SUBGROUPS WITH TRIVIAL MAXIMAL INTERSECTION

BY

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In a group G, let $\Phi(G)$ be the intersection of all maximal subgroups. If $H \leq G$, then it is clear that $H \leq \Phi(G)$ if and only if $H \leq M$ for every maximal subgroup M of G. It is well known that if G is finite then $\Phi(G)$ is a nilpotent group. It follows that if $H \cap M = H$ for all maximal subgroups M of a finite group G, then H is nilpotent. In this note we will consider a similar situation.

DEFINITION. A subgroup H of G is said to satisfy $\mathscr{P}(G)$ if for any maximal subgroup M of G either $H \cap M = H$ or $H \cap M = \langle 1 \rangle$.

It is proved in [1] that if G is finite and solvable then if H satisfies $\mathscr{P}(G)$, H is nilpotent. In this note we provide more information about H. In particular, we say something of the embedding of H in G when H satisfies $\mathscr{P}(G)$.

All groups will be finite and most notations standard. We use $M <\cdot\, G$ for M being a maximal subgroup of G.

LEMMA 1. *Let $K \leq H < G$ with H satisfying $\mathscr{P}(G)$. If $N \lhd G$ then K satisfies $\mathscr{P}(G)$ and HN/N satisfies $\mathscr{P}(G/N)$.*

Proof. The statement about K is clear. Let $M/N <\cdot\, G/N$. Then Dedekind's theorem yields

$$\frac{HN}{N} \cap \frac{M}{N} = \frac{(H \cap M)N}{N}.$$

Since H satisfies $\mathscr{P}(G)$ the result follows.

There are some particular situations where subgroups H satisfying $\mathscr{P}(G)$ arise. For example, if $H \leq \Phi(G)$ or $H \leq N$ where N is a minimal normal subgroup of a solvable group G, then H satisfies $\mathscr{P}(G)$. Let G be a Frobenius group with kernel N and complement M. If N is minimal normal in G and $H \leq \Phi(M)$, then H is easily seen to satisfy $\mathscr{P}(G)$. Thus Frobenius actions sometimes give rise to subgroups satisfying $\mathscr{P}(G)$.

DEFINITION. A group H is said to be of Frobenius type if it has Sylow p-subgroups which are cyclic for $p > 2$ and cyclic or generalized quaternion for $p = 2$.

LEMMA 2. *Let H satisfy $\mathscr{P}(G)$ in a solvable group G. If N is a minimal normal complemented subgroup of G with $(|H|, |N|) = 1$, then either*

(1) $[H, N] = 1$ *or*
(2) *H is of Frobenius type.*

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Proof. Let $M <\cdot G$ be a complement to N in G. Since $(|H|, |N|) = 1$, by choosing conjugates, we may assume that $H \leq M$. Suppose that for some $n \in N$ we have $H \cap H^n \neq \langle 1 \rangle$. By Lemma 1 we know that H^n satisfies $\mathscr{P}(G)$ and thus $H^n \leq M$. Therefore $H^n \leq HN \cap M = H$ and by comparing orders we see that $H^n = H$. It follows that $[H, n] \leq H \cap N = \langle 1 \rangle$ and $n \in C(H)$. Thus if $H \cap H^n \neq 1$ then $n \in C_N(H)$. Since $(|H|, |N|) = 1$, Fitting's theorem implies that $N = [N, H] \times C_N(H)$. If $[N, H] = \langle 1 \rangle$ then we have (1). Thus assume $[N, H] \neq 1$. Let $X = [N, H]H$. Since no nonidentity element of $[N, H]$ centralizes H, it follows that $N_X(H) = H$ and that the conjugates of H are a TI set. Thus X is a Frobenius group with complement H. It follows that H is of Frobenius type.

LEMMA 3. *If H satisfies $\mathscr{P}(G)$ and N is a minimal normal abelian subgroup of G with $H \cap N \neq \langle 1 \rangle$ then either*

(1) *$H \leq \Phi(G)$ or*
(2) *HN is a p-group.*

Proof. If $N \leq \Phi(G)$ then since $H \cap N \neq \langle 1 \rangle$ and H satisfies $\mathscr{P}(G)$ we have that $H \leq \Phi(G)$. Thus assume N is complemented in G by the maximal subgroup M. Since $H \cap N \neq \langle 1 \rangle$ we must have that $H \cap M = \langle 1 \rangle$. However it also follows that $H \cap M^g = \langle 1 \rangle$ for any $g \in G$. Since $(G\colon M)$ is prime power, Sylow's theorem therefore forces H to be a p-group for some prime p. Since $H \cap N \neq \langle 1 \rangle$ the proof is complete.

