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STABILITY OF GAUSS MAPS

BY

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1. Introduction

In this article we prove that certain geometrical properties of immersions of differentiable surfaces M (without prescribed metric) into Euclidean space $\mathbf{E}^3$ are generic. Our results concern properties related to the set of *parabolic* points (points where the Gaussian curvature, of the metric induced by the immersion, vanishes). Observe that all the interesting properties (in the differentiable category) of the Gauss normal map occur in a neighborhood of this set. In some sense our results form an extension of a small, but significant, part of Feldman's (et. al.) work on geometric transversality (see for example [3]–[7], [12], [13]). Although we could use much of the machinery and many of the constructions in these works, there seems to be little advantage to doing so in the low-dimensional case at hand. In fact, we discovered the results before reading these papers. We are however indebted for the idea of applying transversality theory to questions of geometrical genericity. Before stating our main result, we introduce some definitions and notation.

Let $C^\infty(M, N)$ be the set of all C^∞ maps from a compact manifold M to an arbitrary manifold N. We give $C^\infty(M, N)$ the topology of uniform convergence of each k-jet ($k = 0, 1, 2, \ldots$). Given an open subset $S \subset C^\infty(M, N)$, we call a property of maps in S *generic* if the subset of maps in S having that property is open and dense in S. We are mostly concerned with the case where M is a compact, orientable surface, N is $\mathbf{E}^3$, and $S = I(M, \mathbf{E}^3)$ is the set of immersions of M into $\mathbf{E}^3$. We define the following properties (P1), (P2), and (P3) of maps $f \in I(M, \mathbf{E}^3)$.

(P1) The Gaussian curvature K from the metric induced on M by f has the property that K and dK do not vanish simultaneously. Hence the parabolic set ($K = 0$) consists of a finite disjoint collection of smoothly embedded circles, the normal derivative of the Gaussian curvature on these circles is nonzero, and none of these points are extrinsically planar: for $0 \neq dK = d(k_1 k_2) = k_1 dk_2 + k_2 dk_1$, where k_1 and k_2 are the principal curvatures at a parabolic point, which implies that k_1 and k_2 are not both zero.

(P2) Property (P1) holds and the zero principal curvature direction field (corresponding to the principal curvature which is zero) along the parabolic curves is transverse to those curves except at a finite number of points. At these points, the derivative of the angle α of transversality is nonzero as one moves

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through the point along the curve with nonzero velocity (i.e., $\alpha(t)$ and $\alpha'(t)$ have no common zero for t a regular parameter, say arclength, for any of the parabolic curves).

(P3) Properties (P1) and (P2) hold and the normal vector to $f(M)$ in $\mathbf{E}^3$ at any point $f(x)$ for which $\alpha = 0$ does not have the same direction as the normal vector to $f(M)$ at $f(y)$ for any other parabolic point y. Moreover, there are at most finitely many pairs (and no triplets, etc.) of parabolic points (necessarily $\alpha \neq 0$) which have parallel normal vectors. Identifying the tangent planes of each such pair by a translation of $\mathbf{E}^3$, we have that the zero principal curvature directions are transverse.

THEOREM 1.1. *Properties* (P1), (P2), *and* (P3) *are generic.*

It was already known (see [7]) that property (P1) is generic. Our contribution is that properties (P2) and (P3) are generic. An example of an immersed surface having (P1) but not (P2) is the standard torus (or any cyclide of Dupin). In fact, the angle α is zero everywhere along the parabolic curves. However, by introducing suitable corrugations with troughs running along the vertical circles of the torus (which has its hole facing upward), we can arrange that α is nowhere zero on the many parabolic curves which are nearly vertical circles each lying between consecutive maximum and minimum circles of the corrugation. Another example of a (P2) immersion is the surface of a banana or cashew with the negatively curved part facing up so that the parabolic curve is the boundary of the saddle of negative curvature. In this case we have two isolated points where $\alpha = 0$, namely the low points of the boundary of the saddle.

In a sense (to be made precise in Section 3) the points in (P2) where $\alpha = 0$ cannot be eliminated by an arbitrarily small perturbation of the immersion. These exceptional points have interesting properties. If C denotes a parabolic curve on M, then these points are precisely where the osculating plane of $f(C)$ (as a space curve) either fails to exist or coincides with the tangent plane of M along C. If $G\colon M \to S^2$ is the Gauss map of a (P2)-immersion, then $G(C)$ is a smooth curve in S^2 except for a finite number of cusps which look like $u^2 = v^3$ in suitable coordinates on S^2. These cusps are precisely the images of the exceptional points on C. Even though the cusps are on $G(C)$ instead of C itself, we will call the exceptional points of C *cusps*. This is in conformity with Whitney's notion of cusps of differentiable maps between surfaces [15].

