/4,6,7		5	5		.			→ 42-1
5/4r,6r,7r					5	5		→ 41-10
5/9		5	.	5	5	.		impossible since $q_{TI}(V) \leq 0$ by Lemmas (5,L,..,L,5), (5,L,T2)
5/9r		.	5	5	.			
		.	5	5	.	L5+		→ 11 or 441n at E → 1-20
			L4			L4		impossible by Lemma (5,L,..,L,5)
5/10		5		5	.	5		→ 41-5
5/15		.	5	5	5			impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,..,L,5), (5,L,T2)

Proof of $q_{TL}(\Gamma_{10})$-Lemma

Case v = 6 or 5, $\bar{v}$ = 5

Sub-case $\bar{v}$ = 110 and $\bar{L}$ = 820 or [illegible]



We have $c_*(F) = 10$. Moreover $c_{**}(F) \leq 5$ since no T2-discharging can go across E-F by Lemma(5,.,T2);

no L5+ can go from E by Lemma (5,L,.,60 or 50)$^+$ and sub-case hypothesis and no L4 can go from E if deg(D) = 5, by Lemma(5, ,.,L,5). Thus, by Restriction (111), v $\geq$ 6 and we have to consider only those arrangements which provide precisely three V_5's.

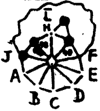
Arrangement check list

	J	A	B	C	D	E	F	
5/4		5	5	5	.	.		→ 1-11
5/6		5	5	.	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,.,L,5), (5,L,T2), and (5,L,.,60 or 50)$^+$
5/7		5	5	.	.	5		→ 15-20
5/9		5	.	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,.,60 or 50)$^+$, (5,$\dot{L}$,T2)
5/10		5	.	5	.	5		no L5+ at A or E by Lemma(5,L,.,60 or 50)$^+$ and sub-case hypothesis
		→ L4	.	L6	.	L4		→ 411 at C → 15-30
5/15		.	5	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,T2)

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, [illegible] = 5

Sub-case $\bar{v}$ = 110 or 105 and $\bar{L}$ = 821,...,826, 834, or 835



We have c(H) = 15 and thus, by Restriction (iii), v $\geq$ 6 and we have to consider only those arrangements which provide precisely three V_5's.

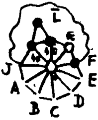
Arrangement check list

	J	A	B	C	D	E	F	
5/4		5	5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2)
5/4r		.	.	5	5	5		
5/6		5	5	.	5	.		recall Lemma(5,L,..,60 or 50)$^+$ and sub-case hyp.
		R	R	.	L6	6^{T2}		$\rightarrow$ 411 at D $\rightarrow$ 15-10
5/6r			5	.	5	5		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,..,60 or 50)$^+$ and Sub-case hypothesis
5/7		5		.	.	5		
		5	5	6^T	6	L6		$\rightarrow$ 411 at E $\rightarrow$ 15-16
5/7r		5	.	.	5	5		impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2)
5/9		5	.	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,..,L,5)
5/9r		.	5	5	.	5		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,..,L,5), (5,L,..,60 or 50)$^+$
5/10		5	.	5	.	5		
		L4	.	L5+	.	5		impossible by Lemma (5,L,..,60 or 50)$^+$ and sub-case hypothesis
		L4	..	L4	.	L6		$\rightarrow$ 411 at E: if $\bar{L}$ = 821, 823, or 825 $\rightarrow$ 1-21 if $\bar{L}$ = 822, 824, or 826 $\rightarrow$ 6-11 if $\bar{L}$ = 834 or 835 $\rightarrow$ 6-11
		R	.	L5+	.	L5+		impossible by Lemma(60 or 50, . . 60 or 50)
5/15			5	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,..,L,5)

Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, [illegible] = 5

Sub-case $\bar{v}_4$ = 1$\bar{7}$0 and $\bar{L}$ = '8'28,...,5''



We have $c_*(G)$ = 10. Moreover, $c_{**}(G) \leq 5$ since no L5+ can go from F to V by Lemma (5,L,.,60 or 50)$^+$ and sub-case hypothesis, and if deg(Ê) = 5 then no L4 can go from F to V by Lemma(5,L,.,L,5). Thus by Restriction (iii), v $\geq$ 6 and we have to consider only those arrangements which provide exactly three V_5's..

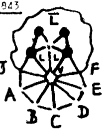
Arrangement check list

	J	A	B	C	D	E	F	
5/4		5	5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,.,T2)$_1$ (5,L,.,L,5), (5,.,.,60 or 50)$^+$ and sub-case hypothesis
5/4r		.	.	5	5	5		impossible since $q_{TL}(V) \leq 0$
5/6		5	5	.	5	.		impossible since $q_{TL}(V) \leq 0$
5/6r		.	5	.	5	5		impossible since $q_{TL}(V) \leq 0$
5/7		5	5	.	.	5		impossible since $q_{TL}(V) \leq 0$
5/7r		5	.	.	5	5		impossible since $q_{TL}(V) \leq 0$
5/9		5	.	5	5	.		impossible since $q_{TL}(V) \leq 0$
5/9r		.	5	5	.	5		impossible since $q_{TL}(V) \leq 0$
5/10		5	.	5	.	5		impossible since $q_{TL}(V) \leq 0$
5/15		.	5	5	5	.		impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(\dot{V},_{C})$ - Lemma

Case ν = 6 or 5, $\bar{\nu}$ = 5

Sub-case $\bar{v}$ = 100 and $\bar{L}$ = 841, 842, or 843



We have to consider only those arrangements which provide precisely three or four V_5's.

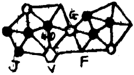
Arrangement check list

	J	A	B	C	D	E	F	
		5	5	5	5			if $\bar{L}$ = 841 or 842 → 15-9 . if $\bar{L}$ = 843 → impossible since $q_{TL}(\bar{V}) \leq 0$ by Lemmas (5,L,T2), (5,L,.,60 or 50)$^+$ and sub-case hypothesis
5/4		5	5	5		5		
		5	5	5	.	L5+ → 411 or 441n at E		if $\bar{L}$ = 841n → 1-21 if $\bar{L}$ = 842n → 6-11 if $\bar{L}$r → 1-20
		L4	5	L4	.	L4 → $\bar{L}$ = 841r or 842r by Lemma (5,L,.,60 or 50)$^+$ → 15-30		
5/4		5	5	5				impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,T2)
5/5		5	5	.	5	5		impossible since $q_{TL}(\bar{V}) \leq 0$ by Lemmas (5,L,.,L,5), (5,L,.,60 or 50)$^+$
5/6		5	5		5	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,.,60 or 50)$^+$, (60 or 50,T,60 or 50)
5/7		5	5	.	.	5		impossible since $q_{TL}(V) \leq 0$ by Lemmas (5,L,.,60 or 50)+, (5,L,F2), (60 or 50,.,60 or 50)
5/9		5	.	5	5			impossible since $q_{TL}(V) \leq 0$
5/10		5	.	5	.	5		
		L5+	.	5	.	5 → 411 or 441r at A		if $\bar{L}$n → 1-20 if $\bar{L}$ = 841r → 1-21 if $\bar{L}$ = 842r → 6-11
5/15		.	5	5	5	.		impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_{10})$ - Lemma

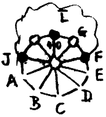
Case v = 6 or 5, w = 5

Sub-case v = 100 and L = 5[illegible]1



We have c(G) = 10 and c(F) $\leq$ 15, thus by Restriction (iv), v $\geq$ 7 contradicting the case-hypothesis. Thus this sub-case is impossible.

Sub-case $\bar{v}$ = 100 or 45 and L = 5 [illegible],...,5[illegible]



We have c(G) $\leq$ [illegible] and thus by Restriction (iii), v $\geq$ 6. Thus we have to consider only those arrangements which provide precisely three V_5's.

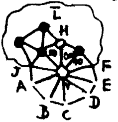
Arrangement check list

	J	A	B	C	D	E	F	
5/4		5	5	5	.	.		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,.,T2)
5/6		5	5	.	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemmas (60 or 50, . ,60 or 50), $(5,L,.,60 \text{ or } 50)^+$ and sub-case hypothesis
5/7		5	5	.	.	5		impossible since $q_{TL}(V) \leq 0$ by Lemma (5,.,T2)
5/9		5	.	5	5			impossible since $q_{TL}(V) \leq 0$ by Lemma (5,L,.,L,5)
5/10		5	.	5		5		impossible since $q_{TL}(V) \leq 0$ by Lemma $(5,L,.,60 \text{ or } 50)^+$ and sub case hypothesis
5/15			5	5	5	.		impossible since $q_{TL}(V) \leq 0$ by Lemma $(5,L,.,60 \text{ or } 50)^+$ and sub-case hypothesis

Proof of $T_L(V,0)$ - Lemma

Case v = 6 or [illegible], $\bar{V}$ = [illegible]

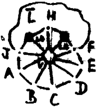
L = [illegible]r,...,854n, 549r, or 550r



Sub-case $\bar{V}$ = [illegible] or [illegible]

We have c(H) = 10 and thus by Restriction (iii), v $\geq$ 6. Thus we have to consider only those arrangements which provide precisely four V's. But in each of these arrangements at least one of the following two cases holds. (1) deg(A) = deg(B) = 5 and thus by Lemma(7,L,..,60 or [illegible]) and sub-case hypothesis, no L-discharging goes from A to V, thus c(J) $\leq$ 15. (2) deg(D) = deg(E) = 5 and thus either an R- or an L4-discharging goes from E to V, thus c(F) $\leq$ 15. In each case it follows by Restriction (iv) that v $\geq$ 7 contradicting the case-hypothesis. Thus this sub-case is impossible.

Sub-case $\bar{V}$ = 80 or 75 and L = 873,...,894

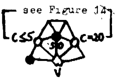


We have c(H) = 10 and thus by Restriction (iii), v $\geq$ 6. Thus we have to consider only those arrangements which provide precisely four V's. But in each of those arrangements we have deg(A) = deg(B) = 5 (L may be reflected or non-reflected) and thus we have either an R- or an L4-discharging from A to V. Thus c(J) $\leq$ 15 and by Restriction (iv), v $\geq$ 7 contradicting the case-hypothesis. Thus this sub-case is impossible.

Sub-case v < 85 and L = 551,...,691, or 861,...,872

L = 551,...,553

see Figure 14

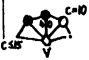


L = 561,...,617 or 632,...,691

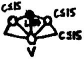
L = 618,619 or 651

L = 620 or 861,...,872

by Lemmas and sub-case hyp.



L = 621,...,624



In every case, we have by Restriction (iv), v $\geq$ 7. Thus this sub-case is impossib

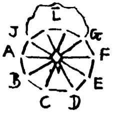
Proof of $q_{TL}(V_{10})$ - Lemma

Case v = 6 or 5, $\bar{v}$ = 4

In this case, no L-discharging can go from a boundary-V_5 of $\bar{L}$ to since otherwise a pair of -situations would violate the case-hypothesis.

Sub-case $\bar{v}$ = 100 or 95 (i.e., $\bar{L}$ = 431,...,437)

In this sub-case, no L5+ can go from F or A to V by Lemma (5,L), (5,L,..,60 or 50)$^+$; no L4 can go from A to V if deg(B) = 5 and no L4 can go from F to V if deg(E) = 5 by Lemmas (5,L,5), (5,L,..,L,5); no T2-discharging can go across J-A or F-G by Lemma(5,L,T2). Consequently, the non-5 vertex of $\bar{L}$ which is adjacent to V has a c-value ≤ 15 and thus by Restriction (iii), v ≥ 6. Thus we have to consider only those arrangements which provide precisely three V_5's.



Arrangement check list

	J	A	B	C	D	E	F	G	
6/8		5	5	5	.	.	.		impossible since $q_{TL}(V) \leq 0$
6/11		5	5	.	5	.	.		
		R	R	.	L6	6^{T2}	6^{T1}	6	→ $\bar{L}$ = 431n or 432n and 411 at D → 15-10
6/13		5	5	.	.	5	.		
		R	5	6^{T}	6	L6	6^{T1}	6	→ $\bar{L}$ = 431n or 432n and 411 at E → 15-16
6/14		5	5	.	.	.	5		impossible since $q_{TL}(V) \leq 0$
6/18		5	.	5	5	.	.		impossible since $q_{TL}(V) \leq 0$
6/19		5	.	5	.	5	.		
		L4	.	L4	.	L6	6^{T1}	6	→ $\bar{L}$ = 431n or 432n and 411 at E → 15-16
6/20		5	.	5	.	.	5		
	6	L4	.	L6	6^{T2}	6	R •	5	impossible since the pair at A,C violates the sub-case hypothesis
	5	R	.	L6	6^{T2}	6	L4	6	→ $\bar{L}$ = 431n or 432n and 411 at C → 15-10

continued next page

Proof of $q_{TL}(V_{10})$ - Lemma

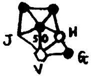
Case v = 6 or 5, [illegible] = 4

Sub-case $\bar{v}$ = 90 or 85

Arrangement check list, finished

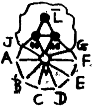
	J	A	B	C	D	E	F	G	
6/27		5	.	.	5	5	.		impossible since $q_{TL}(V) \leq 0$
6/28		.	5	5	5	.	.		impossible since $q_{TL}(V) \leq 0$
6/29		.	5	5	.	5	.		impossible since $q_{TL}(V) \leq 0$

Sub-case $\bar{v}$ = 80 and $\bar{L}$ = 441



We have $c(H) = 5$, $c(J) \leq 20$, and thus by Restriction (iv), $v \geq 7$ contradicting the case-hypothesis. Thus this sub-case is impossible.

