ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

PERIODICITY IN GROUPS

BY

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1. Introduction

The phenomenon of periodicity of group cohomology has long been understood; we shall investigate a more general phenomenon which has arisen in our work, namely, the periodicity of a module. Therefore, we fix G, a finite group, and F, a field of prime characteristic p, and we denote the corresponding group algebra by FG. We shall implicitly assume that all FG-modules are finitely generated, that is, in view of the finiteness of G, finite dimensional over F. We say that an FG-module M is *periodic* if there is an exact sequence

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where each P_i, $0 \leq i < n$, is a projective FG-module. In particular, a projective FG-module P is periodic because of the following exact sequence: $0 \to P \to P \oplus P \to P \to 0$. However, there are many more periodic modules.

If M is periodic as above, then it follows that there is a projective resolution $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$ where P_i and P_j are isomorphic if n divides $i - j$ and a similar statement holds for the maps in the resolution. Hence, if V is any FG-module, then

$$\mathrm{Ext}^i_{FG}(M, V) \cong \mathrm{Ext}^j_{FG}(M, V)$$

when i and j are positive integers whose difference is divisible by n. So if m is the maximal dimension over F of the vector spaces $\mathrm{Ext}^k_{FG}(M, V)$, $0 \leq k \leq n$, then $\mathrm{Ext}^s_{FG}(M, V)$ has dimension at most m for all $s \geq 0$. With this in mind, we say that an FG-module U is *bounded* if, for any FG-module V, there is an integer m, depending on V, such that for all $s \geq 0$, $\mathrm{Ext}^s_{FG}(U, V)$ has dimension at most m over F. Hence, we have seen that a periodic module is bounded, which is half of our first result.

THEOREM 1. *If F is an algebraic extension of its prime subfield then an FG-module is periodic if, and only if, it is bounded.*

Presumably, the hypothesis on the field is unnecessary. Our next result shows that there are many such modules.

Received March 24, 1976.

[1] Supported in part by a National Science Foundation grant.

THEOREM 2. *If the Sylow p-subgroups of G are not cyclic, then there are infinitely many isomorphism classes of indecomposable periodic FG-modules.*

This is analogous to a theorem of D. G. Higman [3] on the existence of infinitely many isomorphism classes of indecomposable FG-modules. Our last result shows that periodicity of a module does force restrictions on its structure.

THEOREM 3. *If G has p-rank exceeding one then every periodic FG-module has dimension divisible by p.*

As usual, the p-rank of G is the rank of the largest elementary abelian p-subgroups that G contains. We suspect that even stronger results on the dimensions of periodic modules do hold.

The rest of this paper is divided into three more sections, devoted, in turn, to some preliminary results, the proofs of the theorems, and to comments and open questions.

2. Preliminary results

First, let's recall a few properties of the syzygy functor, which attaches to each FG-module U another FG-module $S(U)$. This allows us to set $S^1(U) = S(U)$ and to define $S^n(U)$ recursively, $n \geq 1$. As is well known [1], [2], S satisfies the following assertions:

(1) If $0 \to K \to P \to U \to 0$ is an exact sequence of FG-modules, with P projective, then K is isomorphic with the direct sum of $S(U)$ and a projective FG-module;

(2) $S(U)$ has no nonzero projective direct summand;

(3) If U and V are FG-modules then $S(U \oplus V) \cong S(U) \oplus S(V)$;

(4) There is a projective module P and an exact sequence

$$0 \to S(U) \to P \to U \to 0;$$

(5) If U is an indecomposable and nonprojective FG-module then so is $S(U)$;

(6) If U and V are FG-modules and $S(U) \cong S(V)$ then there are projective FG-modules P and Q with $U \oplus P \cong V \oplus Q$.

LEMMA 1. *If U is an FG-module and V is an irreducible FG-module then for* $n > 0$, $\mathrm{Ext}^n_{FG}(U, V) \cong \mathrm{Hom}_{FG}(S^n(U), V)$.