Whitney also uses the terms "good" and "excellent" for maps between surfaces. We give Whitney's original definitions for these terms in Section 3. Here it suffices to say that a map $g\colon M \to N$ between surfaces is good iff, in local coordinates, the Jacobian and the differential of the Jacobian do not vanish simultaneously. The map g is excellent iff it is good and, at each point p, is either nonsingular or, in some pair of local coordinate charts about p and $g(p)$, is given by

$$(x, y) \to (x^2, y) \quad \text{or} \quad (x, y) \to (xy - x^3, y).$$

(In the first case p is called a fold point and in the second p is called a cusp point.) Whitney proved that for maps in $C^\infty(M, N)$ the property of excellence is generic. The map g is *stable* iff there is an open neighborhood U of g in $C^\infty(M, N)$ such that $g_1 \in U$ implies $g_1 = k \circ g \circ h^{-1}$ for diffeomorphisms $h\colon M \to M$ and $k\colon N \to N$. Using this terminology, we establish in Section 3 that, for the Gauss map $G\colon M \to S^2$ of an immersion $f\colon M \to \mathbf{E}^3$, f has (P1) iff G is good, f has (P2) iff G is excellent, and f has (P3) iff G is stable.

The basic tool used in the proofs of genericity of (P1), (P2), and (P3) is the Thom Transversality Theorem. Roughly speaking, we show that for f to have any of these properties the jet extension of f must be transverse to certain submanifolds of an appropriate jet space. A key step in the proof is Proposition 2.2, that the map grad: $C^\infty(\mathbf{E}^2, \mathbf{E}^1) \to C^\infty(\mathbf{E}^2, \mathbf{E}^2)$ given by $F \mapsto (F_x, F_y)$ sends a dense set of functions in $C^\infty(\mathbf{E}^2, \mathbf{E}^1)$ to excellent maps in $C^\infty(\mathbf{E}^2, \mathbf{E}^2)$. This map is essentially the map which sends a function to the Gauss map of its graph followed by a central projection onto the tangent plane of S^2 at the north pole.

There is also a topological consequence of our result. We show in Section 3 that each cusp of a (P2)-immersion can be assigned a value $\pm\frac{1}{2}$. The algebraic sum of the cusps is then $\chi(M^-)$ where $M^- \subset M$ is the subsurface of negative Gaussian curvature.

The reader may proceed to Section 3 where there is additional motivation and also an explicit example of a family of quartic surfaces with cusps.

2. Genericity of stability of Gauss maps

Suppose f is an immersion of a closed, oriented surface M into $\mathbf{E}^3$. The Gauss map $G(f)\colon M \to S^2$ associates to each $x \in M$ the unit outward normal vector to $f(M)$ at $f(x)$.

LEMMA 2.1. *The transformation $G\colon I(M, \mathbf{E}^3) \to C^\infty(M, S^2)$ is continuous.*

Proof. Let $I^1(2, 3)$ denote the space of rank 2 jets in $J^1(2, 3)$ and $I^1(M, \mathbf{E}^3)$ the corresponding subbundle of $J^1(M, \mathbf{E}^3)$ (for a thorough discussion of jet spaces see [9]). Let u_x denote the partial derivative of a function $u\colon \mathbf{E}^2 \to \mathbf{E}$ with respect to a coordinate x in $\mathbf{E}^2$. We can choose as coordinates of $j^1f(0) \in J^1(2, 3)$, where $f = (u, v, w)$, the sextuple

$$(u_x, v_x, w_x, u_y, v_y, w_y)(0).$$

If f is an immersion, then $v = (u_x, v_x, w_x) \times (u_y, v_y, w_y)(0)$ is a nonzero vector normal to the image of f; as v never vanishes, ρ defined by $\rho(j^1f(0)) = v/|v|$ is a C^∞ function on $I^1(2, 3)$. Each coordinate patch on M has an associated coordinate patch on $J^1(M, \mathbf{E}^3)$ on which we can define ρ; restricting ourselves to oriented patches, the local ρ's agree on overlaps yielding a C^∞ map $\rho\colon I^1(M, \mathbf{E}^3) \to S^2$. Then G equals $\rho_* \circ j^1$ restricted to $I(M, \mathbf{E}^3)$; G is continuous by II.3.4 and II.3.5 of [9]. □

