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MULTIPLIER SEQUENCES FOR FIELDS

BY

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1. Introduction

Let F be a field and $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements in F. If for every polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k \in F,$$

which splits over F, the polynomial $\Gamma[f(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also splits over F, then Γ is called a *multiplier sequence* for F. In the case when F is the field $\mathbf{R}$ of real numbers this concept was first introduced in 1914 by Pólya and Schur in their celebrated paper [9] entitled *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen.* This beautiful paper has been the fountainhead of numerous later investigations. The main result of this work has been hailed by R. P. Boas [1, p. 418] as a "key result on the boundary between Algebra and Analysis." Pólya and Schur have shown that all the multiplier sequences for $\mathbf{R}$ are generated by entire functions which can be uniformly approximated in a neighborhood of zero by polynomials with only real (negative) zeros. (For a precise statement of this result see Section 3.) In subsequent developments, these entire functions found important applications in other fields: for example, in the theory of integral transforms [3], approximation theory [11], the theory of total positivity [4] and probability theory [5].

In this paper, inspired by the work of Pólya and Schur, we investigate and characterize the multiplier sequences for more general fields. In Section 2 we describe some of the intrinsic properties of multiplier sequences and establish the main techniques for analyzing multiplier sequences. Our results primarily concern the algebraic and arithmetic properties a field must have in order to possess a multiplier sequence of a prescribed form. In Section 3 we show that several properties of multiplier sequences for $\mathbf{R}$ are also enjoyed by multiplier sequences of an arbitrary ordered field. Moreover, with the aid of a theorem of Tarski, we are able to provide a particularly useful necessary and sufficient condition for a sequence to be a multiplier sequence for a real closed field (and for certain somewhat more general fields). In fact, it is shown that $\Gamma[f(x)]$ splits for all polynomials $f(x)$ which split and have degree less than or equal to n if and only if $\Gamma[(x+1)^n]$ splits and has all its roots of the same sign. Section 4 is devoted to the complete characterization of multiplier sequences for all finite fields. In the final section we provide a list of open questions.

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2. Intrinsic properties of multiplier sequences

We begin by considering properties of multiplier sequences which apply to any field. In later sections we shall apply these results to more restricted classes of fields. In this section, F will denote an arbitrary field and F^2 will denote the set of all squares in F (including 0).

DEFINITION 2.1. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements of F. By a *shift* of Γ, we mean a sequence $\{\gamma_s, \gamma_{s+1}, \gamma_{s+2}, \ldots\}$ for some nonnegative integer s. We shall refer to a finite sequence $\{\gamma_r, \gamma_{r+1}, \ldots, \gamma_{r+s}\}$, for some nonnegative integers r and s, as a *segment* of Γ.

We begin with several easy observations concerning multiplier sequences.

PROPOSITION 2.2. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements of F.*

(a) *If Γ is a multiplier sequence, then any shift of Γ is a multiplier sequence.*
(b) *If $\gamma_k = 0$ for $k \neq n$, $n+1$ (some fixed $n \geq 0$), then Γ is a multiplier sequence; if F is a perfect field of characteristic $p > 0$ and $\gamma_k = 0$ for $k \neq n$, $n + p^m$ (n, m fixed nonnegative integers), then Γ is a multiplier sequence.*
(c) *If $\gamma_k = cr^k$, $k = 0, 1, 2, \ldots$ for some $c, r \in F$, then Γ is a multiplier sequence.*
(d) *If $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ and $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ are multiplier sequences, then $\Gamma\Lambda = \{\gamma_k \lambda_k\}_{k=0}^{\infty}$ is also a multiplier sequence.*

DEFINITION. Sequences as in (c) above will be called *exponential sequences.* These, together with sequences as in (b), will be called *trivial multiplier sequences.*

PROPOSITION 2.3. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for F. Then for all $k \geq 0$, we have $\gamma_k \gamma_{k+2} \in F^2$.*

Proof. Apply Γ to the polynomial $x^k(x+1)(x-1)$.

PROPOSITION 2.4. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for F and suppose the characteristic of F is not 2. Then for all $k > 0$, we have $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \in F^2$.*

Proof. Apply Γ to the polynomial $x^{k-1}(x+1)^2$.

The next proposition says that any segment of a multiplier sequence, if written in reverse order, will still act as a multiplier sequence for polynomials of appropriately small degree. Combined with Proposition 2.2 (d), this provides a powerful tool for analyzing multiplier sequences, as will be seen in Section 4.

PROPOSITION 2.5 (Reversing segments). *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for F. Then any segment $\{\gamma_r, \gamma_{r+1}, \ldots, \gamma_{r+s}\}$ has the property that if $f(x) = \sum_{i=0}^{n} a_i x^i$ splits and $n \leq s$, then the polynomial $\sum_{i=0}^{n} a_i \gamma_{r+s-i} x^i$ also splits.*

Proof. By Proposition 2.2 (a), we may assume $r = 0$. For any polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$, let

$$f^*(x) = x^n f(1/x) = \sum_{i=0}^{n} a_{n-i} x^i,$$

and note that $f(x)$ splits if and only if $f^*(x)$ splits. Thus if $f(x)$ splits, so does

$$\begin{aligned}\Gamma[x^{s-n} f^*(x)] &= \sum_{i=0}^{n} a_{n-i} \gamma_{s-n+i} x^{s-n+i} \\ &= x^{s-n} \sum_{i=0}^{n} a_{n-i} \gamma_{s-n+i} x^i \\ &= x^{s-n} g(x),\end{aligned}$$

where $g(x) = \sum_{i=0}^{n} a_{n-i} \gamma_{s-n+i} x^i$. Therefore $g^*(x) = \sum_{i=0}^{n} a_i \gamma_{s-i} x^i$ also splits as desired.