THEOREM 1. *If G is solvable and H satisfies $\mathscr{P}(G)$ then one of the following is true:*

(1) *$H \leq \Phi(G)$.*
(2) *H is elementary abelian of prime power order.*
(3) *H is of Frobenius type.*

Proof. Let G be a minimal counterexample to the theorem. Thus G contains a subgroup H satisfying $\mathscr{P}(G)$ but not satisfying (1) or (2) or (3). We will show this leads to a contradiction. Since H satisfies $\mathscr{P}(G)$ it is clear that $H \cap \Phi(G) = 1$. Let N be a minimal normal subgroup of G.

Case 1. Suppose $H \cap N = \langle 1 \rangle$. Since HN/N satisfies $\mathscr{P}(G/N)$ the minimality of G forces $HN/N \leq \Phi(G/N)$. If $N \leq \Phi(G)$, it follows that $\Phi(G/N) = \Phi(G)/N$. This is a contradiction to $H \not\leq \Phi(G)$. Thus N is complemented. Let M be a maximal subgroup complementing N. Suppose core $_G(M) \neq \langle 1 \rangle$ and let T be a minimal normal subgroup of G inside of M. Since $HN/N < \Phi(G/N)$ and $H \not\leq \Phi(G)$, we may choose M so that $H \cap M = \langle 1 \rangle$. Therefore $H \cap T =$

$\langle 1 \rangle$. Knowing that $H \nleq M$ we have $HT/T \nleq M/T$. Therefore $HT/T \nleq \Phi(G/T)$. Since $HT/T \cong H$ and G is a minimal counterexample we must have a contradiction. Thus core $_G(M) = \langle 1 \rangle$ and $C(N) = N$. It follows that $O_p(G/N) = \langle 1 \rangle$ and thus $(|\Phi(G/N)|, |N|) = 1$. Since $HN/N \leq \Phi(G/N)$ we have that $(|H|, |N|) = 1$. Lemma 2 yields a contradiction.

Case 2. If $H \cap N \neq \langle 1 \rangle$ then, by Lemma 3, HN is a p-group. Since $H \cap \Phi(G) = 1$, N is complemented. Suppose M is a maximal subgroup complementing N. Since $H \cap N \neq \langle 1 \rangle$, then $H \cap M = \langle 1 \rangle$. If core $_G(M) \neq \langle 1 \rangle$, then we can produce a minimal normal subgroup T of G such that $H \cap T = \langle 1 \rangle$. This situation was argued in Case 1. Thus $C(N) = N$ and $O_p(G/N) = \langle 1 \rangle$. Since $H \cap N \neq \langle 1 \rangle$ and H satisfies $\mathscr{P}(G)$ then $HN/N \leq \Phi(G/N)$. Thus HN/N is a subnormal p-group of G/N. This forces $H \leq N$. This final contradiction completes the proof of Theorem 1.

Putting Theorem 1 together with the result of [1] demonstrating that H is nilpotent gives us more information on the structure of H. In fact, if H is not abelian, nor a subgroup of $\Phi(G)$, then it must be a direct product of a quaternion group and a cyclic group of odd order. Such nonabelian groups may occur as the following example will show. Let Q be a quaternion group of order 8 with generators a and b of order 4. Let C_2 be a cyclic group of order 2 with generator t. Consider X as the wreath product of Q by C_2. Denote the base group of X by $Q \times \bar{Q}$ where $a^t = \bar{a}$, $b^t = \bar{b}$. It is not difficult to show that $\mathbf{Z}(X) = \langle a^2\bar{a}^2 \rangle$. If $H = \{u\bar{u} \mid u \in Q\}$, then $\Phi(X) = H\langle a^2 \rangle = H\langle \bar{a}^2 \rangle$. It is clear that $\mathbf{Z}(X) < H < \Phi(X) < X$. Since $\mathbf{Z}(X)$ is cyclic it is well known that for any $p > 2$, X admits a faithful irreducible representation over $GF(p)$. Let N be a representation module and let $G = N \rtimes X$ with the natural action of the representation. We claim that H satisfies $\mathscr{P}(G)$. If $H < \Phi(X)$ then $HN/N < \Phi(G/N)$. Therefore if $N < M <\cdot\, G$ it follows that $H \leq M$. Since N is minimal normal and $C(N) = N$, it follows that the only other maximal subgroups of G are the conjugates of X by elements of N. Let $z \in \mathbf{Z}(X)$ be the unique involution in H. It is clear that $X = C(z)$. Suppose that $H \cap X^n \neq \langle 1 \rangle$ for some $n \in N$. Since $H \cong Q$ we must have $z \in X^n$. Thus z and nzn^- are both in $X \cap N\langle z \rangle$. It follows that $z = nzn^-$ or that $n = 1$. Therefore if $H \cap X^n \neq \langle 1 \rangle$ then $H \leq X^n$. Note again that $H \cong Q$ and thus H need not be abelian.