For each pair of manifolds M, N and nonincreasing, finite sequence

$$\omega = (i_1, \ldots, i_k)$$

of nonnegative integers there is a fiber subbundle S^ω of $J^k(M, N)$ called a Thom-Boardman singularity (see [1], [8], and [10]). If we let $S^\omega(f) = \{x \in M: j^kf(x) \in S^\omega\}$, then $S^{i_1}(f)$ is the set of points x at which dim (ker $T_x f) = i_1$, $S^{i_1,i_2}(f)$ is the set of points x at which dim (ker $T_x(f\,|\,S^{i_1}(f))) = i_2$, etc. In particular, we have partitions $J^3(\mathbf{E}^2, \mathbf{E}^2) = S^0 \cup S^1 \cup S^2$ (S^0 is short for $\pi^{-1}S^0$, where π: $J^3(\mathbf{E}^2, \mathbf{E}^2) \to J^1(\mathbf{E}^2, \mathbf{E}^2)$ is the natural projection, etc.), $S^1 = S^{1,0} \cup S^{1,1}$, and $S^{1,1} = S^{1,1,0} \cup S^{1,1,1}$. Let 1_k denote $1, 1, \ldots, 1$ k-times; k will always vary from 1 to 3. The codimensions of S^2 and the S^{1_k} are 4 and k respectively. We say $f \in C^\infty(\mathbf{E}^2, \mathbf{E}^2)$ is *good* (respectively *excellent*) if j^1f (respectively j^3f) intersects all Thom-Boardman singularities transversally; actually f is excellent iff j^3f intersects S^2 and the S^{1_k} transversally. If f is excellent, then $S^{1,0}(f)$ is the set of *fold points* and $S^{1,1,0}(f)$ is the set of *cusp points* ($S^{1,1,1}(f) = S^2(f) = \emptyset$). A mapping between surfaces is good (respectively excellent) if its coordinate representatives are.

The Gauss map of the graph of $F \in C^\infty(\mathbf{E}^2, \mathbf{E}^1)$ is a map from $\mathbf{E}^2$ to the unit sphere S^2 and, by choosing the coordinates on S^2 given by radial projection from the center of the sphere onto the tangent plane $z = 1$ followed by 180° rotation in that plane, we represent this Gauss map as grad $F = (F_x, F_y)$.

PROPOSITION 2.2. *For a dense set of* $F \in C^\infty(\mathbf{E}^2, \mathbf{E}^1)$, grad F *is excellent.*

Proof. The transformation $F \to$ grad F induces a map

$$\Gamma: J^4(\mathbf{E}^2, \mathbf{E}^1) \to J^3(\mathbf{E}^2, \mathbf{E}^2).$$

Let $T^\omega = \Gamma^{-1}S^\omega$ for each ω. The Jacobian matrix of grad F is the Hessian matrix of F; thus T^2 is a codimension 3 submanifold of $J^4(\mathbf{E}^2, \mathbf{E}^1)$ and T^0 is open. We will show that Γ intersects the S^{1_k} transversally, whence the T^{1_k} are codimension k submanifolds and $j^4F \cap_t T^{1_k}$ iff $j^3(\text{grad } F) \cap_t S^{1_k}$ (where $\cap_t$ means "intersects transversely"). The proposition will then follow from Thom's Transversality Theorem.

A *cochart* for a codimension p submanifold P of a manifold N is a pair (U, ϕ) with U open in N and ϕ: $U \to \mathbf{E}^p$ a submersion such that $\phi^{-1}(0) = U \cap P$. If $f \in C^\infty(M, N)$ and P is covered by cocharts (U, ϕ) for each of which $\phi \circ f$ is a submersion, then $f \cap_t P$ (see II.4.3 of [9]). We apply this to the case $M = J^4(\mathbf{E}^2, \mathbf{E}^1)$, $N = J^3(\mathbf{E}^2, \mathbf{E}^2)$, $P = S^{1,1,1}$ and $f = \Gamma$, the cases $P = S^1$ and $P = S^{1,1}$ being simpler.