Proof. It suffices, in view of the recursive definition of S^n, to show that

$$\mathrm{Ext}^{i+1}_{FG}(U, V) \cong \mathrm{Ext}^i_{FG}(S(U), V)$$

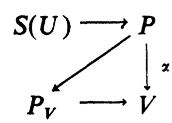
for all $i \geq 0$. Choose an exact sequence $0 \to S(U) \to P \to U \to 0$ with P a projective FG-module. The corresponding long exact sequence is as follows:

$$0 \to \mathrm{Hom}_{FG}(U, V) \to \mathrm{Hom}_{FG}(P, V) \to \mathrm{Hom}_{FG}(S(U), V)$$
$$\to \mathrm{Ext}^1_{FG}(U, V) \to \mathrm{Ext}^1_{FG}(P, V) \to \cdots$$
$$\cdots \to \mathrm{Ext}^i_{FG}(P, V) \to \mathrm{Ext}^i_{FG}(S(U), V)$$
$$\to \mathrm{Ext}^{i+1}_{FG}(U, V) \to \mathrm{Ext}^{i+1}_{FG}(P, V) \to \cdots.$$

Since P is projective, $\mathrm{Ext}^j_{FG}(P, V) = 0$ for all $j > 0$ so our assertion holds for $i > 0$. To see that it holds for $i = 0$ it is enough to see that the map

$$\mathrm{Hom}_{FG}(U, V) \to \mathrm{Hom}_{FV}(P, V)$$

is onto. Suppose that there is a homomorphism α of P onto V which is not in the image. Thus, α maps P onto V as V is an irreducible module. Let P_V be a projective cover of V, so P_V is an indecomposable projective FG-module, has a unique maximal submodule and the corresponding quotient module is isomorphic with V. We therefore have a commutative diagram



The image of $S(U)$ in P_V is all of P_V; otherwise the image would have zero image in V. But then the map α would be a map of $P/S(U)$, against our hypothesis. The lemma is proved.

LEMMA 2. *If U and V are periodic FG-modules then so is* $U \oplus V$.

Proof Since U and V are periodic there are exact sequences

$$0 \to U \to P_{m-1} \to \cdots \to P_0 \to U \to 0,$$
$$0 \to V \to Q_{n-1} \to \cdots \to Q_0 \to V \to 0,$$

where all the P_i and Q_j are projective. By "splicing" of exact sequences, it follows that there are similar exact sequences for U and V with mn projectives. Addition of these two new sequences gives one demonstrating the periodicity of $U \oplus V$.

LEMMA 3. *A summand of a periodic FG-module is periodic.*

Proof. Let U be a periodic FG-module and express

$$U = U_1 \oplus \cdots \oplus U_k \oplus Q$$

where each U_i is an indecomposable and nonprojective FG-module and Q is a projective FG-module. By the Krull-Schmidt theorem, any summand of U is isomorphic with a direct sum of certain of the U_i and a summand of Q. Therefore, since projective modules are periodic, it suffices, in view of the preceding lemma, to show that each U_i is a periodic module.

The properties of the syzygy functor and the periodicity of U yield immediately that there is a projective module P and a positive integer n with $S^n(U) \oplus P \cong U$. But

$$S^n(U) \cong S^n(U_1 \oplus \cdots \oplus U_k) \cong S^n(U_1) \oplus \cdots \oplus S^n(U_k).$$

Hence, these isomorphisms, plus the fact that $S^n(U)$ has no nonzero projective summand, imply that

$$U_1 \oplus \cdots \oplus U_k \cong S^n(U_1) \oplus \cdots \oplus S^n(U_k).$$

Again, by the properties of the syzygy functor and the Krull-Schmidt theorem, there is a permutation α of $\{1, \ldots, k\}$ such that $S^n(U_i) \cong U_{i\alpha}$. Hence, for a suitably large integer m, $S^m(U_i) \cong U_i$ for all i, $1 \leq i \leq k$. Therefore, each U_i is periodic as desired.

LEMMA 4. *An FG-module U is periodic if, and only if,* $S^m(U) \cong S^n(U)$ *for some* $m > n > 0$.

Proof. If U is periodic then $S^n(U) \oplus P \cong U$, as above, with P projective, so that $S^{n+1}(U) \cong S(S^n(U) + P) \cong S(U)$, as desired. On the other hand, suppose that $S^m(U) \cong S^n(U)$, with $m > n > 0$. Then $S^{m-n}(U) \oplus R \cong U$ for some projective FG-module R. Hence, if $U = U' \oplus Q$, where U' has no nonzero projective summand and Q is projective, then $S^{m-n}(U') \cong U'$. Hence, U' is periodic and so is U, by Lemma 2.