If H is a cyclic group, find a cyclic group C such that $H = \Phi(C)$. By the Dirichlet theorem on primes in arithmetic progressions we can choose a prime p such that $p \equiv 1$ $(|C|)$. Thus C acts faithfully on a cyclic group N such that $|N| = p$ Let $G = N \rtimes C$ with this natural action. It is not hard to see that H satisfies $\mathscr{P}(G)$.

In the examples above, the groups H satisfying $\mathscr{P}(G)$ behave in the following manner. There is in each case a normal subgroup N of G such that $H \cap N = \langle 1 \rangle$ and $NH/N < \Phi(G/N)$. The following theorem shows that this is not a random occurrence.

THEOREM 2. *Let G be solvable and H satisfy $\mathscr{P}(G)$. There exists an $N \lhd G$ (perhaps trivial) with $H \cap N = \langle 1 \rangle$ such that one of the following occurs:*

(1) *$|H|$ is a prime.*
(2) *$HN/N \leq \Phi(G/N)$.*
(3) *HN/N is contained in a minimal normal subgroup of G/N.*

Proof. As usual the proof will proceed by induction on $|G|$. Let T be minimal normal in G and assume $H \cap T = \langle 1 \rangle$. By induction on HT/T in G/T, there is an $N/T \lhd G/T$ such that $|HN/N|$ is a prime, $HN/N \leq \Phi(G/N)$ or HN/N is contained in a minimal normal subgroup of G/N. Further, since $HT/T \cap N/T = T/T$, it follows that $HT \cap N = (H \cap N)T = T$. Thus $H \cap N \leq T$ and $H \cap N = \langle 1 \rangle$. If $H \cap T \neq \langle 1 \rangle$, we may assume that T is complemented and by Lemma 3 that HT is a p-group. Let M be any complement to T in G. Since $H \cap T \neq \langle 1 \rangle$, it follows that $H \cap M = \langle 1 \rangle$. If $\text{core}_G(M) \neq 1$, we can find a minimal normal subgroup of G which intersects H trivially. This case has been handled. Thus $C(T) = T$ and $O_p(G/T)$ is trivial. Since $H \cap T \neq \langle 1 \rangle$ and H satisfies $\mathscr{P}(G)$, it follows that $HT/T \leq \Phi(G/T)$. Since HT/T is a p-group we have $H \leq T$. This completes the proof.

We may say something a little more about the embedding in Theorem 2 if option (3) occurs.

THEOREM 3. *Let G be solvable and H satisfy $\mathscr{P}(G)$. Suppose there is a minimal normal subgroup N with $(|N|, |H|) = 1$ such that HN/N is contained in a minimal normal subgroup of G/N. Then either*

(1) *$|H|$ is prime or*
(2) *H is contained in a minimal normal subgroup of G.*

Proof. If N is complemented then Lemma 2 implies that H is of Frobenius type or $[H, N] = \langle 1 \rangle$. Since H is elementary abelian, by hypothesis we may assume the second alternative. Suppose $HN/N \leq K/N$ where K/N is a chief factor of G/N. We may also assume $|HN/N| > 1$. Since $C_K(N) > N$ it follows that $N < \mathbf{Z}(K)$. The hypothesis also implies that $(|K/N|, |N|) = 1$. Thus the Schur splitting theorem yields that $K = N \times L$, where L is a minimal normal subgroup of G and $H \leq L$. This completes the proof.

COROLLARY. *Let H satisfy $\mathscr{P}(G)$ in a solvable group G, and N a normal subgroup of G with $(|H|, |N|) = 1$. Suppose HN/N is contained in a minimal normal subgroup of G/N. Then either*

(1) *$|H|$ is prime or*
(2) *H is contained in a minimal normal subgroup of G.*

Proof. Use Theorem 3 and work down a chief series of G in N.

In this section we consider the case in which $\mathscr{P}(G)$ on H is relaxed to where $H \cap M \lhd H$ for all maximal subgroups of G. We prove the following theorem.