Sub-case $\bar{v}$ = 80 and $\bar{L}$ = 801, 802, or 803



In this sub-case, no L5+ can go from A or F to V by Lemma (5,L,.,60 or 50)$^+$, no L4 can go from A to V if deg(B) = 5 and no L4 can go from F to V if deg(E) = 5 by Lemma(5,L,.,L,5); [illegible] across J-A or F-G by Lemma(5,L,T2). Consequently $c(G) \leq 15$ and thus by Restriction (iii), $v \geq 6$. Thus we have to consider only those arrangements which provide precisely four V_5's.

Arrangement check list

	J	A	B	C	D	E	F	G	
6/4		5	5	5	5	.	.		→ 15-5
6/6		5	5	5	.	5	.		impossible since $q_{TL}(V) \leq 0$
6/7		5	5	5	.	.	5		impossible since $q_{TL}(V) \leq 0$
6/9		5	5	.	5	5	.		impossible since $q_{TL}(V) \leq 0$
6/10		5	5	.	5	.	5		impossible since $q_{TL}(V) \leq 0$
6/12		5	5	.	.	5	5		impossible since $q_{TL}(V) \leq 0$

continued next page

Proof of $q_{TL}(V_{10})$ - Lemma

Case $v = 6$ or 5, $W = 4$

Sub-case $\mathcal{L} = 801, 802,$ or 803

Arrangement check list, finished

	J	A	B	C	D	E	F	G	
6/16		5	.	5	5	5	.		impossible since $q_{TL}(V) \le 0$
6/17		5	.	5	5	.	5		impossible since $q_{TL}(V) \le 0$
6/27		.	5	5	5	5	.		impossible since $q_{TL}(V) \le 0$

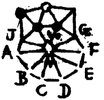
Sub-case $\bar{v} < 80$ and $\mathcal{L} \ne 465$

$\mathcal{L} = 461, \ldots, 464$

$\mathcal{L} = 466, \ldots, 522$

In both cases, we have by Restriction (iv), $v \ge 7$. Thus this sub-case is impossible

Sub-case $\bar{v} = 70$ and $\mathcal{L} = 465$



No L5+ can go from F to V by Lemma (5,L,.,60 or 50)†; no L4 can go from F to V if deg(E) = 5 by Lemma(5,L,.,L,5); no T2 can go across F-G by Lemma(5,L,T2). Consequently, $c(G) \le 15$. We have also $c(J) \le 15$. By Restriction (iii), $v \ge 6$. Thus we have to consider only those arrangements which provide precisely four V_5's and $c(G) = c(J) = 15$ (since otherwise by Restriction (iv), $\dot{v}$ would be ≥ 7). Thus we must have either a T1 across F-G, or an L4 at F and deg(E) > 5; further we must have an L5+ at A; (a T2 across J-A and an L5+ at B would also yield $c(J) = 15$ but this is impossible since it does not allow for four V_5-neighbors of V outside $\mathcal{L}$). Thus we have one of the following two cases.

J	A	B	C	D	E	F	G	
	L5+	.	5	5	5	6^{T1}		$\rightarrow$ 411 at A $\rightarrow$ no L at C $\rightarrow$ $q_{TL}(V) \le 0$ $\rightarrow$ impos.
	L5+	.	5	5	.	L4		$\nearrow$

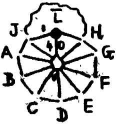
Proof of $q_{TL}(V_{10})$ - Lemma

Case $\nu = 6$ or 5, [illegible] = 3

In this case, any two V_5's which L-discharge to V must be at a distance of at least three (on the ring about V) since otherwise a pair of L-situations would violate the case-hypothesis. Thus at most three L-dischargings can go to V.

Sub-case $\bar{v} = 70$ (i.e., $\bar{L}$ = 401, 402, or [illegible])

In this sub-case, no T2-discharging can go across J-A or G-H by Lemma(5,L,T2). No L-discharging can go from A or G to V. Thus if $\bar{L}$n then c(H) ≤ 15, and if $\bar{L}$ r^{-1} then c(J) ≤ 15. Thus by Restriction (iii), $\nu \geq 6$ and we have to consider only those arrangements which provide precisely four V_5's.



Arrangement check list

	J	A	B	C	D	E	F	G	H	
7/8		5	5	5	5	.	.			impossible since $q_{TL}(V) \leq 0$ by Lemma(5,L,T2)
7/12		5	5	5	.	5	.	.		
		R	R	R	.	L6	6^{T2}	6^{T1}	6	$\to \bar{L}$n and 411 at E $\to$ 15-10
7/14		5	5	5	.	.	5	.		
		R	R	5	6^{T}	6	L6	6^{T1}	6	$\to \bar{L}$n and 411 at F $\to$ 15-16
7/15		5	5	5	.	.	.	5		impossible since $q_{TL}(V) < 0$
7/19		5	5	.	5	5	.	.		impossible since $q_{TL}(V) < 0$
7/21		5	5	.	5	.	5	.		impossible since $q_{TL}(V) \leq 0$
7/22		5	5	.	5	.	.	5		impossible since $q_{TL}(V) \leq 0$
7/24		5	5	.	.	5	5	.		impossible since $q_{TL}(V) < 0$
7/25		5	5	.	.	5	.	5		impossible since $q_{TL}(V) \leq 0$
7/27		5	5	.	.	.	5	5		impossible since $q_{TL}(V) < 0$

continued next page

Proof of $q_{TL}(V_{10})$ - Lemma

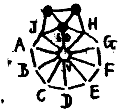
Case $\nu = 6$ or 5, [illegible] = 3

Sub-case $\tilde{v} = 70$

Arrangement check list, finished

	J	A	B	C	D	E	F	G	H
7/33		5	.	5	5	5	.	.	impossible since $q_{TL}(V) < 0$
7/34		5	.	5	5	.	5	.	impossible since $q_{TL}(V) \leq 0$
7/35		5	.	5	5	.	.	5	impossible since $q_{TL}(V) < 0$
7/37		5	.	5	.	5	5	.	impossible since $q_{TL}(V) \leq 0$
7/38		5	.	5	.	5	.	5	impossible since $q_{TL}(V) < 0$
7/43		5	.	.	5	5	5	.	impossible since $q_{TL}(V) < 0$
7/54		.	5	5	5	5	.	.	impossible since $q_{TL}(V) < 0$
7/55		.	5	5	5		5	.	impossible since $q_{TL}(V) \leq 0$
7/57		.	5	5	.	5	5	.	impossible since $q_{TL}(V) \leq 0$

Sub-case $\tilde{v} = 60$ (i.e., $\mathcal{L} = 411$)



In this sub-case, no L-discharging can go from a V_5 to V if the V_5 is adjacent to another degree-5 neighbor of V. No L-discharging can go from A or G to V.

Now suppose that $\nu = 5$. Then by Restriction (iii), every non-5 neighbor of V must have a c-value of 20. In order to obtain c(H) = c(J) = 20 we must have T-dischargings across both J-A and G-H; thus deg(A) = deg(G) = 6. Now, in order to obtain c(A) = 20, we must have either another T-discharging across A-B or an L-discharging from B to V; thus one of B, C is a non-5 vertex. Consequently, since $\nu = 5$, D, E, and F are V_5's. Thus $c_{**}(G) =$ [illegible] and $c(G) \leq 15$ which violates the above condition. Thus $\nu = 5$ is impossible.

It remains to consider all those arrangements which provide precisely five V_5's (outside of $\mathcal{L}$ on the ring about $\tilde{V}$).

Proof of $q_{TL}(V_{10})$ - Lemma

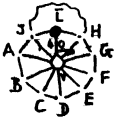
Case ν = 6 or 5, [illegible] = 3

Sub-case $\bar{v}$ = 60

Arrangement check list

	J	A	B	C	D	E	F	G	H	
7/4,6,7		5	5	5	5					→ 15-10
7/10		5	5	5	.	5	5	.		impossible since $q_{TL}(V) < 0$
7/11		5	5	5	.	5	.	5		impossible since $q_{TL}(V) \leq 0$
7/13		5	5	5	.	.	5	5		impossible since $q_{TL}(V) < 0$
7/17		5	5	.	5	5	5	.		impossible since $q_{TL}(V) < 0$
7/18		5	5	.	5	5	.	5		impossible since $q_{TL}(V) < 0$
7/20		5	5	.	5	.	5	5		impossible since $q_{TL}(V) \leq 0$
7/31		5	.	5	5	5	5	.		impossible since $q_{TL}(V) < 0$
7/32		5	.	5	5	5	.	5		impossible since $q_{TL}(V) < 0$
7/53		.	5	5	5	5	5	.		
		$^{T2}6$	5	5	5	5	5	6^{T2}		→ 15-16

Sub-case $\bar{v}$ = 40 (i.e., $\bar{L}$ 421,...,428)



In this sub-case, no L5+ can go to V. Thus we have c(H) $\leq$ 15, c(J) $\leq$ 15 and hence by Restriction (iii), $\nu \geq 6$.

Now suppose that ν = 6. Then, because of Restriction (iv), we must have c(H) = c(J) = 15. But this requires T2-dischargings to go across both J-A and G-H (since no L-discharging can go from A or G to V), which is impossible by Lemma(T2,L,T2). Thus this sub-case is impossible.

Case ν = 6 or 5, [illegible] = 0

This case is impossible since at most 9 - ν T-dischargings can go to V and thus $q_{TL}(V) \leq 0$.

This finishes the proof of the $q_{TL}(V_{10})$ - Lemma. ■

Proof of the $q_{TL}(V_{11})$ - Lemma.

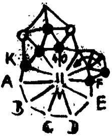
We apply the Upper Bound Lemma for $d(V_k)$ (as proved in Section 3 of the paper) to our q_{TL}-positive vertex V of degree k = 11. Since $q_o(V) = -300$ and $q_{TL}(V) = q_o(V) + d(V) > 0$, we have $d(V) > 300$. Thus the inequalities (3.1) and (3.2) give us the following restrictions for the number v of degree-5 neighbors of

(I) $v \geq 8$.

(II) If the configuration of Figure 13 does not occur then

$v \geq 9$.

Case 5,L,T7 (i.e., the configuration of Figure 13 occurs with pivot identified to V).



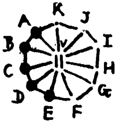
For the vertices A, B,...,E, we have to consider only those arrangements which provide at least four V_5's.

Arrangement check list

	K	[illegible]	B	C	D	E	F	
5/1		5	5	5	5	5		→ 15-34
5/2		5	5	5	5	.		
		R	R	R	L4	.	L6	— impossible by Lemma(5,L,.,60 or 50)$^+$
5/2r		.	5	5	5	5		— impossible since $q_{TL}(V) \leq 0$
5/3		5	5	5	.	5		> impossible since $q_{TL}(V) \leq 0$
5/3r		5	.	5	5	5		
5/5		5	5	.	5	5		— impossible since $q_{TL}(V) \leq 0$

Proof of $q_{TL}(V_{11})$ - Lemma

Case "not 5,L,T2" (i.e., the configuration of Figure 1 does not occur)



In this case we have $v \geq 9$ by Restriction (II). We denote the neighbors of V in consecutive order by A, B,...,K so that the corresponding reading is maximal (among the 22 possible readings which can be obtained by cyclic permutations and reflections).

Arrangement check list

A	B	C	D	E	F	G	H	I	J	K	
5	5	5	5	5	5	5	5				→ 15-34
5	5	5	5	5	5	5	.	5	5	.	
5	5	5	5	5	5	5	6	5	5	.	→ 15-35
5	5	5	5	5	5	5	U	5	5	U	impossible since $q_{TL}(V) < 0$ by Lemma(5,L,..,L,5)
5	5	5	5	5	5	.	5	5	5	.	
5	5	5	5	5	5	6	5	5	5	.	→ 15-35
5	5	5	5	5	5	U	5	5	5	U	impossible since $q_{TL}(V) < 0$ by Lemma(5,L,..,L,5)
5	5	5	5	5	.	5	5	5	5	.	
5	5	5	5	5	6	5	5	5	5	.	→ 15-35
5	5	5	5	5	U	5	5	5	5	U	impossible since $q_{TL}(V) < 0$ by Lemma(5,L,..,L,5).