With the Green correspondence in mind, we make the following definition which will be useful in what follows. Let U and V be indecomposable FH and FK modules, respectively, where K is a subgroup of the group H. We say that U and V are *related* provided V is isomorphic with a direct summand of the restriction U_K of U to K, written $V \mid U_K$, and U is isomorphic with a summand of the FH-module V^H induced from V, written $U \mid V^H$.

LEMMA 5. *If U and V are related then U is bounded if, and only if, V is bounded.*

Proof. First, suppose that V is bounded. Let W be any FH-module; we must obtain a bound for $\dim_F \operatorname{Ext}^n_{FH}(U, W)$ independent of n. But U is a summand of V^H so

$$\dim_F \operatorname{Ext}^n_{FH}(U, W) \leq \dim_F \operatorname{Ext}^n_{FH}(V^H, W) = \dim_F \operatorname{Ext}^n_{FK}(V, W_K)$$

and the boundedness of V yields the desired conclusion.

On the other hand, suppose that U is bounded and that W is any FK-module. We have

$$\dim_F \operatorname{Ext}^n_{FK}(V, W) \leq \dim_F \operatorname{Ext}^n_{FK}(U_K, W) = \dim_F \operatorname{Ext}^n_{FH}(U, W^H)$$

so we are done.

LEMMA 6. *If K is a normal subgroup of the group H, U is an FH-module with U_K projective, and H/K has p-rank at most one, then U is bounded.*

Proof. Let V be any FH-module; we must bound all the dimensions of the $\operatorname{Ext}^n_{FH}(U, V)$. However, if $W = U^* \otimes V$, where U^* is the dual of U, then $\operatorname{Ext}^n_{FH}(U, V) \cong \operatorname{Ext}^n_{FH}(F, W)$, so it suffices to bound all the dimensions of these latter vector spaces.

The restriction U^*_K is also projective and therefore so is the restriction $W_K \cong U^*_K \otimes V_K$, since the tensor product of any module with a projective module is again projective. Therefore, for $n > 1$, $\operatorname{Ext}^n_{FK}(F, W_K) = 0$. The Lyndon spectral sequence [4] (over F instead of over the integers) yields

$$\operatorname{Ext}^n_{FH}(F, W) \cong \operatorname{Ext}^n_{F(H/K)}(F, \operatorname{Hom}_{FK}(F, W)) \quad \text{for all } n \geq 0.$$

But H/K has periodic cohomology, by assumption, so the vector spaces on the right-hand side are of bounded dimension.

Now let E be an elementary abelian group of order p^2 with generators a and b. Let V_n be a vector space over F with basis consisting of elements v_{ij}, $1 \leq i \leq n$, $1 \leq j \leq p$. If we define

$$v_{ij}a = \begin{cases} v_{ij} + v_{i,j+1}, & j < p \\ v_{ij}, & j = p, \end{cases}$$

$$v_{ij}b = \begin{cases} v_{ij}, & j > 1 \text{ or } j = 1 \text{ and } i = n \\ v_{i1} + v_{i+1p}, & j = 1 \text{ and } i < n, \end{cases}$$

this defines the structure of an FE-module on V_n, as is easy to check.

LEMMA 7. *V_n is a bounded indecomposable FE-module.*

Proof. The module V_n is free as an $F\langle a\rangle$-module, inasmuch as $v_{11}, v_{21}, \ldots, v_{n1}$ are free generators. This has several consequences. First, V_n as an FE-module is certainly bounded, by the previous lemma. Second, $v_{1p}, v_{2p}, \ldots, v_{np}$ span the socle of V_n as $F\langle a\rangle$-module. Since the element b leaves fixed each of these n vectors, it follows that these vectors are a basis of the socle of the FE-module V_n. Hence, if we let U be a nonzero proper direct summand of V, it follows, perhaps after replacing U by a complementary summand, that there are elements $\alpha_2, \ldots, \alpha_n$ of F such that $v = v_{1p} + \alpha_2 v_{2p} + \cdots + \alpha_n v_{np}$ lies in U.