We next obtain a strong theorem regarding the occurrence of "embedded" zeros in a multiplier sequence, where by an *embedded zero* we mean $\gamma_k = 0$ for some k and $\gamma_s \gamma_t \neq 0$ for some s, t such that $s < k < t$.

THEOREM 2.6. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for F. If $\gamma_l \gamma_m \neq 0$ and $\gamma_k = 0$ for all k, $l < k < m$, then F is closed under the operation of taking $(m-l)$th roots and F contains all $2(m-l)$th roots of unity.*

Proof. Apply Γ to the polynomial $x^l(x + a)(x + 1)^{m-l-1}$, where a is an arbitrary element of F.

Finally, we shall obtain a necessary condition for a field to have nonzero multiplier sequences which are not exponential sequences.

LEMMA 2.7. *If a is a nonzero element of a field F and $z^2 + a$ is a square in F for all $z \in F$, then F is a pythagorean field; that is, every sum of squares is already a square.*

Proof. If $z = 0$, the hypothesis implies that a is a square, say $a = c^2$. Let x, $y \in F$, $y \neq 0$. Then

$$x^2 + y^2 = (yc^{-1})^2((cxy^{-1})^2 + a) = (yc^{-1})^2(z^2 + a)$$

where $z = cxy^{-1}$. Since $z^2 + a$ is a square by hypothesis, the element $x^2 + y^2$ is a square, and thus F is pythagorean.

THEOREM 2.8. *Assume $\{\gamma_k, \gamma_{k+1}, \gamma_{k+2}\}$ is a segment of a multiplier sequence for some field F, where $\gamma_k \gamma_{k+1} \gamma_{k+2} \neq 0$ and $\gamma_{k+1}^2 - \gamma_k \gamma_{k+2} \neq 0$.*

(a) *If the field F has characteristic different from 2, then F is a pythagorean field.*

(b) *If F has characteristic 2, then the additive subgroup $S = \{y^2 + y \mid y \in F\}$ is equal to F.*

Proof. Replace the multiplier sequence by a shift if necessary in order to assume that $k = 0$, and apply the sequence to the polynomial $(x + 1)(x + b)$, where b is an arbitrary element of F. Then $b\gamma_0 + (b + 1)\gamma_1 x + \gamma_2 x^2$ must split for all b in F.

First we consider case (a). Then the discriminant

$$D = \gamma_1^2(b + 1)^2 - 4\gamma_0\gamma_2 b$$

must be a square in F for all $b \in F$. Completing the square in terms of b, we obtain

$$D = (\gamma_1 b + \gamma_1^{-1}(\gamma_1^2 - 2\gamma_0\gamma_2))^2 + 4\gamma_0\gamma_2(1 - \gamma_0\gamma_2\gamma_1^{-2}) = z^2 + a,$$

where $z = \gamma_1 b + \gamma_1^{-1}(\gamma_1^2 - 2\gamma_0\gamma_2)$ ranges through all the elements of F as b does, and $a = 4\gamma_0\gamma_2(1 - \gamma_0\gamma_2\gamma_1^{-2})$ is nonzero by hypothesis. By Lemma 2.7, F is pythagorean.

In case (b), the characteristic of F is 2, and so the polynomial $b\gamma_0 + (b + 1)\gamma_1 x + \gamma_2 x^2$ splits if and only if

$$\gamma_1^{-2}\gamma_2(b + 1)^{-2}(b\gamma_0 + (b + 1)\gamma_1 x + \gamma_2 x^2)$$

splits for any $b \neq 1$. Replacing $\gamma_1^{-1}\gamma_2(b + 1)^{-1}x$ by x, we see that the above polynomial splits if and only if $b(b + 1)^{-2}d \in S$ for all $b \neq 1$, where $d = \gamma_0\gamma_2\gamma_1^{-2}$ is not equal to 0 or 1 by hypothesis. Now $b(b + 1)^{-2}$ ranges through all elements of S since $b(b + 1)^{-2} = s^2 + s$ where $s = (1 + b)^{-1}$ if $b \neq 0$ and $s = 0$ if $b = 0$. Therefore $Sd \subseteq S$, and so $Sd^2 \subseteq S$. Now let $s \in F$ be arbitrary and set $t = sd$. Then $(s^2 + s)d^2 = t^2 + td \in S$ and hence $t^2 + td = r^2 + r$ for some r. But then

$$(t + r)^2 = t^2 + r^2 = r + td = (t + r) + t(1 + d),$$

or equivalently, $t(1 + d) = (t + r)^2 + (t + r)$, which is an element of S. Since s, and hence t, was arbitrary, and $d \neq 1$, the element $t(1 + d)$ ranges over all of F as t does. Therefore $F = S$ and the theorem is proved.

LEMMA 2.9. *Assume $\{0, c, cr, cr^2\}$ or $\{c, cr, cr^2 0\}$ is a segment of a multiplier sequence for a field F, where c and r are nonzero elements of F.*

(a) *If the characteristic of F is not 2, the field F is quadratically closed.*

(b) *If the characteristic of F equals 2, then F^2 is contained in the additive subgroup $S = \{y^2 + y \mid y \in F\}$.*

Proof. By Propositions 2.2 (a) and 2.5, it will suffice to consider a multiplier sequence Γ with $\gamma_0 = c$, $\gamma_1 = cr$, $\gamma_2 = cr^2$, and $\gamma_3 = 0$. Since Γ is a multiplier sequence,

$$\Gamma[(x + 1)^2(x + b)] = cr^2(2 + b)x^2 + cr(2b + 1)x + cb$$

must split for all b in F. If F has characteristic other than 2, then the discriminant $c^2r^2(1 - 4b) \in F^2$ for all $b \in F$; that is, every element of F is a square. If the characteristic of F is 2, the polynomial becomes $c(r^2bx^2 + rx + b)$. Multiplying by b and making the change of variables $z = rbx$, we see that this splits if and only if $z^2 + z + b^2$ splits; that is, if and only if the element b^2 is in S for all b in F.