This finishes the proof of the $q_{TL}(V_{11})$ - Lemma and finishes the proof of the $q_{TL}(V_k)$ - Lemma. ■

(10) Proof of the S-Lemma. Many CTL-situations (## 1,2,6,7,9,11,...,41,43,...,4
49,...,54,56,65,69,70,72,81,...,94,99,111,112,113,116,119,123,...,127,133,135, 137,138,141,145,150,151, and 152) are to be regarded as single configurations rather than configuration-classes and the corresponding S-situations occur as sub-configurations of the drawings in Table 4. In these cases, the numbers of the S-situations are written at the corresponding places in the check list pp. C 194, ..., C 199.

In several other CTL-situations (## 5,9,10,47,48,55, and 66) some vertex with degree-specification "." must be specified more in detail as "6 " or "U " in order to obtain a configuration which contains some S-situation as a sub-configuration. These cases are also treated on pp. C 194,..., C 199 where the two S-situations (one of them corresponding to "6 " and the other to "U ") are joined by a brace.

In case of CTL # 3 it is necessary to specify for some outside vertices "5 ", "6 ", or "U " in order to obtain some required S-situation as a sub-configuration. This is done in detail on p. C 200. In case of CTL # 4, some "." must be specified "5 " or "6 ", see p. C 201.

The remaining CTL-situations are treated as configuration-classes on pp. C 202 ...,210 where the parameter ranges over the set of L-situations as specified in Table 4. The only 2-parameter classes are CTL # 121 and 122. In some of the 1-parameter classes (## 74,101,102,103,128,129,132) the symbol  is used in Table 4 in such a way that in the class check lists it must be specified more in detail as  or  in order to obtain configurations which contain the required S-situations as sub-configurations. ■

This finishes the proof of the Discharging-Theorem for $\mathcal{P}(\mathcal{T}, \mathcal{S}, \mathcal{L}), \mathcal{U}$.

■

Every planar map is four colorable
Part II: reducibility
Supplement: The reducers and the n-decreased extensions of the configurations of $\mathcal{U}$ and some remarks

(a) The immersion reducibility of the configurations of $\mathcal{U}$.

For the configurations of the unavoidable set $\mathcal{U}$, as presented in Part II of the paper, we establish the following properties.

(i) If C is a configuration of $\mathcal{U}$ of ring size n and containing precisely vertices then $6 \leq n \leq 14$ and $n + m \leq 28$.

(ii) If C is a configuration of $\mathcal{U}$ then C is D-reducible or is C-reducible with a fine reducer (as defined in Section 3 of Part II of the paper).

(iii) If C is a configuration of $\mathcal{U}$ and if C_1 is an n-decreased extension of C which is obtained from C by adding precisely one vertex, then C_1 is of ring-size ≥ 6 and one of the following two cases applies.

(iii.a) C_1 contains a configuration, say C_1', of $\mathcal{U}$ so that the ring size of C_1' is not greater than the ring size of C_1 (it may be that $C_1' = C_1$)

(iii.b) C_1 is D-reducible or C-reducible with a fine reducer and if C_2 is an n-decreased extension of C_1 which is obtained from C_1 by adding precisely one vertex, then C contains a configuration of $\mathcal{U}$ which is of no greater ring size than C_2

Property (i) is easily established by ocunting. In Section (b) of this supplement the numbers of configurations in with particular n- and m-values are given.

Property (ii) is established by the machine computations described in Section 2 of Part II of the paper. For all those configurations of $\mathcal{U}$ which are not D-reducible and of ring-size $n \geq 11$ the C-reducers are presented in Section (β) of this supplement.

Property (iii) is established in Section (γ) of this supplement. For all but 55 configurations of $\mathcal{U}$, Case (iii.a) applies for every n-decreased extension C_1 which is obtained by adding precisely one vertex. For each configuration C the number of "first generation" n-decreased extensions (i.e., n-decreased extensions which are obtained by adding precisely one vertex to C) is equal to the number of 1-legger outer sectors of C (a 1-legger vertex of C lies at precisely one 1-legger outer sector, but an articulation vertex of C lies at precisely two 1-legger outer sectors; the vertex to be added to C in order to obtain an n-decreased extension must have at least three neighbors in G and thus must be a vertex on the ring about C which is adjacent to at least one 1-legger or articulation vertex of C, see Section 3 of Part II of the paper and in particular, Figure A) This number is between zero and five for all configurations in $\mathcal{U}$. For each (first generation) n-decreased extension C_1 we give in Section (γ) the number of the configuration C'_1 of $\mathcal{U}$ which is contained in C_1 (and of no greater ring-size than C). The 55 cases in which such a configuration C'_1 does not exist are explicitely drawn. In all but 19 of these cases, the D-reducibility of C_1 follows immediately from the corollary to Lemma I (see Section 3 of Part II of the paper). In the 19 remaining cases, a machine computation for C_1 has been carried out; in these cases we have given C_1 the same number as C (the configuration of which it is an extension), followed by an asterisk. All but three of these configurations are D-reducible. For the three D-irreducible configurations, 22-7*, 29-2*, and 53-31* the C-reducers are presented at the end of the table in Section (β) of this supplement. Moreover, in all 55 cases the first generation n-decreased extensions of C_1 (which are "second generation" n-decreased extensions of C) are taken care of by giving the numbers of sub-configurations of $\mathcal{U}$.

Note that it is practically much more convenient to establish Property (iii) than it would be to explicitely consider all n-decreased extensions (of arbitrarily high generation) of all configurations of $\mathcal{U}$.

Now we claim the following.

Immersion reducibility of the configurations in $\mathcal{U}$. As a consequence of Properties (i), (ii), and (iii) above and of the Theorem proved in Section 3 of Part II of the paper, no configuration of $\mathcal{U}$ can be immersed into a minimal five-chromatic planar triangulation.

Proof. We need the following.

Corollary to the Theorem (p. 498 of the paper). Let C be a configuration which contains precisely m vertices and is of ring-size n so that $n \leq 14$ and $n + m \leq 28$, and let $C'_{(1)}, \ldots, C'_{(p)}$ be all n-decreased extensions of C. Suppose that

(a) C is D-reducible or is C-reducible with a fine reducer, and that

(b) none of $C'_{(1)}, \ldots, C'_{(p)}$ can be immersed into a minimal five-chromatic triangulation.

Then C cannot be immersed into a minimal five-chromatic triangulation.

The proof of the corollary is obtained from the proof of the theorem (as given on pp. 498 to 503 of the paper) by deleting paragraphs (1) and (2) and all references to the induction hypothesis (2). (Note that the hypothesis of the corollary is stronger than the hypothesis of the theorem and thus no induction on n is necessary; otherwise the proof remains verbally the same.)

Now let $\mathcal{U}^*$ be the set which consists of the configurations of $\mathcal{U}$ and of all those first generation n-decreased extensions of configurations of $\mathcal{U}$ for which Case (iii.b) (p. 198) applies. We prove by induction on n that if C is a configuration of $\mathcal{U}^*$ of ring-size n then C cannot be immersed into a minimal five-chromatic triangulation.

Note that because of Properties (i) and (iii), all configurations of $\mathcal{U}^*$ have ring-sizes ≥ 6 and if $n = 6$ then C has no n-decreased extensions. Thus, if $n = 6$ then C fulfills the hypothesis of the above corollary and cannot be immersed into a minimal five-chromatic triangulation.

From now on we assume $n > 6$ and we assume by induction that no configuration of $\mathcal{U}^*$ whose ring-size is smaller than n can be immersed into a minimal five-chromatic triangulation. It remains to prove that C cannot be immersed into a minimal five-chromatic triangulation.

We claim that no n-decreased extension of C can be immersed into a minimal five-chromatic triangulation. For if C_* is an n-decreased extension of C then there is a first generation n-decreased extension, say C_1, of C such that either $C_* = C_1$ or C_* is an n-decreased extension of C_1. In every case there is an immersion $f_1 : C_1 \to C_*$ (which respects the degree-specifications). But because of Property (iii), C_1 contains a configuration, say C', of $\mathcal{U}$ whose ring-size is smaller than n, i.e., there is an immersion $f' : C' \to C_1$. Now suppose that there is an immersion $f_* : C_* \to \Delta$ of C_* into a planar triangulation Δ (which respects the degree-specifications). Then the composite mapping $f' \circ f_1 \circ f_*$ is an immersion of C' into Δ (which respects the degree-specifications). But by induction hypothesis, C' cannot be immersed into a minimal five-chromatic triangulation. Thus C_* cannot be immersed into a minimal five-chromatic triangulation as asserted. Thus C fulfills Hypothesis (b) of the above corollary. But the other hypotheses of the corollary are fulfilled because of Properties (i), (ii), and (iii). Thus C cannot be immersed into a minimal five-chromatic triangulation and the proof of the immersion reducibility is finished. ∎

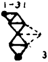
(β) **The reducers.** On the following 15 pages the reducers are presented for those configurations of $\mathcal{U}$ which are not D-reducible and are of ring size 11 or greater. Also presented are the reducers for the three configurations 22-7*, 29-2*, and 53-31* which do not belong to $\mathcal{U}$ but are required for the immersion reducibility of the corresponding configurations 22-7, 29-2, and 53 of $\mathcal{U}$. For an explanation of the diagrams see p. 504 of the paper.

1-25

7/11
3707
I 1,3
I 5,7,9,11

1-30

8/12
11968
I 1,3,6,11
I 7,9

1-31

8/13
39448
I 1,3,10,12
I 5,7,9

2-28

8/12
6655
I 3,5 D 2,6
I 7,9,11 D 6,12
D 1,6

3-6

10/14
7290
I 1,3,12
I 6,8

3-10

6/11
3309

3-11

8/12
2471
I 1,4,9,11
I 5,7

3-13

7/12
7738
I 1,5 D 1,7
I 9,11 D 2,11
D 7,12

3-21

7/11
1377
I 1,3
I 5,7,9,11

3-22

8/12
4158
I 3,5,8,10,12

3-23

8/12
145
I 1,3,5,11
I 7,9 D

3-27

6/11
1344
I 1,10
I 2,4,6,8

4-2

7/12
6232
I 1,4,6,8
I 9,11

4-3

8/13
8316
I 1,10,12
I 2,4,6,8

4-5

8/13
8289
I 1,3,5,7
I 8,10

4-25

12393
I 1,3,5,7,9

5-3

7/11
3830
I 1,3,10
I 5,7,9

5-14

7/11
3680
I 1,4,6
I 8,10
D 1,8

5-29

9/14
11665
I 1,3,10
I 5,7,9

5-30

9/12
10048
I 1,3,6,11
I 7,9

5-31

9/12
10093
I 1,3,6,11
7,9

5-32

11/14
89015
I 1,8,13 D 1,11
I 9,11 D 1,6
D 1,5

6-13

8/12
11189
I 1,3,10
I 5,7,9

6-16

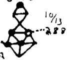
8/12
2471
I 1,8,10
I 2,4,6

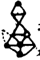
6-17

9/13
7371
I 5,7,9
I 2,4,13

6-21

9/13
32588
I 1,3,8,10,12
I 5,7

6-30

10/13
28860
I 2,4,6,11,13
I 7,9

6-31

10/13
29594
I 3,4 I 6,8
I 9,11 I 1,12
D 1,8 D 2,6

7-4

7/11
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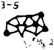

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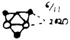
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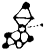
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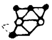

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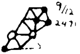

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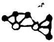

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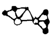

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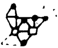

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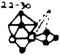

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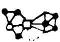

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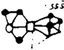

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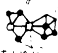

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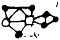

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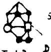

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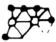

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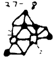

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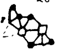

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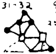
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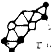
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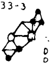
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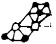
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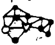
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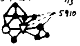

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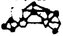

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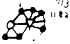

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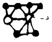

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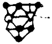

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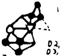

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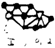

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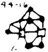

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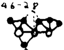

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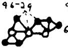

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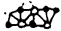

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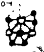

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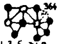

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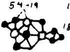

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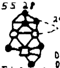

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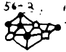

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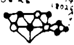

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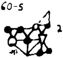

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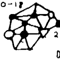
I 5,10 D8,10