We conclude the proof by showing that this leads to a contradiction. Since U is free as an $F\langle a\rangle$-module, being a summand of V_n, there is u in U with $u(a-1)^{p-1} = v$. Hence, $u(b-1)$ is in U. But we must have that

$$u = v_{11} + \alpha_2 v_{21} + \cdots + \alpha_n v_{n1} + w$$

where w is a linear combination of the v_{ij} with $j > 1$. Hence,

$$u(b-1) = v_{2p} + \alpha_2 v_{3p} + \cdots + \alpha_{n-1p} v_{np}$$

is an element of U. Continuing in this way we will finally get that v_{np} is in U and then, since we will have $v_{n-1p} + \alpha_2 v_{np}$ in U, it follows that v_{n-1p} is in U. This leads to $v_{n-2p}, \ldots, v_{1p}$ all being in U so U contains the entire socle of V_n, contradicting the assumption that U is a proper direct summand of V_n.

3. Proofs

To conclude the proof of Theorem 1, we let U be a bounded FG-module and we assume that F is an algebraic extension of its prime subfield. Since G is finite, it follows that there is a finite extension F_0 of that prime subfield and an F_0G-module U_0 such that $F \otimes U_0 \cong U$. If V_0 is any F_0G-module and $V = F \otimes V_0$ then

$$F \otimes \operatorname{Ext}^n_{F_0G}(U_0, V_0) \cong \operatorname{Ext}^n_{FG}(U, V)$$

so that U_0 is a bounded F_0G-module. Hence, it suffices to prove that U_0 is a periodic F_0G-module, inasmuch as the sequence exhibiting the periodicity of U_0, when tensored with F, will yield a sequence giving the periodicity of U. Therefore, by Lemma 4, it suffices to show that $S^m(U_0) \cong S^n(U_0)$ for some $m > n > 0$. But F_0 is a finite field so there are up to isomorphism only finitely many F_0G-modules of any given dimension. Hence, to conclude the proof, we need only show there is a bound for all the dimensions of all the modules $S^k(U_0)$.

Let $S_1, \ldots, S_r$ be irreducible F_0G-modules, one of each possible isomorphism type. Since U_0 is bounded, it follows from Lemma 1, that there is an integer N such that $\dim_{F_0} \operatorname{Hom}_{F_0G}(S^k(U_0), S_j) \leq N$, for all k and j. If P_j is any projective module with S_j as an image then it follows that every $S^k(U_0)$ is a homomorphic image of the direct sum of each of the P_j taken N times. This gives the needed bound on the dimensions of the $S^k(U_0)$.

We now turn to the proof of Theorem 2. First, we shall deal with the case that G contains an elementary abelian p-subgroup E of order p^2. Let K be the prime subfield of F and let V_n be the KE-module discussed in Lemma 7. The induced module V_n^G, when restricted to E, has V_n as a direct summand, by Mackey's Theorem, so there is an indecomposable summand U_n of V_n^G whose restriction to E contains a summand isomorphic with V_n. Hence, U_n and V_n are related in the sense we defined above, so that U_n is a bounded KG-module, by Lemma 5. Therefore, by Theorem 1, U_n is a periodic KG-module.

Now $F \otimes V_n$ is indecomposable, by Lemma 7, so $F \otimes U_n$ has an indecomposable direct summand W_n whose restriction to E has a summand isomorphic with $F \otimes V_n$; in particular, W_n is of dimension at least np, the dimension of V_n. Moreover, $F \otimes U_n$ is certainly a periodic FG-module, so W_n is also, by Lemma 3. Since the dimensions of the W_n go to infinity there must be infinitely many isomorphism types represented by these modules and so the theorem is proved in this case.

Now since G does not have cyclic Sylow p-subgroups by assumption and since we may now assume that G does not contain an elementary abelian

p-subgroup of order p^2, it follows that $p = 2$ and that a Sylow 2-subgroup S is quaternion or generalized quaternion. Therefore, G has periodic cohomology, that is, F is a periodic FG-module [5]. The sequence giving the periodicity of F can be tensored with any module so every FG-module is periodic. Since there are infinitely many isomorphism types of indecomposable FG-modules [3] the proof of the theorem is complete.