COROLLARY 2.10. *Assume F is not pythagorean and has characteristic different* from 2. *If Γ is a multiplier sequence for F without embedded zeros, then Γ is a trivial multiplier sequence.*

Remark 2.11. The above results show that many common fields such as number fields and function fields (finite extensions of the field of fractions of a polynomial ring over a field) have only trivial multiplier sequences. Indeed, Theorem 2.6 implies that their multiplier sequences cannot contain any embedded zeros, and so the above corollary implies they must all be trivial.

3. Ordered fields

We shall begin this section with a brief review of the known algebraic and transcendental characterizations of multiplier sequences for the field $\mathbf{R}$ of real numbers. In order to facilitate our discussion we shall present here a definition which was first introduced by Pólya and Schur [9].

DEFINITION. A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence of the first kind* if Γ takes every polynomial $f(x)$, $f(x) \in \mathbf{R}[x]$, which has only real zeros into a polynomial, $\Gamma[f(x)]$, of the same class. A sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence of the second kind* if Γ takes every polynomial $f(x)$, $f(x) \in \mathbf{R}[x]$, all of whose zeros are real and of the same sign into a polynomial all of whose zeros are real.

With the aid of theorems on the composition of polynomials, Pólya and Schur [9, p. 100] proved the following algebraic characterization of these sequences.

THEOREM 3.1. *A real sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind if and only if the zeros of the polynomials*

$$\Gamma[(1 + x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k, \quad n = 1, 2, 3, \ldots,$$

are all real and of the same sign.

Mutatis mutandis an analogous result holds for multiplier sequences of the second kind. The difference between the two kinds of multiplier sequences is brought into sharper focus by Pólya and Schur's transcendental criteria.

THEOREM 3.2. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 \neq 0$, be a sequence of real numbers. Then in order that Γ be a multiplier sequence of the first kind it is necessary and* *sufficient that the series*

$$f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

converge in the whole plane, and that the entire function $f(z)$ can be represented in the form $f(z) = ce^{\sigma z} \prod_{n=1}^{\infty} (1 + z/z_n)$, where $\sigma \geq 0$, $z_n > 0$, $c \in \mathbf{R}$, and $\sum_{n=1}^{\infty} z_n^{-1} < \infty$.

We shall present here an observation in regard to the proof of Theorem 3.2. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence of the first kind. For $n = 1, 2, 3, \ldots$, let $g_n(z) = \Gamma[(1 + z)^n]$ and set $G_n(z) = g_n(z/n)$. The standard proofs (see, for example, Levin [6, p. 346]) which show that the function

$$f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

is entire, apply normal family arguments to the sequence $\{G_n(z)\}$ of polynomials. These proofs can be simplified as shown by the following elementary considerations. Since the sequence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ satisfies the *Turán inequalities*, that is,

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 1, 2, 3, \ldots,$$

it follows that the power series $\sum_{k=0}^{\infty} \gamma_k z^k$ has a positive radius of convergence. Hence the function $f(z)$ is an entire function, and thus the sequence $\{G_n(z)\}$ converges uniformly on compact subsets, to $f(z)$.

The analogue of Theorem 3.2 for multiplier sequences of the second kind is the following theorem of Pólya and Schur [9, p. 105].

THEOREM 3.3. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 \neq 0$, be a sequence of real numbers. Then in order that Γ be a multiplier sequence of the second kind it is necessary and sufficient that the series*

$$f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$$

converge in the whole plane, and that the entire function $f(z)$ can be represented in the form

$$f(z) = e^{-\alpha z^2 + \beta z} \prod_{n=1}^{\infty} (1 - z/z_n)e^{z/z_n},$$

where $\alpha \geq 0$, β, and z_n are real and $\sum_{n=1}^{\infty} z_n^{-2} < \infty$.

A comprehensive treatment of multiplier sequences of $\mathbf{R}$ may be found in the original paper on the subject by Pólya and Schur [9] (see also Levin [6, pp. 340–347] and Obreschkoff [7, Chapter 2]). For the significance of the Turán inequality mentioned above we refer the reader to a recent paper by Csordas and Williamson [2] and to the references contained therein.

In the sequel we shall be dealing with multiplier sequences of arbitrary ordered fields. While even in this general setting it is possible to introduce the notion of a multiplier sequence of the second kind, we shall not do so. Thus the term "multiplier sequence" will be used in the sense of the definition introduced in Section 1. Our next result shows that several properties of multiplier sequences of **R** are also enjoyed by multiplier sequences of an arbitrary ordered field.

THEOREM 3.4. *Let F be an arbitrary ordered field. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in F$, be a multiplier sequence for F.*

(a) *The relations $\gamma_k \gamma_m \neq 0$ and $\gamma_l = 0$, for any l, $k < l < m$, cannot hold at the same time.*

(b) *$\gamma_k \gamma_l \in F^2$ whenever $k \equiv l$ (mod 2).*

(c) *For all $k \geq 1$, $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$.*

Proof. Since no ordered field can contain all the nth roots of 1 if $n > 2$, part (a) of the theorem follows immediately from Theorem 2.6.

We now turn to the proof of the second part of the theorem. Suppose $k \equiv l$ (mod 2) and $\gamma_k \gamma_l \neq 0$. Since Γ contains no embedded zeros by part (a), the result follows from Proposition 2.3.

To prove the third assertion we make use of Proposition 2.4. Thus $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \in F^2$, $k \geq 1$, and *a fortiori* the Turán inequality $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$ holds for all $k \geq 1$. This completes the proof of the theorem.

Remarks. We note that part (b) of Theorem 3.4 asserts in particular that the elements in the sequence $\gamma_0, \gamma_1, \ldots$ either all have the same sign or they have alternating signs. Part (c) is of special interest when it is applied to fields with many square classes. Also we observe that the Turán inequality imposes a strong growth condition on the sequence $\{\gamma_k\}$. Indeed, if $|\gamma_m| > |\gamma_{m+1}|$, then the sequence $\{|\gamma_k|\}_{k=m}^{\infty}$ is strictly decreasing, where the absolute value is defined relative to the given ordering.