60-18 11/14 22159



I 1 8,11,13 D1,5 D1,6

61-7 10/13 12042

I 1,6,8,10,12

▼61-11 10/13 9450

I 1,6,8,10,12

▼61-17 11/14 19548

I 1,11,13 D1,7
I 5,7 D1,8
D1,9

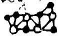
6 28 11/14 40992

I 1,11,13 D7,10
I 5,10 D8,10

5 31 11/14 7371

I 2,4,6
I 1,12
I 9,11

▼62 11 10/14 4374

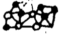


I 2,4,6,8
I 1,13
I 10,12

62 15 11/14 4374

I 2,4,6,8
I 1,13
I 9,11

62-17 11/14 19008



I 1,9,11,13
I 2,4,6,8

63-1 10/14 22154

I 1,11,13 D6,9
I 4,9 D1,9 D7,9

63-2 11/14 22154

I 1,5,11,13
D1,9 D1,8

63-6 11/14 22154

I 1,9,11,13 D1,5 D1,6 D1,7

62-11 11/14 21065

I 1,3,5,7 D1,9 D1,10 D1,11

16-20 9/13 9504

I 5,7,9,11,13

13-24 10/13 13440

I 1,3,5
I 6,8,10

5-35 8/12 12492

I 1,6,11
I 2,4 I 1,9

13-29 8/12 704

I 2,4,6,9,11

6-34 8/11 591

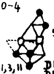
I 1,8,10
I 2,4, I 5,7

13-8 8/12

4158

I2,5,7,9,11

30-4 9/12



5456

I1,3,11
I4,6

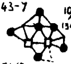
D1,7
D1,8
D1,9

34-1 8/12

2471

I1,6,9,11 I3,5

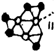
43-7 10/13



13440

I1,10,12
I5,7,9

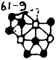
48-10 9/13



11988

I2,7,9,11,13

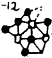
61-9 11/13



6048

I2,4,6,8,10

61-12 10/14



36801

I1,6,8,10,12

62-3 10/13

7371

I1,3,5
I9,11,13

7-34 9/13

7371

I1,3
I7,9,11,13

9-23 9/13

9639

I1,6,8,10,12

29-6 10/14

34047

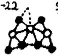
I1,4,6,8,10

56-10 10/14

22154

I1,4,9,11,13

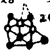
52-22 9/13



13440

I1,3,5
I6,8,10

50-28 9/12



2471

I1,4,9,11
I5,7

50-27 10/13

8478

I1,3,5
I8,10,12

50-29 10/13

7371

I1,8,10,12
I5,7

34-21 10/13

7371

I1,8,10,12
I5,7

23-17 10/14

20628

I1,13 I2,4
I6,8,10,12

17-14 9/13

0
3187

I1,3,5
I9,11,13

22-7* 9/12

4704

I2,4,6,9,11

53-31* 11/13

6264

I2,4,6
I1,10,12

29-2* 9/12

2471

I1,3,5,8
I9,11

(γ) The n-decreased extensions. On the following 30 pages the first generation n-decreased extensions of the configurations of $\mathcal{U}$ are treated. Configurations of $\mathcal{U}$ which do not have any 1-legger outer sectors need not be considered. If a configuration C of $\mathcal{U}$ has more than one 1-legger outer sector then its 1-legger outer sectors are treated in clockwise order (going around the configuration), beginning with that 1-legger outer sector which is in the highest position in the drawing in the $\mathcal{U}$-table; if two 1-legger outer sectors are in equally high positions then we begin with the left one of them.

If a 1-legger outer sector of C is not consecutive with another 1-legger outer sector then it corresponds one-to-one to the n-decreased extension of C which is obtained by adding a V_5 to C at the end of the leg. If precisely two 1-legger outer sectors of C are consecutive then they correspond to the two n-decreased extensions of C which are obtained by adding a V_5 or a V_6, respectively to C at the common end of the two legs. If three 1-legger outer sectors are consecutive then they correspond to the three n-decreased extensions of C which are obtained by adding a V_5, a V_6, or a V_7, respectively, to C at the common end [illegible]

In most cases, every first generation n-decreased extension of C contains some configuration of $\mathcal{U}$ of no greater ring-size. In these cases we have listed the numbers of the sub-configurations after the number of C with an arrow from C to each one of its extensions. The (number of the sub-configuration of the) n-decreased extension which corresponds to the first 1-legger sector is written in the highest position, then the others follow below in order. If two extensions correspond to two consecutive sectors then the one obtained by adding a V_5 is listed above the one obtained by adding a V_6 and the two entries are joined by a brace (and only one arrow is drawn from C to the pair of extensions); likewise of triplets of consecutive 1-legger sectors.

If not every first generation n-decreased extension of C contains a configuratic of $\mathcal{U}$ of no greater ring-size then both C and the exceptional extension(s) are explicitely drawn and the first generation n-decreased extensions of the exceptional extension(s) are also treated. In the drawings, the 1-legger outer sectors are indicated by dashed lines.

There is only one case (9b-7) in which a configuration of $\mathcal{U}$ has more than one exceptional n-decreased extension. But there are several cases in which two different configurations of $\mathcal{U}$ have the same exceptional n-decreased extension. Numbers are given only to those exceptional n-decreased extensions which required machine computations, i.e., are not n-decreased extensions of any irreducible configuration of $\mathcal{U}$.

1-4 → 1-1

1-9 → 1-2

1-14 → 1-6

1-15 → 1-6

1-17 → 1-9

1-19 → 1-6

1-20 ↗ 1-2
↘ 1-2

1-21 ↗ 1-2
↘ 1-2

1-22 ↗ 1-2
↘ 1-2

1-23 ↗ 1-
↘ 1-3

1-24 ↗ 1-
↘ 1-3

1-25 ↗ 1-6
↘ 1-6

1-26 ↗ 1-2
↘ {1-2, 1-3}

1-27 ↗ 1-2
↘ 1-2

1-28 ↗ 1-3
↘ 1-1

1-29 ↗ 1-2
↘ 1-7

1-30 ↗ 1-6
↘ 1-6

1-31 ↗ **1-6**
↘ 1-6

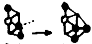
1-32 ↗ 1-9
↘ 1-11

2-3 → 1-1

2-4 → 1-1

2-5 → 1-2

2-7 → 1-2

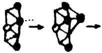


2-12 C → 2-12* D

2-14 ↗ 1-2
↘ 1-2

2-15 → 1-1

2-16 → 1-2



2-18 D → 1-

2-19 → {1-2, 1-7}

2-20 → 1-22

2-21 → 1-1

2-22 → 1-2

2-23 → 1-2

2-24 → 1-4

2-25 ↗ 1-2
↘ 1-2

2-26 → 1-6

2-27 → 1-11

2-28 → 1-2

2-29 → 1-2

2-31 → 1-24

2-33 → 1-5

2-34 ↗ 1-2
↘ 1-6

2-35 ↗ 1-2
↘ 1-2

3-1 → 1-6

3-2 → 1-6

3-3 → 1-7

3-5 → 1-6

3-6 ↗ 1-6
↘ 1-6

-7 → 1-2

3-8 → 1-6

3-9 → 1-6

3-11 → 1-2

3-12 → 2-4

3-13 C → 3-13 D.