We choose as coordinates of $j^4F(x, y)$ the tuple

$$(x, y, F_x, \ldots, F_{yyyy})$$

and as coordinates of $j^3(u, v)(x, y)$ the tuple

$$(x, y, u, v, u_x, \ldots, v_{yyy}).$$

Let p denote the polynomial $u_x v_y - u_y v_x$ on $J^3(\mathbf{E}^2, \mathbf{E}^2)$. We designate by p_x the polynomial $u_{xx} v_y + u_x v_{xy} - u_{xy} v_x - u_y v_{xx}$. In like manner, we have polynomials p_y, $q = \operatorname{jac}(u, p) = u_x p_y - u_y p_x$, $r = \operatorname{jac}(v, p)$, $s = \operatorname{jac}(u, q)$ and $t = \operatorname{jac}(v, r)$ on $J^3(\mathbf{E}^2, \mathbf{E}^2)$. Gaffney in [8] shows that $S^{1,1,1}$ is contained in

$$(p, q, s)^{-1}(0) \cup (p, r, t)^{-1}(0).$$

Straightforward calculation yields $\partial p/\partial v_y = u_x$, $\partial q/\partial v_{yy} = u_x^2$, $\partial s/\partial v_{yyy} = u_x^3$, and $\partial p/\partial v_{yy} = \partial p/\partial v_{yyy} = \partial q/\partial v_{yyy} = 0$. Thus $(\{u_x \neq 0\}, (p, q, s))$ is a cochart for $S^{1,1,1}$. Similarly,

$$(\{u_y \neq 0\}, (p, q, s)) \quad \text{and} \quad (\{v_x \neq 0 \text{ or } v_y \neq 0\}, (p, r, t))$$

are cocharts. Together these cocharts cover $S^{1,1,1}$.

Note that $p \circ \Gamma$ equals $F_{xx}F_{yy} - F_{xy}^2$, a polynomial on $J^4(\mathbf{E}^2, \mathbf{E}^1)$. If $F_{xx} \neq 0$, then

$$\partial(q \circ \Gamma)/\partial F_{yyy} = F_{xx}^2 \neq 0 \quad \text{and} \quad \partial(s \circ \Gamma)/\partial F_{yyyy} = F_{xx}^3 \neq 0.$$

Since $p \circ \Gamma$ only depends on the 2-jet and $q \circ \Gamma$ on the 3-jet, partials of these with respect to the higher jet variables vanish. Thus $(p, q, s) \circ \Gamma$ is a submersion on $\{F_{xx} \neq 0\}$. Similarly, $(p, r, t) \circ \Gamma$ is a submersion on $\{F_{yy} \neq 0\}$. If $F_{xy} \neq 0$ and $p \circ \Gamma = 0$, then $F_{xx} \neq 0$. Thus $T^{1,1,1}$ is covered by $\{F_{xx} \neq 0\}$ and $\{F_{yy} \neq 0\}$. Thus $\Gamma \pitchfork_t S^{1,1,1}$. □

Remark. The analogous proposition for grad: $C^\infty(\mathbf{E}^m, \mathbf{E}) \to C^\infty(\mathbf{E}^m, \mathbf{E}^m)$ is not true. While $\Gamma \pitchfork_t S^{1^k}$ for all k, Γ is not transverse to S^2 and $\Gamma^{-1}S^2$ has codimension 3; so if $m \geq 3$, we cannot in general avoid nontransverse corank 2 singularities.

THEOREM 2.3. *Let M be a closed, oriented surface. The set of immersions whose Gauss maps are excellent is open and dense in $I(M, \mathbf{E}^3)$.*

Proof. Openness follows from Lemma 2.1. $I^4(M, \mathbf{E}^3)$ is the union of open sets O_1, O_2, and O_3, where

$$O_1 = \{z = j^4(f_1, f_2, f_3)(x) \colon H = (f_2, f_3) \text{ is nonsingular at } x\}$$

and O_2 and O_3 are defined analogously. For each z in O_i define $\pi_i(z)$ to be $j^4(f_i \circ H^{-1})(y)$, where $y = H(x)$. Each

$$\pi_i \colon O_i \to J^4(\mathbf{E}^2, \mathbf{E})$$

is a submersion. The fibers of the submersion π_i are the orbits in O_i under the pseudo-group of local diffeomorphisms on M, i.e., each fiber is the set of jets of immersions having a given image germ. For each ω, $\pi_i^{-1}T^\omega$ and $\pi_j^{-1}T^\omega$ agree in $O_i \cap O_j$ because: if $j^4F(0)$ is in T^ω and if G is obtained by interchanging the ith and jth component functions of F, then $j^4G(0)$ is also in T^ω. Thus, for each ω, we have a submanifold $W^\omega = \bigcup \pi_i^{-1}T^\omega$ whose codimension equals that of T^ω.