THEOREM 4. *Let G be solvable and $H \leq G$ such that $H \cap M \lhd H$ for every $M <\cdot G$. If the quarternion group is not involved in H then $H/H \cap \Phi(G)$ is supersolvable.*

In proving Theorem 4 we consider groups which are not supersolvable but in which every proper subgroup is supersolvable. Such groups have been studied in [2] by Doerk. We list the results needed in the next lemma.

LEMMA 4. *If H is a nonsupersolvable group all of whose proper subgroups are supersolvable, then H contains a normal p-Sylow subgroup H_p for some prime p. It also follows that:*

(1) *$H_p/\Phi(H_p)$ is a noncyclic chief factor of H.*
(2) *Chief factors of H above H_p and below $\Phi(H_p)$ are all cyclic.*
(3) $\Phi(H_p) \leq Z(H_p)$.
(4) *If $p > 2$,* $\exp(H_p) = p$ *and if $p = 2$ then* $\exp(H_p) \leq 4$.
(5) *H/K is supersolvable if and only if $H_p \leq K$.*

Proof. The proofs of (1)–(4) appear in [2]. To prove (5) note that (2) implies that H/H_p is supersolvable. Thus if $H_p \leq K$, H/K is also supersolvable. Conversely, since H/H_p is supersolvable, $H/H_p \cap K$ is also. Since $H_p/\Phi(H_p)$ is a chief factor of H, $\Phi(H_p)\cdot(H_p \cap K)$ equals $\Phi(H_p)$ or H_p. The first alternative contradicts the supersolvability of $H/H_p \cap K$ while the second yields the result by using the nongenerating property of the Frattini subgroup.

Proof of Theorem 4. We proceed by induction on $|G|$ and $|H|$. We note that the hypothesis which H satisfies in G inherits to HN/N in G/N for all $N \lhd G$ and to X in G where $X \leq H$.

(1) *$\Phi(G)$ is trivial.* If not, choose $N \cdot\!\lhd G$ where $N \leq \Phi(G)$. By induction on HN/N in G/N, and the fact that $\Phi(G/N) = \Phi(G)/N$, it follows that $H/H \cap \Phi(G)$ is supersolvable.

(2) *G is primitive.* Let $N \cdot\!\lhd G$ and $\Phi_N/N = \Phi(G/N)$. It is clear that $\Phi_N = \bigcap M$ where the intersection runs over all maximal subgroups of G containing N. As in (1), it follows that $H/H \cap \Phi_N$ is supersolvable, and thus $H/H \cap M$ is supersolvable for all $M <\cdot G$ containing N. If $\operatorname{core}_G(M) \neq 1$ for all $M <\cdot G$, then by the formation property of supersolvables we find that $H/H \cap \Phi(G)$ is supersolvable. Thus we may assume that there is an $M <\cdot G$ in which $\operatorname{core}_G(M)$ is trivial.

By induction on proper subgroups of H, and noting that $\Phi(G) = 1$, we may assume that H is a minimal nonsupersolvable group. Thus all the notation and results of Lemma 4 apply to the subgroup H. Further, if N is the unique minimal normal subgroup of G, then, by induction on G/N and Lemma 4(5), it follows that $H_pN/N \leq \Phi(G/N)$.

(3) $|N|$ *is relatively prime to p.* If $|N| = p^\alpha$ then, since G is primitive, $O_p(G/N)$ is trivial. Since $H_pN/N \leq \Phi(G/N)$ it follows that $H_p \leq N$. Let Q be a complement to H_p in H. We may choose $M <\cdot\, G$ such that $M \cap N$ is trivial and $Q \leq M$. It follows that $H \cap M = Q$. Thus $Q \lhd H$ and this forces $H = H_p \times Q$ to be supersolvable. This contradiction assures the result.

Since $(|N|, p) = 1$ we have by Fitting's theorem that $N = [N, H_p] \times C_N(H_p)$. Let X be the first factor. Since $C(N) = N$, X is not trivial. Choose $M <\cdot\, G$ with $M \cap N$ trivial and $H_p \leq M$. By (3) this is possible. Suppose that for some $n \in N$, $H_p \leq M^n$. It follows that $H_pN \cap M^n = H_p = H_p^n$. Therefore $H_p \leq M^n$ if and only if $n \in C_N(H_p)$.