3-16 → 1-6

3-17 → 1-6

3-**18** → 1-1

3-19 ↗ 1-6
↘ 1-2

3-20 → 1-2

3-22 → 1-2

3-25 → 2-9

3-30 → 3-1

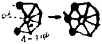
4-3 → 3-10

4-8 → 1-2

4-10 → 1-2

4-11 → 3-18

4-14 → 2-2



4-19

D

4-20 → 2-2

4-21 → 3-26

4-22 ↗ 3-26 ↘ 1-2

4-23 → 3-26

4-24 → 4-28

4-25 → 4-28

4-26 → 4-29

4-27 → 4-29

5-1 ↗ 2-1 ↘ 2-1

5-2 ↗ 2-1 ↘ 2-1

5-3 ↗ 2-1 ↘ 2-1

5-4 ↗ 2-1 ↘ {2-1 2-2

5-5 ↗ 2-1 ↘ 2-1

5-6 ↗ 2-4 ↗ 2-1 ↘ 1-2 ↘ 1-2

5-7 ↗ 2-1 ↗ 2-1 ↘ 1-3 ↘ 1-2

5-8 ↗ 2-1 ↘ 2-1

5-9 ↗ 2-1 → 2-2 ↘ 1-1

5-10 ↗ 2-1 ↘ 3-26

5-11 ↗ 2-6 ↘ 3-10

5-12 ↗ 2-2 ↘ 2-2

5-13 ↗ 2-1 ↘ 2-2

5-14 ↗ 2-2 ↘ 2-2

5-15 ↗ 3-28 → 2-2 ↘ 1-1

5-16 ↗ 3-28 ↘ 2-2

5-17 ↗ 2-2 ↘ 2-1

5-18 ↗ 2-2 ↘ 2-2

5-19 ↗ 3-28 ↘ 2-2

5-20 ↗ 2-2 ↘ 3-26

5-21 ↗ 2-10 ↘ 2-3

5-22 ↗ 2-16 → 2-3 ↘ 1-2

5-23 ↗ 2-2 ↘ 3-26

5-24 ↗ 2-16 ↘ 2-3

5-25 ↗ 2-9 ↘ 2-3

5-26 ↗ 3-10 ↘ 2-6

5-27 ↗ 2-2 ↘ 3-26

5-28 ↗ 3-21 ↘ 3-10

5-29 ↗ 2-3 → 2-2 ↘ 1-1

5-30 ↗ 2-6 ↘ 3-21

5-31 ↗ 3-21 ↘ 3-21

5-32 ↗ {1-1 3-10 ↘ 2-9 ↘ 1-1

5-33 ↗ 3-28 ↘ 2-2

5-34 ↗ 2-2 → 3-26 ↘ 1-2

5-35 ↗ 2-6 ↘ 2-6

6-1 → 2-2, 2-3, 1-1

6-2 → 2-3, 2-2, 1-1

6-3 → 2-2, 2-2

6-4 → 2-2, 2-2

6-5 → 2-2, 2-7

6-6 → 2-8, 2-7

6-7 → 2-12, 2-12

6-8 → {2-2, 2-9}, 2-12

6-9 → {2-1, 2-2}, 2-2

6-10 → {2-2, 2-9}, 2-2

6-11 → 1-2, 2-2

6-12 → 1-2, 2-2

6-13 → 1-6, 2-2

6-14 → 2-3, 2-2

6-15 → 2-3, 2-2

6-16 → 1-2, 1-3

6-17 → 1-6, 1-6

6-18 → 1-5, 1-6

6-19 → {1-2, 1-3}, 1-2

6-20 → {1-3, 1-6}, 1-3

6-21 → {1-6, 1-11}, 1-6

6-22 → {1-9, 2-5}, 1-11

6-23 → 1-4, 1-3

6-24 → {1-2, 1-3}, 1-3

6-25 → 1-2, 1-2

6-26 → 1-2, 1-2

6-27 → {1-2, 1-3}, 1-2

6-28 → {1-3, 1-6}, 1-4

6-29 → 1-4, 1-3

6-30 → {1-6, 1-11}, 1-6

6-31 → 1-6, 1-6

6-32 → {1-9, 1-14}, 1-11

6-33 → 1-11, 1-9

6-34 → 1-2, 1-2

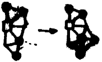
6-35 → {1-2, 1-3}, 1-3

7-4 → 1-1

7-5 → 1-2

7-6 → 1-2

7-9 → 1-1



7-10
D

7-12 → 1-1

7-14 → 1-2

7-16 → 1-6, 1-2

7-18 → 1-1, 1-6

7-20 → 1-2

7-21 → 1-1

7-23 → 1-1

7-26 → 1-2

7-29
D

7-30 → 1-2
C

7-31 → 1-1

7-32 → 1-2

7-34 7-34*
C D

7-35
C D

8-2 → 7-7

8-4 → 1-2

8-5 ↗ 1-2
↘ 1-2

8-6 ↗ 1-1
↘ 8-8

8-7 → 1-2

8-9 → {1-1
1-5

8-10
D

8-11 ↗ 1-1
↘ 1-6

8-14 → 1-2

8-15 → 1-2

8-16 → 1-22

8-18 ↗ 1-2
↘ 1-1

8-19 ↗ 1-2
↘ 1-6

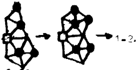
8-23 → 1-2

8-24 ↗ 1-2
↘ 1-2

8-26 → 1-5

8-27 ↗ 1-11
↘ 1-2

8-28 → 1-1



8-29
D

8-30 → 1-2

8-31 ↗ 1-2
↘ 1-2

8-32 ↗ 1-2
↘ 1-2

9-4 → 10-1

9-6 → 1-12

9-8 → 1-2

9-11 → 1-1

9-12 → 1-2

9-13 → 1-1

9-15 → 8-8

9-17 → 8-8

9-21 → 1-2

9-22 → 1-1

9-23 9-23*
C D

9-26 → 1-1

9-27 → 8-13

9-28 → 1-2

9-30 → 1-6

9-31 ↗ 1-6
↘ 1-2

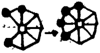
9-32 ↗ 1-2
→ 1-2
↘ 1-2

10-1 → 7-1

10-2 → 7-1

10-3 → 7-2

10-4 → 7-2



10-5 10-5
C

10-6 → 9-10

10-7a → 7-5

10-8 ↗ 9-16
↘ 7-27

10-9 → 10-13

10-10 → 10-13

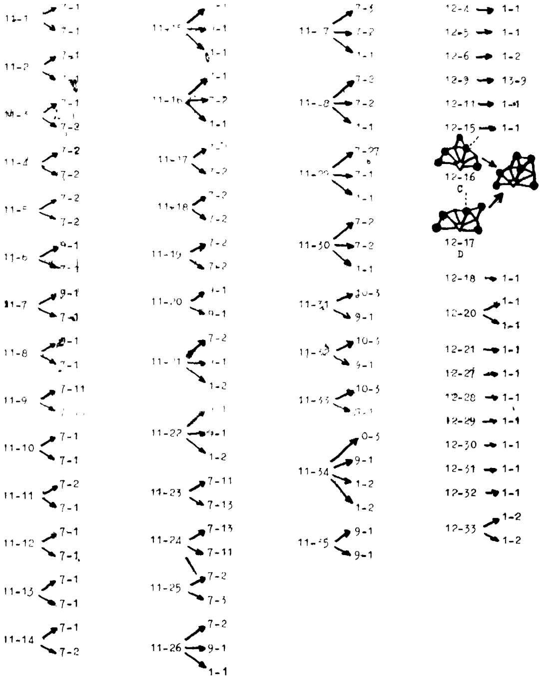
10-11 → 9-9

10-12 → 7-2

10-21 ↗ 7-2
↘ 7-27

10-22 ↗ 7-27
↘ 7-27

10-23 ↗ 9-1
↘ 7-1


11-1
7-1
7-1
11-2
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11-3
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11-4
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7-2
7-2
11-6
9-1
7-1
11-7
9-1
7-1
11-8
9-1
7-1
11-9
7-11
11-10
7-1
7-1
11-11
7-2
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11-16
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11-27
7-3
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7-27
b
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11-32
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11-33
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9-1
11-34
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1-2
1-2
11-35
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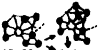
9-1
12-4
1-1
12-5
1-1
12-6
1-2
12-9
13-9
12-11
1-1
12-15
1-1
12-16
C
12-17
D
12-18
1-1
12-20
1-1
1-1
12-21
1-1
12-27
1-1
12-28
1-1
12-29
1-1
12-30
1-1
12-31
1-1
12-32
1-1
12-33
1-2
1-2

13-3 → 1-2
13-4 → 1-1
13-21 → 1-1
13-27 → 12-2
13-28 → 14-4
13-30 → 12-2
13-31 → 12-2
13-32 ↗ 1-1
↘ 12-2
13-33 ↗ 1-1
→ 1-1
↘ 14-4
13-34 → 12-7

14-2 ↗ 13-5
→ 12-2
↘ 1-1
14-3 ↗ 1-1
↘ 12-2
14-17 ↗ 12-1
↘ 12-1
14-18 ↗ 12-2
↘ 12-2
14-19 ↗ 12-1
→ 12-2
↘ 1-1
14-20 ↗ 12-1
→ 12-2
↘ 1-1

14-21 ↗ 1-1
↘ 12-1
14-22 ↗ 12-1
↘ 12-2
14-23 ↗ 12-2
→ 12-2
↘ 1-1
14-24 ↗ 12-1
→ 12-2
↘ 1-1
14-25 ↗ 13-2
↘ 12-12
14-26 ↗ 12-2
→ 13-8
↘ 1-1
14-27 ↗ 13-10
→ 13-10
↘ 1-2
14-28 ↗ 13-10
↘ 13-10
14-29 ↗ 12-1
↘ 12-2

[illegible] → 1-1
15-6 ↗ 1-9
↘ 1-6
15-7 ↗ [illegible]
↘ [illegible]
15-8 → 1-1
15-14 ↗ 1-1
→ 1-1
↘ 1-1
15-15 → 1-1
15-17 ↗ 1-1
→ 1-1
↘ 5-1
15-19 → 1-1
15-21 → 1-1



15-22 ↘ 1-1
D

15-23 ↗ 1-20
↘ 1-2
15-24 → 1-2
15-27 → 1-1
15-28 → 1-1
15-29 ↗ 1-1
↘ 1-1
15-33 ↗ 15-3
↘ 15-3

16-1 → 1-1

16-3 → 16-2

16-6 → 16-2

16-7 ↗ 16-1
↘ 17-1

16-8 → {16-2
1-2}

16-9 → 16-2

16-10 → 16-4

16-11 → 16-17

16-14 → 1-2

16-15 → 18-13

16-18
D

16-21 → 1-1

16-23 → 1-2

16-24 → 17-10

16-25 → 17-10

16-26 → 16-16

16-27 ↗ 16-17
↘ 17-25

16-28 → 2-6

16-29 → 2-3

16-30 ↗ 19-3
↘ 17-8

16-31 ↗ 6-11
↘ 2-1

16-32 → 2-2

16-33 → 16-11

16-34 ↗ {1-2
1-6}
↘ 1-2

16-35 → 16-20

17-4 → 16-2

17-6 → 1-2

17-12 → 2-2

→ 1-1

17-13
D

17-14 → 2-6

17-15 → 16-16

17-19 → 2-6

17-20 → 2-8

17-22 ↗ 16-1
↘ 2-1

17-23 → 1-1

17-24 → 1-1

17-27 → 16-16

17-31 → 2-1

17-32 → 16-16

17-33 → 2-2

17-34 → 16-16

18-1 ↗ 2-1
↘ 2-1

8-2 ↗ 2-3
↘ 2-2

18-3 ↗ 2-2
↘ 2-3

18-4 ↗ 2-6
↘ 2-6

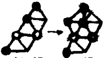
18-5 ↗ 2-7
↘ 2-8

18-6 ↗ 2-8
↘ 2-7

18-7 → 16-1

18-13 → 1-1

18-16 → 1-1



18-17 18-17
C D

18-20 → 1-1

18-23 → 3-26

18-25 ↗ 16-16
↘ 2-7

18-26 → 1-2

18-27 → 3-26

18-35 → {1-2
1-5}

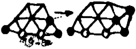
19-1 → 1-1

→ 1-1

19-2 19-2*

C → D

19-7 → 1-2



19-8

D

19-9 → {1-1, 1-2}

19-10 → {1-4, 2-2}, 2-1

19-11 → 2-1, 2-1

19-12 → 2-2, 2-2

19-13 → 2-6, 2-10

19-14 → 1-1, 2-2, 16-2

19-15 → 1-1, 2-2, 16-2

19-16 → 1-1, 3-28, 16-2

19-17 → 1-2, 2-2, 16-2

19-18 → 2-4, 1[illegible]-4, 15-18

19-19 → 1-2, 2-2, 16-2

19-20 → 1-2, 3-28, 16-2

19-21 → 2-2, 2-2, {16-1, 16-12, 48-5}

19-22 → 2-2, 2-12

19-23 → 2-2, [illegible]

19-24 → 2-1, 2-2

19-25 → 2-7, 17-1

19-26 → 3-26, 1-2, 17-1

19-27 → 3-26, 17-1

19-28 → 2-6, 3-10

[illegible] → [illegible], [illegible]

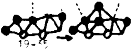
19-30 → {1-1, 2-6}, 17-1

[illegible] → {1-1, 1-4}, 2-12

19-32 → 2-1, 2-1

19-33 → 17-8

19-34 → 17-8



19-35

D

→ 1-1

→ 1-3

20-1 → 16-2, 16-2

20-3 → 16-2

20-5 → {3-26, 4-14}

20-7 → 17-29

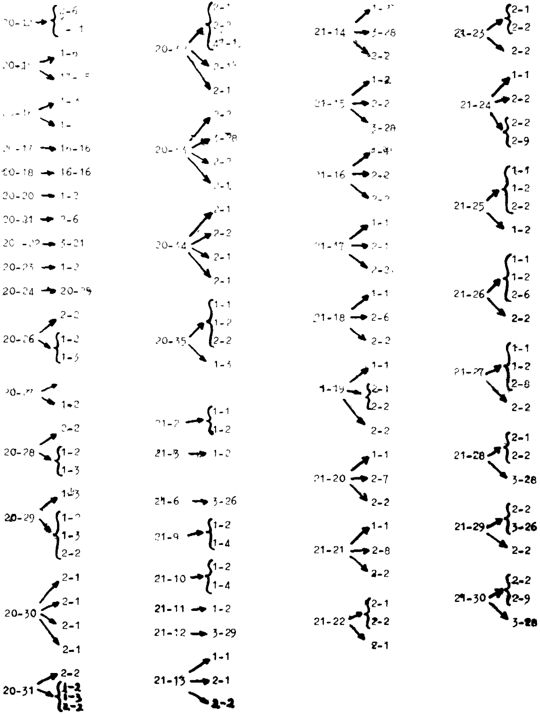
20-9 → 17-29

20-10 → 17-30

20-11 → 17-35

20-12 → 16-2

20-13 → {2-3, 2-12}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1-6
20-17
16-16
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16-16
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1-2
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2-6
5-21
20-23
1-2
20-24
20-29
20-26
2-2
1-2
1-3
1-2
20-28
2-2
1-2
1-3
20-29
1-3
1-2
1-3
2-2
20-30
2-1
2-1
2-1
2-1
20-31
2-2
1-2
1-3
2-2
2-1
2-1
2-28
2-1
20-34
2-1
2-2
2-1
2-1
20-35
1-1
1-2
2-2
1-3
21-2
1-1
1-2
21-3
1-2
21-6
3-26
21-9
1-2
1-4
21-10
1-2
1-4
21-11
1-2
21-12
3-29
21-13
1-1
2-1
2-2
21-14
3-28
2-2
21-15
1-2
2-2
3-28
21-16
2-2
21-17
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21-18
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21-19
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2-2
2-2
21-20
1-1
2-7
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21-21
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1-2
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2-2
21-28
2-1
2-2
3-28
21-29
2-2
3-26
2-2
21-30
2-2
2-9
3-28

21-31
↗ {2-2, 1-9}
↘ 2-2

21-32
↗ {2-1, 2-2}
↘ 3-18

21-33
↗ {2-2, 3-26}
↘ {2-2, 2-9}

21-34
↗ {2-1, 2-2, 17-1}
↘ 2-2

21-35
↗ {1-2, 1-3, [illegible]}
↘ 2-2

22-1 → 22-1

22-2 → 22-14

22-3 → 22-10

22-4 → 22-10

22-5 → 1-11

22-6
↗ 1-20
↘ 1-1

22-7 C → 22-7 C

22-8 → 22-11

22-9 → 22-16

22-11 → 1-1

22-13 → 1-1

22-15 → 1-2

22-16 → 1-2

22-20 → 1-1

22-21 → 22-12

22-22
↗ 22-11
↘ 1-2

22-23
↗ 22-12
↘ 1-1

22-25
↗ 7-1
↘ 7-1

22-26
↗ 1-1
→ 7-1
↘ 25-1

[illegible]
↗ {[illegible]}
↘ [illegible]

[illegible]
↗ [illegible]
↘ [illegible]

[illegible]
↗ [illegible]
↘ [illegible]

[illegible]
↗ [illegible]
↘ [illegible]

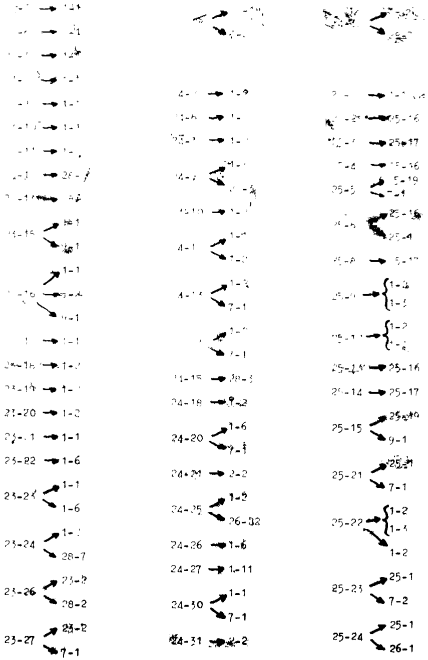
[illegible]
↗ [illegible]
↘ [illegible]