Furthermore, $j^4 f \pitchfork_t W^{1_k}$ iff $j^3 G(f) \pitchfork_t S^{1_k}$. Density follows by Thom's Transversality Theorem. □

See [16] or VII.6.3 of [9] for a proof of the following result.

PROPOSITION 2.4. *If M and N are surfaces, M compact, then $g \in C^\infty(M, N)$ is stable iff*

(a) *g is excellent, and*
(b) *the images of fold curves intersect at most pairwise and transversally, whereas images of cusps do not intersect with images of other singular points.*

Let GS denote the set of those $f \in I(M, \mathbf{E}^3)$ whose Gauss map g is stable.

THEOREM 2.5. *Suppose M is a closed, oriented surface. Then GS is open and dense in $I(M, \mathbf{E}^3)$.*

Proof. First we translate (2.4.b) into conditions on jet extensions. Define

$$M^r = M \times \cdots \times M \quad (r \text{ times})$$

and

$$M^{(r)} = \{(x_1, \ldots, x_r) \in M^r : x_i \neq x_j \text{ whenever } i \neq j\}.$$

Let α: $J^k(M, N) \to M$ and β: $J^k(M, N) \to N$ be the canonical projections. Define α^r: $J^k(M, N)^r \to M^r$ in the obvious fashion. Then

$${}_rJ^k(M, N) = (\alpha^r)^{-1}(M^{(r)})$$

is the *r-fold k-jet bundle.* Each $f \in C^\infty(M, N)$ has a jet extension ${}_rj^k$: $M^{(r)} \to {}_rJ^k(M, N)$ defined by

$${}_rj^k(x_1, \ldots, x_r) = (j^k f(x_1), \ldots, j^k f(x_r)).$$

The diagonal of N^r is $\Delta = \{(y, \ldots, y) \in N^r\}$.

Then $A_1 = (\mathbf{E}^2)^{(3)} \times \Delta \times (S^1)^3$ is a submanifold of ${}_3J^3(\mathbf{E}^2, \mathbf{E}^2)$ of codimension 7, and $A_2 = (\mathbf{E}^2)^{(2)} \times \Delta \times S^1 \times S^{1,1}$ and $A_3 = (\mathbf{E}^2)^{(2)} \times \Delta \times (S^1)^2$ are submanifolds of ${}_2J^3(\mathbf{E}^2, \mathbf{E}^2)$ of codimensions 5 and 4 respectively. By Proposition 2.4 and the proof of VI.5.6 of [9], an excellent map f is stable iff ${}_3j^3 g \pitchfork_t A_1$ and ${}_2j^3 g \pitchfork_t A_i$, $i = 2, 3$. As in the proofs of Proposition 2.2 and Theorem 2.3, we can lift the A_i to submanifolds W_i of ${}_rJ^4(M, \mathbf{E}^3)$, $r = 2, 3$, and 3 respectively, such that W_i and A_i have the same codimension and such that ${}_rj^4 f \pitchfork_t W_i$ iff ${}_rj^3 g \pitchfork_t A_i$. Thus, by the Multijet Transversality Theorem and Theorem 2.3, GS is dense in $I(M, \mathbf{E}^3)$. Openness follows from Lemma 2.1 and the openness of the set of all stable maps. □

3. Geometric and topological consequences

First we wish to prove the equivalence of the properties (P1), (P2), and (P3) to the properties of the Gauss map of the immersion being good, excellent, and stable respectively. The definitions of good and excellent given in Section 2 are not well-suited for this demonstration. By exercise 4 following Section 4 in Chapter 6 of [9], these definitions are equivalent to the following original definitions of Whitney [15].

Let g be a C^∞ mapping from an open set $U \subset \mathbf{E}^2$ to $\mathbf{E}^2$ and let J be its Jacobian matrix. If $\det J(p) \neq 0$, then $p \in U$ is said to be a *regular* point of g; otherwise p is a *singular point*. Whitney calls p a *good point* for g if p is regular or if grad $(\det J)(p) \neq 0$. If g is *good*, that is each point of U is good for g, then the singular set $\det J = 0$ is a 1-manifold (by the implicit function theorem). Suppose p is a singular point of a good map g with $\phi(t)$ a regular parametrization of the singular curve through p. Whitney calls p a *fold point* of g if $d(g \circ \phi)/dt \neq 0$ and a *cusp point* of g if $d(g \circ \phi)/dt = 0$ and $d^2(g \circ \phi)/dt^2 \neq 0$ at p. Cusp points are necessarily isolated. A point p is an *excellent point* of a good map g if it is either regular, or else a fold or a cusp point, and g is *excellent* if each point of U is excellent for g.