Finally, we prove the last theorem. By assumption, G contains an elementary abelian p-subgroup E of order p^2. Let U be the FG-module induced from the trivial FE-module F. Since $\mathrm{Ext}^n_{FG}(F, U) \cong \mathrm{Ext}^n_{FE}(F, F)$ it follows that U is not a bounded FG-module. Since $\mathrm{Ext}^n_{FG}(U, V) \cong \mathrm{Ext}^n_{FG}(F, U^* \otimes V)$ for any FG-module V, it follows that F is not a bounded FG-module.

Now let M be a periodic FG-module and assume that M has dimension d not divisible by p. If V is any FG-module then

$$\mathrm{Ext}^n_{FG}(M \otimes M^*, V) \cong \mathrm{Ext}^n_{FG}(M, M \otimes V)$$

so $M \otimes M^*$ is bounded since M is bounded. Hence, any summand of $M \otimes M^*$ is also bounded because of the additivity of the functors involved.

Let δ be a representation of G in $GL(d, F)$ associated with the module M. The vector space of all d by d matrices over F now becomes a module for G by letting the action of g in G be defined by conjugation by the matrix $g\delta$. As is well known this module is isomorphic with $M \otimes M^*$. However, this module is the direct sum of the scalar matrices and the matrices of trace zero, since d is not divisible by p. This first summand is isomorphic with F as FG-module, so F is a bounded module, which is a contradiction. The theorem is proved.

4. Remarks

A number of questions are suggested by the results we have obtained. Is the restriction on the field necessary in Theorem 1? Is an extension of a periodic module by a periodic module also periodic? Certainly, by the long exact sequence for the functors Ext^n, such an extension is a bounded module. How are all periodic modules constructed? If G has p-rank e does p^{e-1} divide the dimension of each periodic FG-module? If G is a p-group does the "period" of a periodic module have to be two if p is odd and one, two or four if p is two?

There are some other approaches available too. G. Janusz has kindly shown us a proof of the indecomposability of the module V_n of Lemma 7 which calculates the endomorphism ring of V_n. And E. Dade has given a direct proof of the periodicity of that module which carries over to a more general situation.

We shall conclude by pointing out that the ideas of projective modules and bounded modules are just special cases of a more general classification of modules. If the group G has p-rank e then to each FG-module we can attach an integer between zero and e which we call the *complexity* of the module. It is zero if, and only if, the module is projective and it is one if, and only if, the module is bounded and not projective. This, of course, also suggests directions in which this work can be generalized.

To define complexity, recall that if M is any FG-module then there are positive integers n_0 and q such that for each i, $0 \leq i < q$, $\dim_F \mathrm{Ext}_{FG}^{nq+i}(F, M)$ is a polynomial in n for all $n \geq n_0$ of degree at most $e - 1$, where e is the p-rank of G [6, p. 403]. If U and V are FG-modules then

$$\mathrm{Ext}_{FG}^n(U, V) \cong \mathrm{Ext}_{FG}^n(F, U^* \otimes V),$$

where U^* is the dual of U, so that a similar statement holds for $\dim_F \mathrm{Ext}_{FG}^n(U, V)$ as a function of n; let $d(U, V)$ be the maximum of the degrees of the polynomials involved. Now let $d(U)$ be the maximum of all the $d(U, V)$ as V ranges over all FG-modules; it exists as all the $d(U, V)$ are at most $e - 1$. We define the complexity $c(U)$ of U to be $d(U) + 1$.

In particular $c(U)$ is zero if, and only if, for any module V there is an integer n_0 such that $\mathrm{Ext}_{FG}^n(U, V)$ is zero for all $n \geq n_0$. But we can choose n_0 independent of V. In fact, let $S_1, \ldots, S_r$ be irreducible FG-modules, one of each isomorphism type of such modules, and choose n_0 such that $\mathrm{Ext}_{FG}^n(U, S_i)$ is zero for all i and all $n \geq n_0$. The long exact sequence for the functors Ext^n now shows that this bound n_0 works for all modules V. Therefore, U is projective. Similarly, U is bounded if, and only if, $c(U) \leq 1$, as claimed above.

This invariant, the complexity, may be of further interest. It seems to have some interesting properties. For example, the complexity of related modules—in the sense defined above—is the same. We hope to return to this topic later.

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