[illegible]
↗ [illegible]
↘ 22-16

[illegible]
↗ 22-5
↘ 7-1

[illegible]
↗ 12-1
↘ 7-1

[illegible]
↗ 7-3
↘ 26-1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9-1
1-1
23-20
1-2
23-22
1-6
23-23
1-1
1-6
23-24
28-7
23-26
28-2
23-27
23-2
7-1
4-1
4-13
1-3
7-1
7-1
24-15
24-18
24-20
1-6
24-21
2-2
24-25
26-32
24-26
1-6
24-27
1-11
24-30
1-1
7-1
24-31
25-17
25-5
5-19
25-16
25-4
25-14
25-16
25-14
25-17
25-15
9-1
25-21
7-1
25-22
1-2
1-2
25-23
25-1
7-2
25-24
25-1
26-1

25-15 ↗ [illegible]
↘ [illegible]

25-26 ↗ [illegible]-14
↘ 7-1

[illegible] ↗ [illegible]
→ [illegible]
↘ [illegible]

25-28 ↗ [illegible]
→ 7-[illegible]
↘ [illegible]-1

25-30 ↗ 1-1
→ 26-5
↘ 7-

25-31 ↗ 1-1
↗ 26-5
↘ [illegible]
↘ [illegible]-1

26-6 → 1-1

26-7 → 22-10

26-12 → 1-1

26-13 ↗ 1-1
↘ 9-1

26-14 → 7-2

26-15 → 7-1

26-16 → 1-1

26-18 → {1-1
1-3

[illegible] → [illegible]

[illegible] → [illegible]

[illegible] ↗ [illegible]
↘ [illegible]

[illegible] → [illegible]

[illegible] ↗ 1-1
→ [illegible]
↘ [illegible]

[illegible] ↗ 1-
↘ {[illegible]
1-[illegible]

27-2 → 27-5

[illegible] ↗ 27-[illegible]
↘ 1-[illegible]

27-4 → 1-1

27-6 → 1-1

27-8 → 1-1

27-9 → 1-[illegible]

27-11 ↗ 7-1
↘ 27-28

27-12 → 1-1

27-14 → 1-1

27-15 → 1-1
27-18 → 9-1
27-19 → 1-2

27-20 → 9-1

27-21 → 1-1

[illegible] → [illegible]

[illegible] → [illegible]

[illegible] → [illegible]

[illegible] → [illegible]

[illegible] ↗ [illegible]
↘ [illegible]

[illegible] → 25-[illegible]

[illegible] → [illegible]

[illegible] → [illegible]

[illegible] ↗ [illegible]
↘ [illegible]

28-10 → 1-1

28-11 → 1-1

[illegible] ↗ [illegible]
↘ [illegible]

28-12 ↗ 1-[illegible]
↘ 1-1

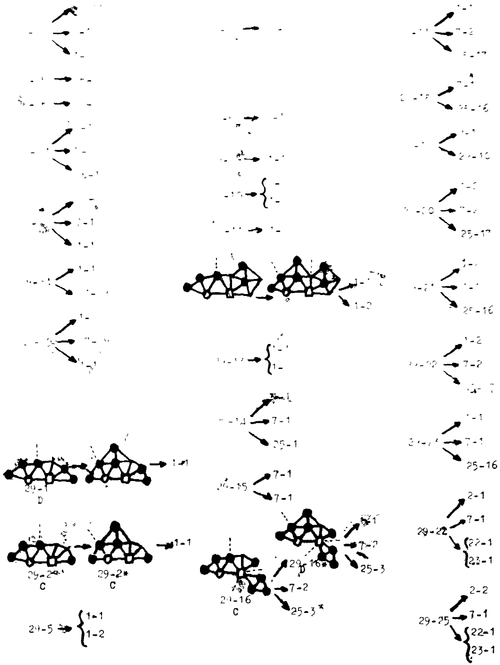
28-14 ↗ 1-2
↘ 1-1

28-15 → 1-1

28-16 ↗ 1-1
↗ 1-2
↘ 27-1
↘ 1-1

28-17 → 1-1

28-18 → 1-1


7-2
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25-16
7-2
25-17
1-1
1-2
25-16
1-2
7-2
7-1
25-1
7-1
25-16
29-1
D
1-1
7-1
7-1
2-1
7-1
29-22
22-1
23-1
29-2
C
29-2*
C
1-1
29-16*
D
7-2
25-3
29-16
C
7-2
25-3*
2-2
29-25
7-1
22-1
23-1
29-5
1-1
1-2

29-26 → [illegible]
→ 7-1
→ { [illegible] / 2[illegible]-5 }

29-27 → 3-28
→ 7-1
→ { 2-1 / 23-2 }

29-28 → [illegible]
→ 7-1
→ { 22-2 / 23-5 }

29-29 → 2-2
→ 7-1
→ { 22-1 / 23-1 }

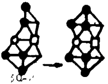
29-30 → [illegible]
→ 7-1
→ 7-1
→ 2-1

29-31 → 2-1
→ 7-1
→ 7-1
→ 2-2

29-32 → 2-1
→ 7-2
→ 7-2
→ 2-1

[illegible] → [illegible]
→ 1-1
→ [illegible]
→ 2-1

[illegible] → [illegible]



30-[illegible]
D

30-11 → 1-6

30-12
D

30-13 → 1-2

30-19 → 1-2
→ 1-2

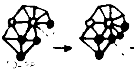
30-20 → 2-2

30-21 → 9-1

30-22 → 1-2

30-24 → 1-2

30-26 → 1-2



30-28
D

→ 1-1

30-[illegible] → [illegible]

30-[illegible] → [illegible]

30-[illegible] → 1-5

30-[illegible] → [illegible]
→ [illegible]

31-[illegible] → 7-2

31-4 → [illegible]-14

31-5 → 10-3

31-8 → 1-2

31-10 → 1-1

31-11 → 7-1

31-12 → 1-1

31-13 → 7-1

31-14 → 7-2

31-15 → 1-2
→ 3-28

31-16 → 1-1

31-22 ↗ 1-
↘ 1-1

31-1[illegible] → 7-16

31-[illegible] → 9-[illegible]

31-25 → 27-5[illegible]

31-[illegible] → 1-1

31-28 → 30-1

31-29 → 32-10

31-30 → 7-2

31-[illegible] → 1-1

31-[illegible] ↗ 1-1
↘ 1-1

32-1 ↗ 1-1
↘ 1-2

32-2 ↗ 1-1
↘ 1-2

32-3 → 1-2

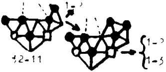
32-4 → 1-2

32-5 → 1-2

32-8 ↗ 1-2
↘ 1-2

32-9 → 7-21

32-10 → {1-2
1-4


32-11

32-12 ↗ 1-1
→ 7-1
↘ 7-2

32-13 ↗ {1-1
1-2
→ 7-1
↘ 7-2

32-14 ↗ {1-1
1-2
→ 7-1
↘ 7-2

32-15 ↗ {1-1
1-2
→ 7-2,
↘ 7-2

32-16 ↗ 1-2
→ 7-2
↘ 7-2

32-17 ↗ 1-1
→ 7-2
↘ 7-3

32-18 ↗ 1-2
→ 1-2
↘ 7-2

32-19 ↗ 1-2
→ 7-2
↘ 7-2

32-20 ↗ {7-1
7-2
↘ 7-2

32-21 ↗ 7-2
↘ 7-2

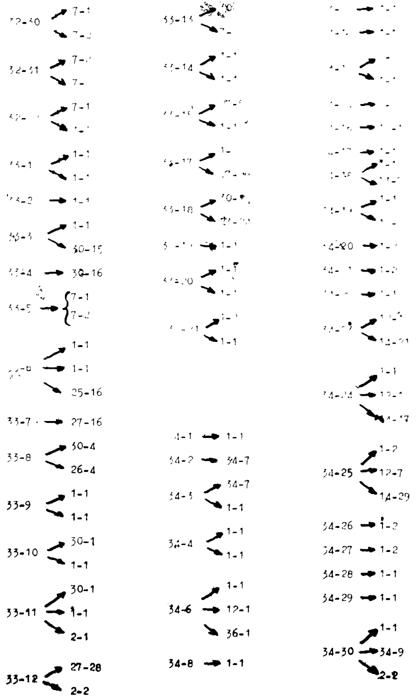
32-23 ↗ {7-2
7-27
↘ 7-3

32-24 ↗ 1-1
→ 7-2
↘ 7-27

32-25 ↗ {7-2
7-7
↘ 7-27

32-27 → 1-1

32-28 → 1-2


32-30
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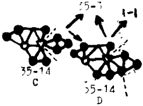
1-2
34-27
1-2
34-28
1-1
34-29
1-1
34-30
1-1
34-9
2-2

[illegible] → -1
→ -1
→ [illegible]

[illegible] → {1-[illegible], 1-[illegible]}

35-1 → 1-1
35-[illegible] → [illegible]
35-[illegible] → 1-1
35-6 → [illegible]-18
[illegible] → 5-1[illegible]
[illegible] → 1-1
35-[illegible] → 1-1
[illegible] → 1-1[illegible]
→ 1[illegible]-[illegible]
35-11 → 1[illegible]-
35-1[illegible] → [illegible]-1[illegible]
35-1[illegible] → 1-1
→ 4-1
→ 36-1



35-15 → 34-1
→ 36-2
35-16 → 35-22
35-17 → 1-1

35-10 → 1-[illegible]
35-[illegible] → 1-1
35-[illegible] → 1-[illegible]
35-[illegible] → 1-1
35-[illegible] → 1-1
35-[illegible] → [illegible]-1[illegible]
→ 34-[illegible]
[illegible] → [illegible]-1[illegible]
→ 17-[illegible]
[illegible]-0 → 12-[illegible]
35-[illegible]1 → 1-[illegible]
35-[illegible] → 1-1
35-[illegible] → 1-1
→ 35-1
→ 35-1
→ 1-1
36-5 → 1-1
36-6 → 34-7
36-7 → 1-1
36-8 → 1-20
→ 1-2
36-10 → 13-10
36-12 → 12-2
→ 36-23
36-13 → 12-2
→ 36-23

36-14 → 1[illegible]-[illegible]
→ 16-2[illegible]
36-15 → [illegible]-1
36-18 → [illegible]
36-17 → 1-1
[illegible] → [illegible]
36-[illegible] → [illegible]-2
→ [illegible]-26
6-20 → 1-1
6-[illegible]1 → 1-1
6-[illegible] → 1-1
6-31 → 1[illegible]-3
36-33 → 1-1
→ 12-7
→ 17-12
6-34 → [illegible]-1
→ 34-1
6-35 → 12-1
→ 34-1

37-2 → 1-1
37-3 → 1-1
37-4 → 1-1
37-5 → 1-1
37-7 → 1-1
→ 12-2
37-8 → 13-7

37-9 ↗ 1-1
↘ 12-1

37-10 → 1-1

37-11 ↗ 1-3
↘ 1-1

37-1 → 1-1

37-13 → 12-1

37-14 ↗ [illegible]-12
↘ 1-1

37-15 → 1-1

37-16 ↗ [illegible]4-15
→ 1 -7
↘ 1-1

37-17 ↗ 1-1
↘ 1-1

37-18 ↗ 1-[illegible]
↘ 1-1

37-19 ↗ 12-2
↘ 1-1

37-20 → 1-1

37-21 → 1-1

37-22 → 1-1

37-23 → 1-1

37-26 ↗ 1-1
↘ 12-2

37-27 → 13-7

37-28 ↗ 1-1
↘ 12-1

37-[illegible]0 → {1-[illegible]
[illegible]

37-[illegible] ↗ 1-[illegible]
↘ [illegible]

[illegible] ↗ [illegible]
↘ [illegible]

[illegible] ↗ 1-[illegible]
↘ [illegible]

38-[illegible] → 15-18

38-[illegible] → 1-1

38-[illegible] → 1-1

[illegible] → 1-[illegible]

38-10 → [illegible]

38-11 → 12-1

38-12 → 13-7

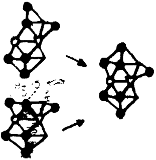
38-13 → 12-2

38-14 → 12-2

38-15 → 12-2

38-16 → {1-1
12-16

38-19 → 1-1

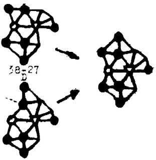


C

[illegible] → 1-[illegible]

[illegible] → [illegible]

[illegible] → 1-[illegible]



38-[illegible]