Whitney's definition of cusp appears different from the definition of "simple cusp" in [9]. The two are seen to be equivalent once it is shown that a point p on a singular curve C of a good map g is a cusp iff the angle α between the tangent to C and the kernel of J is zero at p and has nonzero derivative as one passes through p along C with nonzero velocity. To this end we may select local coordinates about p, say (t, s), such that $\phi(t) = (t, 0)$ is a regular parametrization of the singular curve of g through $p = (0, 0)$. Let (u, v) be local coordinates about $g(p)$, and let $g(t, s) = (u(t, s), v(t, s))$. At each point $(t, 0)$, g has rank 1. So the vectors (u_t, u_s) and (v_t, v_s) are dependent on $(t, 0)$ but not both zero and are perpendicular to the kernel of the derivative of g. Consequently, $\sin \alpha(t, 0)$ can be expressed as at least one of

$$u_t/(u_t^2 + u_s^2)^{1/2} \quad \text{or} \quad v_t/(v_t^2 + v_s^2)^{1/2}.$$

If p is a cusp point, then $(u_t, v_t) = (0, 0)$ at $(0, 0)$ while $(u_{tt}, v_{tt}) \neq (0, 0)$ at $(0, 0)$. Thus

$$\alpha'(0) = (\sin \alpha)'(0) = \partial(u_t/(u_t^2 + u_s^2)^{1/2})/\partial t|_{t=0} = u_{tt}/|u_s| \quad \text{at } (0, 0)$$

or

$$\alpha'(0) = v_{tt}/|v_s| \quad \text{at } (0, 0).$$

One of these is well-defined and nonzero (use the fact that

$$(u_t v_s - u_s v_t)(t, 0) \equiv 0$$

whence $u_{tt} v_s = u_s v_{tt}$ at $(0, 0)$ to dispel the cases $v_{tt} \neq 0$, $v_s = 0$ or $u_{tt} \neq 0$, $u_s = 0$). One can reverse the argument to get the converse.

Since the determinant of the Jacobian of the Gauss map is essentially the Gaussian curvature, it is plain from Whitney's definition that $f: M \to \mathbf{E}^3$ has (P1) iff $G: M \to S^2$ is good. Also from the preceeding paragraph we now have that f has (P2) iff $G: M \to S^2$ is excellent. Note that the singular curves of G are the parabolic curves and the kernels of $J(G)$ make up the zero principal curvature direction line field along the parabolic curves. To show that (P3) is equivalent to the stability of G, we need only show that (P3) is equivalent to (a) and (b) of Proposition 2.4. The derivative of the Gauss map is just the second fundamental form when we identify T_pM with $T_{G(p)}S^2$ via translation. Let p_1 and p_2 be fold points with $G(p_1) = G(p_2)$ and let $\phi_1(t)$ and $\phi_2(t)$ regularly parameterize the fold curves with $\phi_1(0) = p_1$ and $\phi_2(0) = p_2$. Since the matrix Jac $(G)_p$ (considered as second fundamental form) is symmetric for each p, the images of Jac $(G)_{p_1}$ and Jac $(G)_{p_2}$ are transverse iff their kernels (which are perpendicular to their images) are transverse. The kernels are just the zero principal curvature directions. The rest is immediate.

Now we establish the other geometric properties of cusps mentioned in the introduction. Letting T be the unit tangent field along a parabolic curve C of a (P2) immersion, the curvature vector of C (as a space curve) is $T' = \kappa N = k_g t + k_n n$ where k_g is the geodesic curvature of C in M and k_n is the normal curvature of M in the direction of T, n is the unit normal to M and t is the normal to T in M. Since $k_n = 0$ precisely at cusps, it follows that N is tangent to M (i.e., the tangent plane of M and the osculating plane of C coincide) at, and only at, cusps. Note, however, that the osculating plane may not be defined (in the usual way) at cusps where $k_g = 0$. In fact, the map

$$(x, y) \mapsto (x, y, ax^4 + bx^2y + cy^2)$$

has an excellent Gauss map iff $bc \neq 0$ and $b^2 \neq 4ac$ and, in this case, the osculating plane is not defined at the cusp (0, 0, 0) iff $b^2 = 6ac$. Also the parabolic curve is

$$y = (b^2 - 6ac)x^2/bc.$$

The centrally projected image of this curve under the Gauss map is

$$(b^2 - 4ac)[4x^3/c, 3x^2/b],$$

a cusp-like semicubical parabola. The special case when $ax^4 + bx^2y + cy^2$ is of the form $(y - mx^2)(y - x^2)$ is easier to visualize. (Note that the graph is below the plane $z = 0$ between the parabolas $y = mx^2$ and $y = x^2$.)