It follows from Corollary 2.10 that the multiplier sequences of nonpythagorean formally real fields are the trivial sequences. Thus we have the following theorem.

THEOREM 3.5. *Let F be a nonpythagorean formally real field. Then Γ is a multiplier sequence for F if and only if Γ is trivial.*

In view of the above result we shall next consider real closed fields, which possess an abundance of nontrivial multiplier sequences. We shall characterize the multiplier sequences of arbitrary real closed fields by making use of a theorem of Tarski [12], [10, p. 55 and p. 105]. This theorem implies that any two real closed fields satisfy precisely the same elementary sentences involving only elements common to the two fields. Thus it is easily seen that the following well known theorem [6, p. 336] is valid for every real closed field.

THEOREM 3.6 (Schur Composition Theorem). *Let*

$$f_1(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k \quad \text{and} \quad f_2(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^k$$

be two polynomials in $\mathbf{R}[x]$. *If both* $f_1(x)$ *and* $f_2(x)$ *split over* $\mathbf{R}$ *and if the zeros of* $f_2(x)$ *all have the same sign, then the polynomial*

$$g(x) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k x^k$$

also splits over $\mathbf{R}$.

We need one more observation before we can state an analogue of Theorem 3.1 for an arbitrary real closed field. For a fixed positive integer n, let $\mathscr{P}_n$ denote the set of all polynomials of degree less than or equal to n in $F[x]$ which split over F. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of elements in F. Then the following natural question arises. What condition must the sequence Γ satisfy in order that $\Gamma[f(x)] \in \mathscr{P}_n$ for every $f(x)$ in $\mathscr{P}_n$? The remarkable fact is that we need only to examine the action of Γ on a single polynomial. Indeed, let us assume that $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in F$, is a sequence with the property that the polynomial $\Gamma[(1+x)^n]$ splits over F and all its zeros are of the same sign. Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary polynomial in $\mathscr{P}_n$. Then by Theorem 3.6 the polynomial

$$\sum_{k=0}^{n} \binom{n}{k} \gamma_k \binom{n}{k}^{-1} a_k x^k = \sum_{k=0}^{n} \gamma_k a_k x^k$$

is also in $\mathscr{P}_n$. Conversely, suppose that $\Gamma[f(x)]$ is in $\mathscr{P}_n$ for all $f(x)$ in $\mathscr{P}_n$. Then, in particular, $\Gamma[(1+x)^n]$ splits over F. Moreover, if we apply the techniques used in the proofs of Theorem 2.6 and Proposition 2.3 we see that the zeros of $\Gamma[(1+x)^n]$ all have the same sign. Thus we have proved the following.

THEOREM 3.7. *Let F be a real closed field and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence with elements in F. Then the polynomial $\Gamma[(1+x)^n]$ splits over F and all its zeros have the same sign if and only if $\Gamma[f(x)] \in \mathscr{P}_n$ for all $f(x)$ in $\mathscr{P}_n$.*

As an immediate consequence of Theorem 3.7 we have the following characterization of multiplier sequences for a real closed field.

COROLLARY 3.8. *Let F be a real closed field and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence with elements in F. Then Γ is a multiplier sequence for F if and only if for every positive integer n the polynomial $\Gamma[(1+x)^n]$ splits over F and all its zeros have the same sign.*

4. Finite fields

Equipped with the techniques and results which we have established in Section 2, we shall now proceed to describe the multiplier sequences for all finite fields. At a first glance our task seems intractable even in the simplest case when the field F under consideration is $\mathbf{F}_2$; the field with only two elements, 0 and 1. If $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in \mathbf{F}_2$, is a multiplier sequence for $\mathbf{F}_2$, what is the distribution of zeros and ones in Γ? Are there any restrictions as to the number of zeros that can lie between two nonzero entries of Γ? We have formulated these questions in this manner for, as we shall see below, in the absence of embedded zeros the multiplier sequences are in fact all trivial. Thus our program will center around the characterization of the zero-one multiplier sequences for finite fields.

Our endeavor is further simplified by the remarkable fact (Theorem 4.4) that finite fields with at least five elements possess only trivial zero-one multiplier sequences. In contrast, we shall show in Theorem 4.11 that the finite fields $\mathbf{F}_q$, where $q = 2$, 3, or 4, have many nontrivial, periodic, zero-one multiplier sequences. The complete characterization of the multiplier sequences for $\mathbf{F}_q$ ($q = 2$, 3, or 4) will be accomplished by means of a representation theorem (Theorem 4.12).

The lack of nontrivial multiplier sequences for finite fields with at least five elements raises the following question. Why should the number five mark the line of demarcation in these considerations? We have been unable to provide a precise explanation of this phenomenon.

In Section 2 we saw that certain segments of multiplier sequences for fields, F, of characteristic 2 provided important information about the additive subgroup $S = \{y^2 + y \mid y \in F\}$ of F. In the sequel we shall make use of the following fact. If F is a finite field of characteristic 2, that is, $F = \mathbf{F}_{2^n}$, then S has 2^{n-1} elements. To see this, consider the additive homomorphism $\mathbf{F}_{2^n} \to S$, where $y \mapsto y^2 + y$. Let $K = \{0, 1\}$ denote the kernel of this homomorphism. Then the sequence $0 \to K \to \mathbf{F}_{2^n} \to S \to 0$ is exact. Hence $F/K \cong S$ and S has 2^{n-1} elements.

Preliminaries aside, we shall now show that for arbitrary finite fields the multiplier sequences without embedded zeros are the trivial multiplier sequences.