C

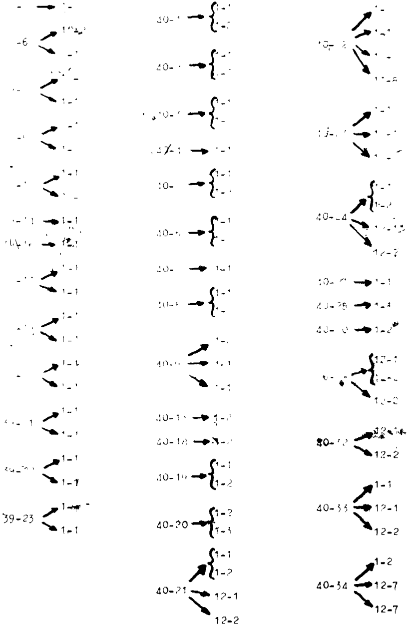
38-28 ↗ 1-1
↘ 13-29

38-30 → 12-29

38-31 see above

38-34 → 38-32

38-35 → 1-2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1-2
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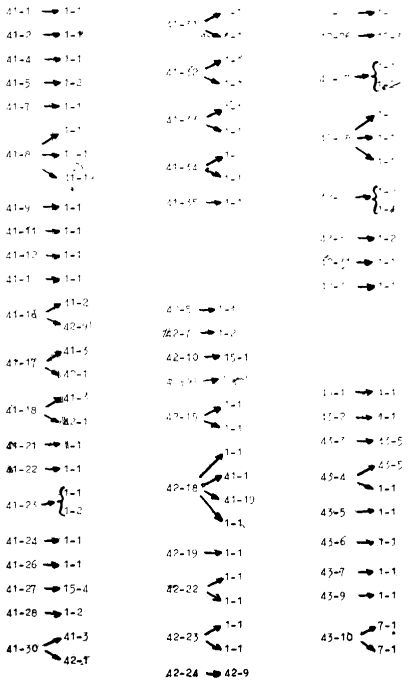
1-2
12-7
12-7


41-1
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41-11
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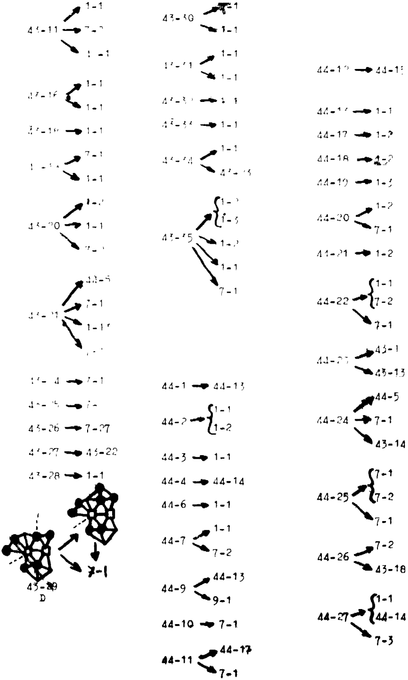
41-12
1-1
41-1
1-1
41-1d
41-2
41-17
41-18
1-1
41-22
1-1
41-23
1-1
1-2
41-24
1-1
41-26
1-1
41-27
15-4
41-28
1-2
41-30
41-3
42-1
42-10
15-1
42-15
1-1
1-1
42-18
1-1
41-1
41-19
1-1
42-19
1-1
42-22
1-1
1-1
42-23
1-1
1-1
42-24
42-9
1-1
1-1
43-4
43-5
1-1
43-5
1-1
43-6
1-1
43-7
1-1
43-9
1-1
43-10
7-1
7-1


43-11
1-1
7-2
1-1
43-16
1-1
1-1
43-18
1-1
43-19
7-1
1-1
43-20
1-2
1-1
7-1
43-21
43-5
7-1
1-1
43-24
7-1
43-25
7-
43-26
7-27
43-27
43-22
43-28
1-1
43-29
D
7-1
43-30
7-1
1-1
43-31
1-1
1-1
43-32
1-1
43-34
1-1
43-34
1-1
43-35
43-35
1-2
1-3
1-2
1-1
7-1
44-1
44-13
44-2
1-1
1-2
44-3
1-1
44-4
44-14
44-6
1-1
44-7
1-1
7-2
44-9
44-13
9-1
44-10
7-1
44-11
44-17
7-1
44-12
44-15
44-16
1-1
44-17
1-2
44-18
1-2
44-19
1-3
44-20
1-2
7-1
44-21
1-2
44-22
1-1
7-2
7-1
44-23
43-1
43-13
44-24
44-5
7-1
43-14
44-25
7-1
7-2
7-1
44-26
7-2
43-18
44-27
1-1
44-14
7-3

↗ 44-17
44-28
↘ 7-1

↗ 44-3
44-29
↘ 7-3

↗ 7-1
44-30
↘ 44-14

↗ 7-1
44-31
↘ 44-16

↗ 1-1
↗ 7-1
44-32
↘ { 43-1
44-5
↘ 7-1

↗ 7-2
44-33 → 7-1
↘ { 43-1
44-5

↗ 7-1
↗ 7-2
44-34
↘ { 43-1
44-5

45-1 → 1-1

↗ 1-1
45-4
↘ 10-3

45-5 → 43-12

45-6 → 43-12

↗ 1-2
45-7
↘ 43-12

↗ 7-1
↗ 7-1
45-8
↘ 7-1
↘ 7-1

45-9 → 1-1

45-10 → 1-1

45-11 → 1-1

45-12 → 1-1

45-13 → 1-1

45-14 → 1-1

↗ 1-1
45-15 → 7-2
↘ 45-21

↗ 1-1
45-16 → 1-1
↘ 12-2

45-18 → 1-1

↗ 1-1
45-19
↘ 1-1

45-20 → 1-1

45-21 → 1-1

45-22 → 1-1

45-23 → 1-1

45-25 → 1-1

↗ 1-1
45-26
↘ 1-1

45-27 → 12-

↗ 14-
45-28
↘ 12-

↗ 1-1
45-30
↘ 1-1

45-31 → 1-1

↗ 1-1
45-32
↘ 1-1

46-4 → 1-1

46-7 → { 1-1
1-2

46-9 → 2-1

46-10 → 1-2

↗ 5-1
46-11
↘ 2-1

46-17 → 2-1

46-18 → 1-1

46-19 → { 2-1
2-2

46-21 → 16-1

46-22 → 16-12
→ 2-1

46-23 → 3-28

46-25 → 2-2

46-27 → 1-1

46-28 → 16-1
→ 17-1

46-29 → 17-1
→ 16-1

46-30 → 17-9

46-31 → 1-2

46-32 → 18-14

46-35 → 1 -14

47 5 → 16-2

47-13 → 16-2

47-14 → 2-1
→ 2-1

47-15 → 46-2

47-16 → 46-2

47-17 → 17-8
→ 18-16

47-18 → 16-2
→ 46-2

47-19 → 46-5

47-20 → 46-5

47-21 → 46-2
→ 46-15

47-22 → 46-5

47-23 → 1-1
→ 46-6

47-24 → 46-2

47-25 → 46-6

47-26 → 1-2

48-1 → 2-1

48-4 → 2-1

48-8 → 1-1

48-9 → 2-1

48-11 → 2-2

48-12 → 2-1

48-15 → 2-2

48-16 → 2-2
→ 2-1

48-17 → 2-2

48-18 → 2-1
→ 48-3

48-19 → 2-1
→ 48-7

48-20 → 2-2
→ 48-6

48-21 → 2-1

48-22 → 2-1
→ 48-1
→ 2-2

48-23 → $\left\{\begin{array}{l}\text{1-1}\\ \text{48-10}\end{array}\right.$
→ 2-2

48-24 → 1-2
→ 1-2

48-25 → 2-1
→ 1-2
→ 1-2

49-6 → 1-3

49-7 → 2-2
→ 17-1

49-8 → 2-2
→ 17-2

49-9 → 3-28
→ 17-1

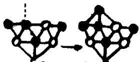
49-10 → $\left\{\begin{array}{l}\text{1-1}\\ \text{1-2}\\ \text{17-1}\end{array}\right.$
→ 2-1

49-11 → 2-2
→ 17-1

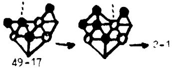
$$\begin{tikzcd}49-12 \arrow[r] \arrow[dr] & 2-2 \\ & 17-2\end{tikzcd}$$

$$\begin{tikzcd}49-13 \arrow[r] \arrow[dr] & 3-28 \\ & 17-1\end{tikzcd}$$

$$\begin{tikzcd}49-14 \arrow[r] \arrow[dr] \arrow[ddr] & \left\{\begin{matrix}1-1 \\ 3-26\end{matrix}\right. \\ & 1-1 \\ & 17-1\end{tikzcd}$$



49-16 C   49-16* D



49-17 D

$$\begin{tikzcd}49-19 \arrow[r] \arrow[dr] \arrow[ddr] & 1-1 \\ & 17-1 \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}49-20 \arrow[r] \arrow[dr] \arrow[ddr] & 1-1 \\ & 2-1 \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}49-21 \arrow[r] \arrow[dr] \arrow[ddr] & 1-1 \\ & \left\{\begin{matrix}2-1 \\ 2-2\end{matrix}\right. \\ & 3-28\end{tikzcd}$$

$$\begin{tikzcd}49-22 \arrow[r] \arrow[dr] \arrow[ddr] & 1-1 \\ & 2-2 \\ & \left\{\begin{matrix}2-2 \\ 3-26\end{matrix}\right.\end{tikzcd}$$

$$\begin{tikzcd}49-23 \arrow[r] \arrow[dr] & \left\{\begin{matrix}2-1 \\ 2-2\end{matrix}\right. \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}49-24 \arrow[r] \arrow[dr] & \left\{\begin{matrix}17-1 \\ 17-8\end{matrix}\right. \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}49-25 \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] \arrow[ddddr] & 1-1 \\ & 2-1 \\ & 2-1 \\ & 2-1 \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}49-28 \arrow[r] \arrow[dr] & \left\{\begin{matrix}1-1 \\ 1-2 \\ 2-2\end{matrix}\right. \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}50-2 \arrow[r] & 1-1\end{tikzcd}$$

$$\begin{tikzcd}50-4 \arrow[r] & 7-2\end{tikzcd}$$

$$\begin{tikzcd}50-5 \arrow[r] & 26-1\end{tikzcd}$$

$$\begin{tikzcd}50-10 \arrow[r] \arrow[dr] & 50-1 \\ & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-15 \arrow[r] & 26-1\end{tikzcd}$$

$$\begin{tikzcd}50-17 \arrow[r] & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-18 \arrow[r] & 1-1\end{tikzcd}$$

$$\begin{tikzcd}50-19 \arrow[r] & 1-1\end{tikzcd}$$

$$\begin{tikzcd}50-20 \arrow[r] & 22-10\end{tikzcd}$$

$$\begin{tikzcd}50-22 \arrow[r] \arrow[dr] & 1-1 \\ & 1-2\end{tikzcd}$$

$$\begin{tikzcd}50-3 \arrow[r] & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-24 \arrow[r] \arrow[dr] \arrow[ddr] \arrow[dddr] & 1-1 \\ & 50-1 \\ & 2-2 \\ & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-25 \arrow[r] \arrow[dr] & 50-1 \\ & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-26 \arrow[r] \arrow[dr] & 1-1 \\ & 1-1\end{tikzcd}$$

$$\begin{tikzcd}50-27 \arrow[r] & 50-2\end{tikzcd}$$

$$\begin{tikzcd}50-28 \arrow[r] & 1-1\end{tikzcd}$$

$$\begin{tikzcd}50-29 \arrow[r] & 2-1\end{tikzcd}$$

$$\begin{tikzcd}50-30 \arrow[r] \arrow[dr] & 51-4 \\ & 2-2\end{tikzcd}$$

$$\begin{tikzcd}50-32 \arrow[r] \arrow[dr] & 51-4 \\ & 2-1\end{tikzcd}$$

$$\begin{tikzcd}51-1 \arrow[r] \arrow[dr] & 1-1 \\ & 1-1\end{tikzcd}$$

$$\begin{tikzcd}51-2 \arrow[r] \arrow[dr] \arrow[ddr] & 1-1 \\ & 7-1 \\ & 1-1\end{tikzcd}$$

$$\begin{tikzcd}51-4 \arrow[r] & 1-1\end{tikzcd}$$

51-5 → 1-1

51-7 ↗ 1-1 ↘ 1-1

51-11 → 1-1

51-12 → 1-1

51-14 ↗ 22-1 ↘ 1-1

51-15 → 22-4

51-16 → 5-1

51-17 → 50-1

51-18 ↗ 2-1 ↘ 5-1

51-19 ↗ 51-8 ↘ 50-27

51-20 ↗ 1-1 ↘ [illegible]