(4) $\Phi(H_p)$ *is trivial.* By Maschke's theorem, $X = \oplus J_i$ where the J_i are irreducible H_p invariant subgroups of X. Let $C_i = C_{H_p}(J_i)$. It is clear that $\bigcap C_i = C_{H_p}(X)$. Since $C(N) = N$ and $N = X \times C_N(H_p)$, it follows that $C_{H_p}(X)$ is trivial. Let $L_i = J_iH_p$. For any $t \in J_i^\#$, consider the group $H_p \cap H_p^t$. Recall that $H \cap M^t \lhd H$ and thus also $H_p \cap M^t \lhd H$. If $H/H_p \cap M^t$ is supersolvable it follows from Lemma 4(5) that $H_p \leq M^t$. By the preceding comment this forces $t \in C_N(H_p)$ which contradicts the fact that $J_i^\# \cap C_N(H_p) = \emptyset$. Since $H_p/\Phi(H_p)$ is a chief factor of H and $H_p \nleq M^t$, we may conclude that $H_p \cap M^t \leq \Phi(H_p) \leq \mathbf{Z}(H_p)$. Therefore $H_p \cap H_p^t \leq \mathbf{Z}(H_p)$. A dual argument shows that $H_p \cap H_p^t \leq \mathbf{Z}(H_p^t)$. Since H_p and H_p^t are distinct maximal subgroups of L_i, it follows that $H_p \cap H_p^t \leq C_i$. Thus if $t \in J_i^\#$, then $H_p \cap H_p^t \leq C_i$. It follows that the group L_i/C_i is a Frobenius group with complement H_p/C_i. By the structure of Frobenius complements and Lemma 4, if $p > 2$ we may conclude that $(H_p\colon C_i) \leq p$. Thus $\Phi(H_p) \leq C_i$ for each i and since $\bigcap C_i = 1$, (4) follows. If $p = 2$, since the quarternions are not present in H, H_p/C_i might be cyclic of order 4. In any case, since $\bigcap C_i = 1$, H_p is abelian of exponent ≤ 4. Let Q be a complement to H_p in H. It follows that

$$H_p = [H_p, Q] \times C_{H_p}(Q).$$

If the first factor is not H_p then $H = [H_p, Q]Q \times C_{H_p}(Q)$ is supersolvable. Thus we may assume that $[H_p, Q] = H_p$ or $C_{H_p}(Q) = 1$. It follows that $\Phi(H_p)$ is a normal elementary abelian subgroup of the supersolvable group $\Phi(H_p)Q$. Therefore $\Phi(H_p) \leq C_{H_p}(Q)$ and again $\Phi(H_p)$ is trivial.

By Lemma 4, H_p is a minimal normal subgroup of H. Let $L = XH_p$. Suppose for $x \in X$, $H_p \cap H_p^x$ is nontrivial. Since $H_p \cap M^x < H$ and is a nontrivial p-group, the minimality of $H_p < H$ forces $H_p \leq M^x$. This implies $x \in C_N(H_p)$. Therefore $x = 1$ and L is a Frobenius group with complement H_p. Since quaternions are not involved in H, this forces H_p to be cyclic and therefore H is supersolvable. This completes the proof of Theorem 4.

Examples. (i) Let $Q = \langle a, b\rangle$ be a quaternion group of order 8 and $C = \langle c\rangle$ be cyclic of order 9. Let Q act on C by $c^a = c^-$ and $c^b = c$. We may form $M = C \rtimes Q$ with this action. It is easy to see that $H = \langle c^3, a\rangle$ is a supersolvable group of order 12. Let F be the field with 37 elements and choose $\varepsilon \in F^{\#}$ where $|\varepsilon| = 9$. It follows that

$$c \to \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon- \end{pmatrix}, \quad a \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \to \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix}$$

gives a faithful irreducible representation of degree 2 of the group M. Letting N be a representation module we can form the group $G = N \rtimes M$. It is routine to check that $H \cap R \lhd H$ for all $R <\cdot\, G$. It is also true that $\Phi(G)$ is trivial and H is *not* nilpotent.

(ii) Let $C = \langle c_1 \rangle \times \langle c_2 \rangle$ where $|C_i| = 4$ and $A = \langle a \rangle$ where $|a| = 3$. Let A act on C according to the following: $c_1^a = c_1 c_2$ and $c_2^a = c_1 c_2^2$. Form $G = C \rtimes A$ with the above action. It is easy to check that $H = \langle c_1^2, c_2^2, a \rangle$ has the property that $H \cap R \lhd H$ for all $R <\cdot\, G$ but H itself is not supersolvable. Of course $H/H \cap \Phi(G)$ is supersolvable.

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