PROPOSITION 4.1. *Let F be a finite field. If $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \in F$, is a multiplier sequence for F without embedded zeros, then Γ is a trivial multiplier sequence.*

Proof. Suppose Γ is a multiplier sequence without embedded zeros. If the characteristic of F is different from 2, then, by Corollary 2.10, Γ is a trivial multiplier sequence. Next we consider the case when the characteristic of F is 2. If Γ has at most two nonzero terms we are done. Thus using the assumption concerning the absence of embedded zeros we can suppose that Γ contains at least three consecutive nonzero terms. Let the first such triple be denoted by γ_k, γ_{k+1}, and γ_{k+2}. Now if $\gamma_{k+1}^2 - \gamma_k \gamma_{k+2} \neq 0$, then by Theorem 2.7 the additive subgroup $S = \{y^2 + y \mid y \in F\}$ of F is equal to F. Since this is impossible, we conclude that $\gamma_{k+1}^2 - \gamma_k \gamma_{k+2} = 0$. Let us set $\gamma_k = c$, $\gamma_{k+1} = cr$, where $c, r \in F$.

Then $\gamma_{k+2} = cr^2$. If $k \neq 0$, then Γ contains a segment of the form $\{0, c, cr, cr^2\}$. But by part (b) of Lemma 2.8 this is impossible. Hence, $\gamma_0 = c$, $\gamma_1 = cr$ and $\gamma_2 = cr^2$. Another application of Lemma 2.8 together with an easy induction argument shows that $\gamma_k = cr^k$, $k = 0, 1, 2, \ldots$. Thus Γ is a trivial multiplier sequence.

Let Γ be a multiplier sequence for a finite field F. How many embedded zeros can there be between two consecutive nonzero entries of Γ? The answer to this question is contained in the following result.

PROPOSITION 4.2. *Let $F = \mathbf{F}_{p^n}$ be the finite field with p^n elements. Let $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$ be a multiplier sequence for F. If $\gamma_k \gamma_m \neq 0$ and $\gamma_i = 0$ for all i, $k < i < m$, then $m - k$ is a power of p.*

Proof. Let $m - k = qp^\alpha$, where $(q, p) = 1$. By Theorem 2.6, F contains all qp^αth roots of 1 and thus we know that $q \mid (p^n - 1)$. Let a be an arbitrary element of F and let $\beta = p^\alpha$. Then Theorem 2.6 implies that the polynomial

$$x^{q\beta} + a^\beta = (x^q + a)^\beta$$

splits. This shows that F contains all qth roots of $-a$.

Now if we choose u to be a generator of the cyclic group F^* of nonzero elements of F, then the order of u will be $p^n - 1$. Since F contains a qth root b of u, we have $b^q = u$. But this implies that

$$u^{(p^n-1)\cdot q^{-1}} = b^{p^n-1} = 1.$$

Therefore $q = 1$ and $m - k = p^\alpha$.

Our next result, when combined with Proposition 4.2, shows that finite fields with at least five elements possess only the trivial zero-one multiplier sequences.

THEOREM 4.3. *Let $F = \mathbf{F}_{p^n}$ be a finite field with at least five elements. If $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$, $\gamma_i \in F$, is a multiplier sequence for F, then for any positive integer r, Γ has no subsequence of the form*

$$\gamma_i = \begin{cases} 1 & \text{if } i = k,\ k + p^r,\ k + 2p^r, \\ 0 & \text{if } k < i < k + 2p^r,\ i \neq k + p^r. \end{cases}$$

Proof. Suppose Γ is a multiplier sequence for F containing the specified subsequence. Without loss of generality we may assume that $k = 0$. By Proposition 2.5 we may also assume that $\gamma_i = 0$ for $2p^r < i < 3p^r$.

We shall first consider the case when $p > 3$. For a in F, consider the polynomial

$$f(x) = (x + 1)^2(x + a)(x^{p-1} - 1)^2.$$

Let $s = p^{r-1}$. Then, since $p > 3$,

$$\Gamma[(f(x))^s] = [(a + 2)x^{2p} - 2(2a + 1)x^p + a]^s.$$

This polynomial splits if and only if the discriminant

$$D = 4(2a + 1)^2 - 4a(a + 2) = 4(3a^2 + 2a + 1)$$

is a square in F for all values a in F. We consider two cases. First, suppose 3 is a square in F. Then by Lemma 2.7 the discriminant

$$D(a) = (2 \cdot 3^{1/2} \cdot a + 2 \cdot 3^{-1/2})^2 + 8/3$$

cannot always be a square. Second, suppose 3 is not a square in F. Then $D(1) = 24$ and $D(-1) = 8$ cannot both be squares in F since their quotient is not a square.

We next consider the case $p = 3$. For a, b in F, consider the polynomial

$$f(x) = (x + 1)^2(x - 1)^2(x + a)(x + b),$$

and let $s = 3^{r-1}$. Then the polynomial

$$\Gamma[(f(x))^s] = [(2a + b)x^6 + (2ab + a^2 + 1)x^3 + a^2b]^s$$

splits if and only if the discriminant

$$D = (2ab + a^2 + 1)^2 - a^2b(2a + b) = b(a - a^3) + (a^2 + 1)^2$$

is a square in F for all a, b in F. Since $F \neq \mathbf{F}_3$, we can find some fixed a in F such that $a - a^3 \neq 0$. Then D ranges over all of F as b does. But $F \neq F^2$, so we have arrived at a contradiction.

Finally, we examine the case when $p = 2$ and $n \geq 3$. Consider the polynomial

$$f(x) = (x + 1)(x + a)(x + a + 1)(x + b)(x + b + 1),$$

where $a, b \in F$, and let $s = 2^{r-1}$. Then the polynomial

$$\Gamma[(f(x))^s] = [x^4 + x^2 + (a^2 + a)(b^2 + b)]^s$$

splits if and only if $(a^2 + a)(b^2 + b)$ is in the additive subgroup $S = \{y^2 + y \mid y \in F\}$ of F for all a and b in F; that is, S is multiplicatively closed. Since F is finite, S must be a subfield of F of order 2^{n-1}. But there is only one field with 2^{n-1} elements, and it is contained in F only if $n - 1$ divides n. Since $n \geq 3$, this cannot happen. Thus the proof of the theorem is complete.