51-21 → 50-1

51-22 ↗ 7-2 ↘ 50-1

51-23 ↗ 23-1 ↘ 50-3

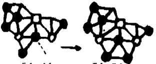
51-25 ↗ 2-1 ↘ 5-1

51-26 → 50-2

51-27 ↗ 1-1 ↘ 50-31

51-29 → 5-12

51-30 ↗ 1-1 → 22-1 ↘ 50-3



51-31 C → 51-31* D

51-32 → 50-28

51-33 ↗ 1-1 ↗ 2-1 ↘ {2-1, 2-2} ↘ 2-1

51-34 ↗ 7-1 → 1-1 ↘ 50-11

52-2 ↗ 1-1 ↘ 7-1

52-3 → 7-1

52-4 → 7-2

52-5 → 7-27

52-6 → 7-1

52-7 ↗ 7-1 ↘ 1-2

52-8 → 1-2

52-9 → 1-2

52-10 ↗ 1-1 ↘ 1-2

52-11 → 1-1

52-12 → 7-1

52-18 → 25-16

52-19 ↗ 1-1 → 1-1 ↘ 1-1

52-20 → 1-1

52-21 ↗ 1-1 ↘ 1-1

52-22 → {1-1, 1-2}

52-23 → 52-24

52-24 → 1-1

52-25 → 1-1

52-26 ↗ 25-16 ↘ 1-1

52-27 → 1-1

52-28 → 1-1

52-29 → 1-1

52-30 → 1-2

52-31 → 1-2

52-32 ↗ 1-1 → 52-24 ↘ 2-2

52-33 ↗ 52-30 → 2-2

53-1 → 22-5

53-2 → 22-11

53-4 → 22-1

53-5 ↗ 26-1
↘ 24-2

53-6 ↗ 1-1
↗ 22-1
↘ 25-17
↘ 8-1

53-7 ↗ 26-1
↘ 25-17

53-8 ↗ 1-1
↘ 53-14

53-9 ↗ 26-1
↘ 53-18

53-10 ↗ 1-1
→ 22-1
↘ 1-2

53-11
D

53-12 → 30-16

53-13 → 1-2

53-14 → 1-1

53-15 → 1-1

53-17 → 1-1

53-18 → 1-2

53-21 ↗ 1-1
↘ 4-1

53-24 ↗ 7-1
↘ 7-1

53-25 → 54-5

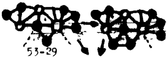
1-1

53-26
C

53-26*
D

53-27 ↗ 16-1
↘ 54-8

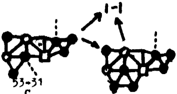
53-28 ↗ 11-10
↘ 7-1



53-29
D

7-1

53-30 ↗ 11-17
↘ 7-2



1-1

53-31
C

53-31*
C

53-32 → 54-5

53-33 ↗ 2-1
↘ 5-1

53-34 ↗ 7-1
↘ {7-1
7-2

53-35 → 54-6

54-1 → 2-2

54-2 → 7-1

54-3 ↗ 7-1
↘ 1-2

54-4 → 1-2

54-6 → 1-1

54-8 → 17-1

54-12 → 9-1

54-13 ↗ 11-10
↘ 7-1

54-16 → 2-1

54-18 ↗ 16-1
↘ 2-1

54-19 ↗ 14-11
↘ 7-2

54-20 ↗ 16-1
↘ 2-1

54-24 → 2-1

54-26 → 1-1

54-27 → 2-2, 55-6, 7-1

54-28 → 55-4, 55-6

54-29 → 2-1, 55-7, 55-1

55-3 → 7-1

55-4 → 1-1, 1-1

55-7 → 2-4

55-8 → 1-1, 1-1, 2-1

55-12 → 2-2, 1-1

55-13 → 2-1

55-14 → 2-2

55-15 → 2-2

55-18 → 2-1

55-19 → 2-2

55-20 → 1-1

55-22 → 1-1

55-29 → 2-1, 7-1

55-30 → 1-2

55-31 → 2-2, 7-1

55-32 → 7-2

55-34 → 2-1

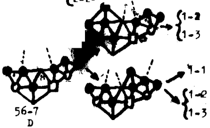
55-35 → 2-2, 1-2

56-1
D

56-3 → 5-1

56-4 → {1-1, 1-2}

56-6 → {1-2, 1-3, 2-2}



56-7
D

56-8 → {1-2, 1-3, 2-2}

56-9 → 22-3

56-10 → 22-9, 2-2

56-14 → 2-1

56-17 → 56-2

56-18 → 1-1, 22-12

56-19 → 1-2

56-20 → 26-1, 7-2

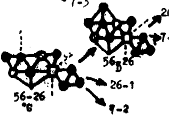
56-21 → 1-1, {1-1, 22-4}, 7-3

56-22 → 1-1, 26-1, 7-3

56-23 → 7-2, 7-2

56-24 → 7-1, 7-2

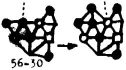
56-25 → 1-1, 22-1, 7-3



56-26
G

56-27 ↗ 7-1
↘ 7-2

56-29 ↗ 2-2
↘ 5-2



→ 2-1

56-30
D

56-31 → 5-1

56-32 → 2-1

57-2 C → 57-2* D

57-4 → 1-3

57-5 → 1-3

57-7 → 7-2

57-8 ↗ 1-3
→ 7-2
↘ 26-1

57-9 ↗ 7-2
↘ 26-1

57-10 ↗ 2-2
↘ 23-3

57-11 → 1-2

57-14 → 2-3

58-2 → 12-2

58-4 → 2-1

58-7 ↗ 1-1
↘ 2-1

58-8 → 1-1

58-9 ↗ 1-1
↘ 1-1

58-10 → 1-1

58-11 ↗ 1-1
↘ 1-1

58-16 ↗ [illegible]
↘ 58-1

58-17 ↗ 1-1
→ 5-1
↘ 2-1

58-18 ↗ 1-1
→ 58-1
↘ 2-1

58-19 → 58

58-20 → 58-1

58-21 ↗ 1-1
↘ 58-1

58-22 ↗ 1-1
→ 34-1
↘ 1-1

58-23 → 1-1

58-25 ↗ 1-1
↘ 34-8

58-28 ↗ 1-1
↘ 1-1

58-30 → 1-1

58-31 → 1-1

58-32 → 1-1

58-33 ↗ 1-1
↘ 1-2

58-34 → 1-2

58-35 → 1-2

59-2 ↗ 34-1
→ 1-1
↘ 1-1

59-3 ↗ 34-1
→ 1-1
↘ 1-1

59-6 ↗ 1-1
→ 34-1
↘ 1-2

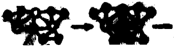
59-7 → 1-1

59-9 → 1-1

59-13 → 2-1

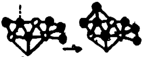
59-14 → 2-1

59-15 → 59-8



→ 1-1

60-2 → 2-1



60-4
D

60-5 → 34-2

60-7 → 35-6

60-10 → 5-1

60-13 → 2-1

60-14 ↗ 12-1
↘ 12-2

60-15 ↗ 2-1
→ 12-1
↘ 12-2

60-16 ↗ 36-1
↘ 34-2

60-17 → 2-1

60-18 → 1-2

60-19 → 1-1

60-20 → 1-3

60-21 → 1-1

60-22 ↗ 2-2
↘ 1-1

60-23 → 2-1

60-24 → [illegible]1

60-25 ↗ 2-2
↘ 1-1

60-26 → 2-2

60-27 → 2-1

60-28 ↗ 2-1
↘ 1-1

60-31 → 2-1

61-1 ↗ 1-1
↘ 7-1

61-2 → 7-1

61-3 ↗ 1-1
→ 7-1
↘ 7-1

61-6 → 1-1

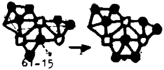
61-7 → 1-1

61-10 ↗ 1-1
→ 5-1
↘ 2-1

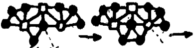
61-11 → 1-1

61-12 → 1-1

61-14 ↗ 1-1
↗ 7-1
↘ 1-1
↘ 61-7



61-15
D

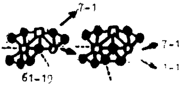


→ 2-1

61-16
D

61-17 ↗ 1-1
↘ 61-9

61-18 ↗ 5-1
→ 2-1
↘ 1-1



61-19
D

61-20 ↗ 22-1
→ 1-1
↘ 61-11

61-21 → 61-8

61-22 → 1-1

61-23 → 1-1

61-24 → 1-1

61-25 → {1-1
1-2

61-26 → 7-1, {1-1, 1-2}

61-27 → 1-1

61-28 → 1-2

61-31 → 7-2, 1-1

61-32 → 12-1, 1-1

61-33 → 1-1

62-3 → 2-1

62-4 → 17-1

62-5 → 17-1

62-6 → 2-1

62-7 → 2-1, 17-8

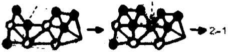
62-9 → 17-1

62-10 → 17-1

62-11 → 62-4, 17-1

62-14 → 46-2

62-15 → 46-2, 46-2



62-17
C

62-17*
D

62-20 → 2-2

62-22 → 2-1

62-23 → 2-2

62-24 → 2-2

63-2 → 2-1

63-3 → 2-1

63-5 → 2-1

63-6 → 7-1

63-7 → {2-1, 2-2}

63-8 → 26-1, 50-2

63-9 → 22-1

63-15 → 1-1

63-16 → 1-1

(6) n- and m-values and major vertices of the configurations of $\mathcal{U}$.

major vertices	# in $\mathcal{U}$
none.	32
7	158
8	150
9	83
10	31
11	2
7-7	194
7-8	340
7-9	212
7-10	61
8-8	74
8-9	20
9-9	5
7-7-7	105
7-7-8	228
7-7-9	72
7-7-10	4
7-8-8	30
7-8-9	3
7-7-7-7	23
7-7-7-8	9
7-7-8-8	2
	1854

	m = n	m = n-1	m = n-2	m = n-3	m = n-4	m = n-5	total
n = 6			1 D				1
n = 7				1 D			1
n = 8		1 D	1 D	2 D	C		5
n = 9			D	3 D	1 D 1 C		8
n = 10		D	7 D	9 D 3 C	5 D 8 C		36
n = 11		6 D	15 D	16 D 8 C	18 D 22 C	4 C	89
n = 12		17 D	51 D 4 C	61 D 45 C	78 D 70 C	1 D 7 C	334
n = 13	2 D	37 D	120 D 8 C	170 D 92 C	146 D 103 C	5 D 20 C	701
n = 14	3 D	52 D	187 D 11 C	316 D 64 C	90 D 24 C	4 D 6 C	660
total					59	40	1854

The above table gives for every n and m the number of D-reducible and the number of D-irreducible, but C-reducible configurations in $\mathcal{U}$ which have ring size n and m vertices.

The table to the left gives for every combination of major vertices the number of configurations in $\mathcal{U}$ which contain precisely that combination of major vertices.

(ε) List of configurations in $\mathcal{U} - \mathcal{U}'$. The 352 configurations listed below may be deleted from $\mathcal{U}$ in order to obtain a set $\mathcal{U}'$ of 1482 configurations which is still unavoidable. (Only the configurations of $\mathcal{U}'$ are used in the argument of Part I.

1-26

4- [illegible]

6-16
17
23

7-[illegible]

8- 2
5
6
10
11
16
18
19
[illegible]
28

[illegible]- 5
11
12
15
27
[illegible]
22

10- 8
14
22

11-31
32
33
34

12- 3
5
6
9
20
33

14- 2
11
15
18
26

15- 2
7
8
12
19
21
22
23
24
25
27
28
31

16-10
24
2

17-1[illegible]
1[illegible]
1[illegible]
[illegible]
24

18-20
2[illegible]
26

19-21

20-16
2[illegible]
24
25
[illegible]

22-21
23
[illegible]

23-13
17

24- 9
13

25- 5
15
26

26-12
25

27- 1
8
16
27
30

28-10
25
26
29

29- 4
5
8
19
31

30-14

31- 2
3
5
12
15
28
30
31
32

[illegible]- 9
21

33- 2
6
7
8
12
13
16
19

34- [illegible]
[illegible]
6
19
20
25
27
28
29

35- 9
11
15

36- 3
5
7
[illegible]
19
21
24
[illegible]

37- 1
8
12
16
18
19
22
23
27
30
34

38-11
16
28
30
31
33

39- 2
12
13
19
20

40-22
30
34

41- 2
4
5
7
8
12
1[illegible]
16
17
19
21
[illegible]
24
27
28
[illegible]
30
31
32
33
34
35

42- 3
4
5
6
7
8
10
11
13
14
15
18
19
20
21
22
23
24
26
27
28
29
30
31
32

43- [illegible]
19
24
25
26
29
32
33

44- 7
11
12
15
17
19
20
21
23
25
32

45- 2
4
6
7
8
9
20
26
27

46- 4
6
7
9
11
14
15

47- 8
13
14
19
21
22
23
25

48- 8
24

49- [illegible]
10

50-11
14
16
17
21
22
25

51- 1
5
12
19
22
26
27
29
34

52- 1
5
7
8
13
19
21
23
25
26

53- 1
4
8
11
13
14
16
19
27
28
33

54- 1
10
12
13
23

55- 9
10
15
20
21

56- 4
10
18
19

57-10
11
13
14

58- 2
16
25
28

59- 6
7
8
16

60- 3
23
28
29
30
31
32

61- 1
2
5
15
16
17
20
24
27
28
30
31

62- 1
2
5
8
10
11
13
18
24

63-1
3
6
14
15