Recall that we also promised to indicate the sense in which cusps cannot be eliminated by arbitrarily small perturbations of a (P2) immersion. The details of the following type of argument can be supplied with the help of [11]. There is a neighborhood O of a given (P2) immersion f_0 such that $f_1 \in O$ implies that

$$f_t = tf_1 + (1 - t)f_0$$

also has (P2) for $0 \leq t \leq 1$. Let g_t be the Gauss map of f_t, p the projection $M \times I \to I$, and G the map (g, p): $M \times I \to S^2 \times I$, i.e., $G(x, t) = (g_t(x), t)$. Let

$$d(x, t) = \det (\text{Jac } G)(x, t) \quad (= \det (\text{Jac } g_t)(x)).$$

G is transverse to all Thom-Boardman singularities (in fact, it is locally stable by the local version of Theorem V.7.1 of [9]). Its singular set S consists of a fold surface F (the union of the fold curves of the g_t) and a cusp 1-manifold C

(consisting of all cusp points of the g_t). Pick a point $(x, t) \in S$ and let $L = p^{-1}(t)$. Since g_t is good, grad $(d \mid L)$ is nonzero at x, hence the tangent space of S at (x, t) is not contained in L. Thus $p \mid S$ is a submersion. The line $K = \ker$ (Jac G) at (x, t) lies in L. If $(x, t) \in C$, then K is tangent to S, but not to C. Thus L does not contain the tangent space to C at (x, t). Thus $p \mid C$ is a local diffeomorphism. Let T be the unit vector field on I; there is a C^∞ vector field X on $M \times I$ such that $(Dp)X = T$ and such that $X \mid S$ is tangent to S and $X \mid C$ is tangent to C. The flow ϕ_t of X maps $p^{-1}(0)$ onto $p^{-1}(t)$, so induces a diffeomorphism from M to M sending the singular set of g_0 to that of g_t, preserving folds and cusps.

Finally we prove that $\chi(M^-)$ is the algebraic number (defined below) of cusps. If the immersion $f\colon M \to \mathbf{E}^3$ is assumed to have (P2), M^- is bounded by perhaps several smooth closed parabolic curves C. The cusps are the points along C where the zero principal direction field Z is tangent to C. Note that Z extends to a line field (in fact a principal direction line field) throughout M^- since M^- has no umbilics. Also since C is compact without any planar points, Z even extends to a collared neighborhood N of M^-. We let $\bar{Z}$ be this extension of Z to N. The integral curves of $\bar{Z}$ (which are lines of curvature) are tangent to C precisely at the cusps. From (P2), the angle α of transversality of Z to C has a nonzero derivative when $\alpha = 0$ as we move along C with nonzero velocity. This implies that at a cusp point p, the integral curve of $\bar{Z}$ through p locally lies in just one of M^- or $M^+ \cap N$. In these cases p is given the value $-\frac{1}{2}$ or $+\frac{1}{2}$ respectively. Now if we "close" the surface M^- by identifying (separately) each of the connected boundary curves C_i of M^- to a point p_i, M^- becomes a compact (possibly not connected) surface $\hat{M}$. Retaining the line field $\bar{Z}$ (on M^-) in the process, we have a line field $\hat{Z}$ on $\hat{M}$ with singularities at the points p_i. Consulting the figures and theorems in [14, p. 324–332], we see that the index of $\hat{Z}$ at the point p_i is one plus the algebraic number of cusps on the boundary curve C_i. Thus $\chi(M^-) = \chi(\hat{M}) - \#\{p_i\} =$ algebraic number of cusps. Note that our proof establishes the desired formula for each component of M^-. As a corollary, we deduce that the number of cusps on the boundary of each component of M^- must be even since χ is integer-valued. Moreover, we can predict the algebraic number of cusps which will result when an immersion having only (P1) is perturbed slightly so as to have (P2). This is because the Euler characteristic of each component of M^- does not change under a small perturbation of a (P1) immersion (use the (P1) version of the homotopy argument used above for (P2)). We may apply this to the standard torus in $\mathbf{E}^3$, or to a torus which has been sliced in half and then smoothly capped twice to form a surface diffeomorphic to a sphere. In the first case, after perturbing, the algebraic number of cusps will be 0 (e.g., the corrugated torus), while in the second case the algebraic number would be 1 (i.e., at least two cusps, as in the "generic" banana).