Combining Proposition 4.2 and Theorem 4.3 we obtain the following result.

THEOREM 4.4. *Let $F = \mathbf{F}_{p^n}$ be a finite field with at least five elements. Then the zero-one multiplier sequences of F are the trivial zero-one multiplier sequences.*

As an immediate consequence of the previous results and the fact that for any multiplier sequence Γ for $\mathbf{F}_{p^n}$, the sequence Γ^{p-1} is a zero-one multiplier sequence, we obtain the following.

COROLLARY 4.5. *Let $F = \mathbf{F}_{p^n}$ be a finite field with at least five elements. Then the multiplier sequences of F are the trivial multiplier sequences.*

In the sequel F will always denote the finite field $\mathbf{F}_q$, where $q = 2$, 3, or 4. Thus the characteristic, p, of F will be 2 or 3. The elements of $\mathbf{F}_4$ will be denoted by 0, 1, ρ, and $\rho + 1$, where $\rho^2 = \rho + 1$. Also, for the sake of convenience we shall introduce here the following definition.

DEFINITION. For each nonnegative integer m we define Γ_m to be the sequence $\{\gamma_k\}_{k=0}^{\infty}$, where

$$\gamma_k = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{p^m} \\ 0 & \text{otherwise.} \end{cases}$$

The action of the sequence Γ_m on certain polynomials is particularly simple. It is an easy exercise to show that if $r = p^m$ and $a \in F$, then

$$\Gamma_m[(a + x)^r] = (a + x)^r \quad \text{and} \quad \Gamma_m[(a + x)^r f(x)] = (a + x)^r \Gamma_m[f(x)],$$

where $f(x)$ is in $F[x]$. This elementary observation simplifies the proof of the following

PROPOSITION 4.6. *Γ_1 is a multiplier sequence for F.*

Proof. By the above observation we need only to consider the action of Γ_1 on polynomials of the form $\prod (x + \alpha_i)^{r_i}$, where α_i ranges over the elements of F and where each $r_i < p$. Let $f(x) = \Gamma_1[\prod (x + \alpha_i)^{r_i}]$. We shall consider three cases.

(a) If $q = 2$, then $f(x)$ is either a constant, x^2 or $x^2 + 1$. Hence $f(x)$ splits.

(b) If $q = 3$ and $f(x)$ is not a constant, then either all $r_i = 2$, so that $f(x) = x^6$, or some $r_i < 2$. In the latter case $f(x)$ has the form $ax^3 + b = (ax + b)^3$ for some $a, b \in F$. Thus, once again we conclude that $f(x)$ splits.

(c) Finally, suppose that $q = 4$. Since in this case all $r_i \leq 1$ and since F has only 4 elements, we have $\deg f(x) \leq 4$. If $\deg f(x) < 4$, then $f(x)$ has the form $ax^2 + b$ which always splits in F. On the other hand, the polynomial $f(x)$ can have degree 4 only if it equals

$$\Gamma_1[x(x + 1)(x + \rho)(x + \rho + 1)] = x^4.$$

Since $f(x)$ splits in all cases, we conclude that Γ_1 is a multiplier sequence for F.

Next, by means of an induction argument we shall show that Γ_m, $m = 0, 1, 2, \ldots,$ is a multiplier sequence for F.

PROPOSITION 4.7. *For any nonnegative integer m, Γ_m is a multiplier sequence for F.*

Proof. The sequence Γ_0 is constantly one, so this case is trivial. Proposition 4.6 shows that Γ_1 is also a multiplier sequence. Now suppose that Γ_m is a multiplier sequence for some $m \geq 1$. We shall demonstrate that Γ_{m+1} is then also a multiplier sequence for F. Let $f(x) = \sum a_k x^k$, $a_k \in F$, be any polynomial which splits over F. Then the polynomial

$$g(x) = \Gamma_m[f(x)] = \sum a_k x^k, \quad p^m \mid k,$$

also splits over F. Let $y = x^s$, where $s = p^m$. Then $g(x)$ becomes $g_1(y) = \sum a_{ks} y^k$. Now it is not difficult to show that $g_1(y)$ splits over F. Hence, it follows that the polynomial

$$g_2(y) = \Gamma_1[g_1(y)] = \sum a_{ks} y^k, \quad p \mid k,$$

also splits over F. Replacing y by x^s, where $s = p^m$, we obtain

$$\begin{aligned} g_2(x^s) &= \sum a_{ks} x^{ks}, \quad p \mid k, \\ &= \sum a_k x^k, \quad p^{m+1} \mid k, \\ &= \Gamma_{m+1}[f(x)]. \end{aligned}$$

Hence $\Gamma_{m+1}[f(x)]$ also splits over F and we are done by induction.

By Proposition 4.2 the number of embedded zeros in a multiplier sequence for F is of the form $p^m - 1$ ($m \geq 0$). Now if a zero-one multiplier sequence for F contains three ones, then the consecutive ones are separated by the same number of zeros. This is the content of the next proposition.

PROPOSITION 4.8 (Periodicity). *Let Γ be a multiplier sequence for F. If Γ has a segment containing three ones, the first two separated by $p^m - 1$ zeros and the second two separated by $p^n - 1$ zeros ($m, n \geq 0$), then $m = n$.*

Proof. Suppose $m \neq n$. Let Λ denote the specified segment and let Λ^* denote the reverse of Λ. By considering a shift of Γ, if necessary, we may assume that Γ begins with the segment Λ. Let $s = p^m$, $t = p^n$ and set

$$f(x) = (x + 1)^{s+t-1}(x + a), \quad a \in F.$$

Then by Proposition 2.5 the polynomial $g(x) = \Lambda^*[f(x)]$ splits and *a fortiori* $\Gamma[g(x)] = x^{s+t} + a$ splits over F for all $a \in F$. But $s + t = p^m + p^n$ is not a power of p, since $m \neq n$. Thus we have arrived at the desired contradiction.

The formation of new multiplier sequences from old ones is one of the leitmotifs that has permeated our discussions. Under somewhat restrictive assumptions, our next result elaborates on this theme and provides us with an important additional device that we shall need in the proofs of Lemma 4.10 and Theorem 4.12.

LEMMA 4.9. *Let K be a perfect field of characteristic p and let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence for K, where, for some fixed m, $\gamma_k = 0$ for all $k \not\equiv 0$ (mod p^m). Then the sequence $\Gamma_* = \{\gamma_k \mid p^m$ divides $k\}$ is a multiplier sequence for K.*

Proof. Let $f(x) = \sum a_i x^i$ be any polynomial which splits over K and let $s = p^m$. Since K is perfect, there exist elements $b_i \in K$ such that $b_i^s = a_i$. Since

$f(x)$ splits and K is perfect the polynomial $\sum a_i x^{is} = (\sum b_i x^i)^s$ also splits over K. Now

$$\Gamma[(\sum b_i x^i)^s] = \sum \gamma_{is} a_i x^{is}$$

splits over K since Γ is a multiplier sequence for K by assumption. But this implies that the polynomial $\sum \gamma_{is} a_i x^i = \Gamma_*[f(x)]$ also splits over K, and thus Γ_* is a multiplier sequence for K.

The purpose of the next lemma is to show that a zero-one multiplier sequence for F with at least three ones cannot begin with too many zeros or terminate in zeros.

LEMMA 4.10. *Let m be a nonnegative integer. Then no zero-one multiplier sequence for F has a segment of the form*

$$\gamma_k = \begin{cases} 1 & \text{if } k = 0,\ p^m,\ 2p^m, \\ 0 & \text{otherwise, } k = 0, 1, \ldots, 3p^m, \end{cases}$$

or

$$\gamma'_k = \begin{cases} 1 & \text{if } k = p^m,\ 2p^m,\ 3p^m, \\ 0 & \text{otherwise, } k = 0, 1, \ldots, 3p^m. \end{cases}$$

Proof. Suppose that a zero-one multiplier sequence Γ contains one of the above segments as its initial segment. Then by Proposition 4.8, $\gamma_k = 0$ for any $k \not\equiv 0 \pmod{p^m}$. Hence by Lemma 4.9 the sequence

$$\Gamma_* = \{\gamma_k \mid k = 0, p^m, 2p^m, \ldots\}$$

is a multiplier sequence. But then Γ_* begins with the segment $\{1, 1, 1, 0\}$ or the segment $\{0, 1, 1, 1\}$. Since this contradicts Lemma 2.8 our proof is complete.

We are now in a position to state the main result of this section; the complete characterization of zero-one multiplier sequences for $F = \mathbf{F}_q$, where $q = 2, 3$, or 4.

THEOREM 4.11 (Main Theorem). *Let Γ be a zero-one sequence. Then Γ is a multiplier sequence for F if and only if one of the following conditions holds:*

(a) *Γ has at most two ones, and if there are two ones they have $p^m - 1$ ($m \geq 0$) zeros between them.*

(b) *Γ is Γ_m or a shift of Γ_m for some nonnegative integer m.*

Proof. Sequences of the form (b) are multiplier sequences by Proposition 4.7. Since F is a perfect field, it follows from Proposition 2.2 (b) that sequences of the form (a) are also multiplier sequences.

Conversely, suppose that Γ is a zero-one multiplier sequence not of the form (a). Then Γ has at least three ones. By Proposition 4.8 any two consecutive ones are separated by the same number of zeros. In view of Proposition 4.2 this number is of the form $p^m - 1$. Lemma 4.10 then implies that Γ has the form (b), since Γ cannot begin with too many zeros or terminate in zeros.

The description of arbitrary multiplier sequences for F is given by the following representation theorem.

THEOREM 4.12 (Representation Theorem). *Let Γ be a multiplier sequence for $F = \mathbf{F}_q$, where $q = 2, 3$, or 4. Then Γ can be written as a product $\Gamma_a \Gamma_b$, where Γ_a is a zero-one multiplier sequence and Γ_b is an exponential sequence. If Γ has more than one nonzero entry, then the representation is unique.*

Proof. Let $\Gamma_a = \Gamma^{q-1}$. Then by Theorem 4.11, Γ_a has either at most two ones or else is a shift of Γ_m. If Γ_a has only one 1, the result is trivial. If Γ_a contains exactly two ones, they are separated by $p^m - 1$ zeros, say

$$\Gamma = \{0, 0, \ldots, 0, \gamma_k, 0, \ldots, 0, \gamma_{k+p^m}, 0, \ldots\}.$$

In this case we take Γ_b to be the exponential sequence $\{cr^k\}_{k=0}^{\infty}$, where $c = \gamma_k^{k+1}\gamma_{k+p^m}^{-k}$ and $r = \gamma_{k+p^m}\gamma_k^{-1}$. Thus it follows that $\Gamma = \Gamma_a \Gamma_b$.

Now suppose that Γ_a is a shift of Γ_m. Without loss of generality we can assume that $\Gamma_a = \Gamma_m$. If $m = 0$, then by Proposition 4.1 we know that Γ is an exponential sequence so that in this case we are done. If $m \geq 1$, we let Γ_* denote the subsequence of Γ obtained by deleting the zero elements from Γ. Then by Lemma 4.9, Γ_* is a multiplier sequence for F. Since Γ_* has no zeros, it follows from Proposition 4.1 that Γ_* is an exponential sequence, say, $\Gamma_* = \{cr^k\}_{k=0}^{\infty}$; that is, $\gamma_{kp^m} = cr^k$ for $k = 0, 1, 2, \ldots$. Let $t = p^{-m}$, $s = r^t$ and let $\Gamma_b = \{cs^k\}_{k=0}^{\infty}$. Then $cs^{kp^m} = cr^k = \gamma_{kp^m}$ and hence we see that $\Gamma = \Gamma_a \Gamma_b$.