4. Lines of further inquiry

The program followed in this paper can also be attempted for m dimensional manifolds M to be immersed in $\mathbf{E}^{m+1}$. The analysis used in this paper will continue to work in the case of the Boardman singularities S^{ω}, where $\omega = (1, \ldots, 1, 0)$; however, it will not work for the higher corank singularities (see the remark after Proposition 2.2). For an immersion f of M into $\mathbf{E}^{m+k}$ one has a Gauss map g from M into $G_{m,k}$, the Grassman manifold of k-planes in $\mathbf{E}^{m+k}$, which has dimension mk. The property of being an immersion (respectively embedding) is generic for $C^{\infty}(M, G_{m,k})$ iff $k \geq 2$ (respectively $k \geq 3$), so our program is not so interesting (if strictly mimicked) for higher codimension immersions.

Now we state some open questions. What are necessary and sufficient conditions for a map from M into S^2 to be the Gauss map of an immersion? For germs of maps, rank 1 is sufficient. Suppose M is the disjoint union of M^+, M^- and C, where M^+ and M^- are open with common boundary C, itself the disjoint union of embedded circles. Suppose there are finitely many given points on C with specified values $\pm\frac{1}{2}$ such that, for each component M^* of M^-, $\chi(M^*)$ equals the sum of the specified values on the boundary of M^*. Is there an immersion with stable Gauss map having C as parabolic set, cusps at the given points with the given values, and M^+ and M^- the regions of positive and negative Gaussian curvature respectively? Such a stable map can be constructed according to Eliasberg [2], but we do not know if it can be chosen to be a Gauss map. One might in addition require the immersion to have given Gaussian curvature functions on M^+ and M^-.

Comment. Since the writing of the above we have learned of a paper, still in the process of revision, of Michael Menn entitled *Generic geometry*. In this paper, he proves our Theorem 2.3, and examines the singularities which occur generically in the higher dimensional codimension 1 case. However his methods are very different and much less elementary than our approach. Moreover, he does not consider the question of stability or geometric interpretation.

Bibliography

1. J. M. Boardmann, *Singularities of differentiable maps*, Publ. Math. l'I.H.E.S., France, vol. 33 (1967), pp. 21–57.
2. J. Eliasberg, *On singularities of folding type*, Math. USSR-Izv., vol. 4 (1970), pp. 1119–1134.
3. E. A. Feldman, *The geometry of immersions. I*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 693–698.
4. ———, *The geometry of immersions. II*, Bull. Amer. Math. Soc., vol. 70 (1964), pp. 600–607.
5. ———, *The geometry of immersions. I*, Trans. Amer. Math. Soc., vol. 120 (1965), pp. 185–224.
6. ———, *Geometry of immersions. II*, Trans. Amer. Math. Soc., vol. 125 (1966), pp. 181–215.
7. ———, *On parabolic and umbilic points of immersed hypersurfaces*, Trans. Amer. Math. Soc., vol. 127 (1967), pp. 1–28.
8. T. Gaffney, *Properties of finitely determined germs*, Thesis, Brandeis Univ., 1975.
9. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York, 1973.
10. J. N. Mather, *On Thom-Boardman singularities*, Proc. Salvador Symposium on Dynamical Systems (1971), Academic Press, New York, 1973.

11. ———, *Stratifications and mappings*, Proc. Salvador Symposium on Dynamical Systems (1971), Academic Press, New York, 1973.
12. W. F. POHL, *Differential geometry of higher order*, Topology, vol. 1 (1962), pp. 169–211.
13. ———, *Connections in differential geometry of higher order*, Applied Mathematics and Statistics Laboratories, Stanford University, Stanford, Calif., 1963.
14. M. SPIVAK, *A comprehensive introduction to differential geometry*, vol. III, Publish or Perish, Inc., Boston, Mass., 1975.
15. H. WHITNEY, *On singularities of mappings of Euclidean spaces, I. Mappings of the plane into the plane*, Ann. of Math., vol. 62 (1955), pp. 374–410.
16. L. WILSON, *Equivalence of stable mappings between two-dimensional manifolds*, Thesis, University of Illinois, Urbana, 1973.

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