Uniqueness is clear because Γ_a is determined by the positions of the nonzero entries in Γ and because any exponential sequence is uniquely determined by two consecutive entries.

Example. The theory of multiplier sequences provides a technique for determining whether or not certain polynomials split. For example, let $F = \mathbf{F}_4$ and consider the polynomial $f(x) = \rho + ax + x^2 + bx^3 + x^4$ for any $a, b \in F$. This polynomial does not split over F for any values of a and b in F, since

$$\Gamma_2[f(x)] = (\rho + 1 + x + x^2)^2$$

and $\rho + 1 + x + x^2$ is irreducible over $\mathbf{F}_4$.

We conclude this section with a result which is an analogue of Theorem 3.2. The periodic nature of the zero-one multiplier sequences of $F = \mathbf{F}_q$, $q = 2, 3$, or 4, makes the transcendental characterization of these sequences particularly simple. In order to expedite our presentation we shall make use of the Mittag-Leffler functions $E_\alpha(z)$, namely

$$E_\alpha(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \quad a_n^{-1} = \int_0^{\infty} e^{-t} t^{\alpha n}\, dt, \quad \alpha > 0,$$

that is, a_n^{-1} is the value of the Gamma function evaluated at $\alpha n + 1$. Thus, as an immediate consequence of Theorem 4.11, we have the following theorem.

THEOREM 4.13. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a zero-one sequence. Let*

$$f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k.$$

Then Γ is a nontrivial zero-one multiplier sequence for $\mathbf{F}_q$, $q = 2, 3$, or 4, if and only if $f(z) = z^r E_{p^m}(z^{p^m})$, where $m = 1, 2, \ldots$, and r is a nonnegative integer less than or equal to $p^m - 1$.

Remark 4.14 (A geometric interpretation of our result). First we recall that Wiman [13] has shown that if $\alpha \geq 2$, then $E_\alpha(z)$ has only real and negative zeros. An elegant proof of this fact using the theory of multiplier sequences for $\mathbf{R}$ may be found in [1, p. 229]. We also note that if α is an integer ($\alpha = 2, 3, \ldots$), then the zeros of

$$E_\alpha(z^\alpha) = \frac{1}{\alpha}[e^z + e^{wz} + \cdots + e^{w^{\alpha-1}z}],$$

where $w = \exp\{2\pi i/\alpha\}$, all lie on the α half-rays given by

$$s_\alpha(t) = t\left(\cos\frac{\pi + 2\pi k}{\alpha} + i\sin\frac{\pi + 2\pi k}{\alpha}\right), \tag{4.15}$$

where $k = 0, 1, \ldots, \alpha - 1$ and $t \geq 0$.

Now let

$$f(z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}, \quad \gamma_0 = 1.$$

If $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is a nonnegative multiplier sequence for $\mathbf{R}$, then by the Pólya-Schur Theorem (Theorem 3.2) all the zeros of $f(z)$ lie on the negative real axis. If, on the other hand, $\Gamma = \Gamma_m = \{\gamma_k\}_{k=0}^{\infty}$ is a zero-one multiplier sequence for $\mathbf{F}_p$ ($p = 2, 3$) with $m = 1$, then by Theorem 4.13, $f(z)$ is $E_2(z^2)$ if $p = 2$, and $f(z)$ is $E_3(z^3)$ if $p = 3$. Thus, when $m = 1$ the zeros of $f(z)$ lie on the $\alpha = p$ ($p = 2$ or 3) half-rays given by (4.15).

5. Open questions

We conclude this paper with a list of open questions.

1. By Proposition 2.2, the set $S(F)$ of all multiplier sequences for a field F is a commutative semigroup containing 0 and 1. Let $H(F)$ be the group of invertible elements of $S(F)$. The group $H(F)$ is isomorphic to $E(F) \times N(F)$, where $E(F)$ is the subgroup of all exponential sequences and $N(F) = \{\Gamma \in H(F) \mid \gamma_0 = \gamma_1 = 1\}$. Are there any fields F, not algebraically closed, for which $N(F)$ is nontrivial?

2. Is there a correspondence between some set of subfields of an algebraically closed field and the subgroups of $N(F)$ or the subsemigroups of $S(F)$?

3. What conditions are necessary and sufficient for a finite sequence to be extensible to a multiplier sequence?

4. When is a multiplier sequence for a field F also a multiplier sequence for an extension of F or for a subfield of F?

5. Let $\{\gamma_k\}_{k=0}^{\infty}$ be a nontrivial multiplier sequence for a field F. Under what conditions on F is the sequence $\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$, $c \in F$, again a multiplier sequence for F? If F is the field of real numbers and if $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, is a multiplier sequence for $\mathbf{R}$, then for any $c \geq 0$,

$$\{c\gamma_k + k\gamma_{k-1}\}_{k=0}^{\infty}$$

is again a multiplier sequence for $\mathbf{R}$. (This observation is an immediate consequence of the results of Section 3.)

6. Let $\{\gamma_k\}$, $\gamma_k \in \mathbf{R}$, be a multiplier sequence of the second kind (see Section 3 for the definition). Is the sequence

$$\{(k+m)(k+m-1)\gamma_{k+m-2} + \gamma_{k+m}\}_{k=0}^{\infty},$$

for m sufficiently large, again a multiplier sequence of the second kind? This question is a reformulation of a famous open problem (see, for example, [8, p. 182]) in the theory of entire functions.

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