ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 2, June 1979

COHOMOLOGY OF FIBER SPACES IS REPRESENTABLE

BY
ROBERT J. PIACENZA

Introduction

In recent years interest in cohomology theories defined on a category of fiber spaces has increased. See for example [5], [17], and [22]. It is the purpose of this paper to show that such theories are representable by a suitable Ω-spectrum.

Sections I, II, and III are devoted to proving a fibered version of E. Brown's representability theorem as formulated in [7] and [13]. In Section IV we give axioms for a cohomology theory over B general enough to include sheaf cohomology, prestack cohomology, and group bundle cohomology. The main difference from other axiom systems as found in [5] or [11] is our weakening of the homotopy axiom. In Section V we define the notions of reduced cohomology theories and Ω-spectra over B. In this section we review Mielke's work on group bundle cohomology and conclude with the representability theorem mentioned before.

The category theory language we use is that of [20]. All spaces considered are of the homotopy type of a C.W. complex or pair and the base space B of the text is assumed to be Hausdorff. We do this since our main result depends on the fundamental theorems of J. H. C. Whitehead. That the various construction in the text do not take us outside the category of C.W.-spaces is a consequence of results found in [19].

Proofs have been omitted (for example 2.3) where they are just fibered versions of standard homotopy theory arguments.

Finally I wish to mention that as this paper was being prepared for publication I learned of the parallel work of Rolf Schön in [25]. Thus, the main result of Section V are independent achievements of Schön and myself by slightly different methods.

I. Preliminaries

Throughout this paper we let B be a fixed connected space.

We let $\downarrow B$ stand for the category of fiber spaces of B defined as follows. An object α of $\downarrow B$ is a triple $\alpha = (X_\alpha, A_\alpha, p_\alpha)$ where A_α is a closed subspace of X_α and p_α is a map from X_α to B. A morphism $f\colon \alpha \to \gamma$ of B is a map

Received December 20, 1977.

$f: X_\alpha \to X_\gamma$ such that:

(1) $f(A_\alpha) \subseteq A_\gamma$.
(2) $p_\alpha = p_\gamma f$.

Composition of morphisms is defined by functional composition. We also let

$$\alpha' = (X_\alpha, p_\alpha) \quad \text{and} \quad \alpha'' = (A_\alpha, p_\alpha \mid A_\alpha).$$

If $\alpha \in \downarrow B$ and Y is a space we may form the object $\gamma = \alpha \times Y$ by setting $\alpha = (X_\alpha \times Y, A_\alpha \times Y, p_\alpha p_1)$ where p_1 is projection on the first factor. We define a fiber homotopy H to be a morphism $H: \alpha \times I \to \gamma$ where I is the unit interval. This gives an obvious meaning to fiber homotopy of maps and defines an equivalence relation on morphisms of $\downarrow B$ in the usual way. If f is a map we let $[f]$ denote its fiber homotopy equivalence class.

There is a full subcategory of $\downarrow B$ called the category of ex-spaces of B. This category is determined by $\alpha \in \downarrow B$ that satisfy:

(1) $\operatorname{card}(p_\alpha^{-1}(b) \cap A_\alpha) = 1$ for each $b \in B$.
(2) The map $s_\alpha: B \to X_\alpha$ defined by $s_\alpha(b) = x$ where x is the unique element determined by (1) is continuous.

The category of ex-spaces of B is denoted $\updownarrow B$. Clearly $p_\alpha s_\alpha = 1_B$ for each $\alpha \in \updownarrow B$ and we call the triple $\alpha = (X_\alpha, p_\alpha, s_\alpha)$ an ex-space of B following James in [16].

It is clear that many of the usual notions of homotopy theory of pairs of spaces or pointed spaces make sense in $\downarrow B$ or $\updownarrow B$. We shall adopt the language of ordinary homotopy theory mutatis mutandis for $\downarrow B$ and $\updownarrow B$. However, to avoid confusion we prefix by "ex-" the word or phrase denoting a construction to be performed in $\updownarrow B$. For example we use ex-coproduct to denote the coproduct construction in $\updownarrow B$. Furthermore, we use the word "ordinary" to indicate that the notion lives in the category of pairs of spaces. For example, "ordinary cofibration" denotes an inclusion having the homotopy extension property and "cofibration" denotes an inclusion in $\downarrow B$ having the fiber homotopy extension property.

Finally, we let $C\downarrow B$ ($C\updownarrow B$) be the full subcategory of $\downarrow B$ ($\updownarrow B$) determined by those α with $\alpha'' \subseteq \alpha'$ a cofibration. The usual arguments give us the following.

1.1. PROPOSITION. *The category $C\downarrow B$ is closed under the formation of*

(1) *arbitrary coproducts,*
(2) *mapping cylinders,*
(3) *mapping torus,*
(4) *finite products,*
(5) *infinite telescope* (*see* [27])

Clearly 1.1 holds for the category $\updownarrow B$.

II. Pair fibrations

An object $\alpha \in \downarrow B$ is a pair fibration if p_α has the covering homotopy property for pairs of spaces. We let $F\downarrow B$ be the full subcategory of $\downarrow B$ determined by the pair fibrations and we set $FC\downarrow B = F\downarrow B \cap C\downarrow B$. An object of $FC\downarrow B$ is called a fiber-cofibration following Heller in [14]. We shall adopt a similar notation for the ex-space categories, i.e., we have the category $FC\updownarrow B$.

The usual properties of fibrations hold for pair fibrations. See [10] and [12]. In particular we have the following.

2.1. Proposition. *Let $f: \alpha \to \gamma$ be a map in $F\downarrow B$. The following are equivalent:*

(1) *f is an ordinary homotopy equivalence.*
(2) *f is a fiber homotopy equivalence.*
(3) *f restricted to each fiber of α is an ordinary homotopy equivalence.*

2.2. Proposition. *Let $\alpha \in FC\downarrow B$ and $f: \alpha'' \to \gamma = (X_\gamma, p_\gamma)$ be a map. If*

$$\begin{array}{ccc} \alpha'' & \xrightarrow{i} & \alpha' \\ {\scriptstyle f}\downarrow & & \downarrow \\ \gamma & \xrightarrow[g]{} & \delta \end{array}$$

is a pushout then g is a fiber cofibration.

Proof. See [14]. ■

Using 1.1 and 2.2 we may verify:

2.3. Proposition. *The category $FC\downarrow B$ is closed under the formation of.*

(1) *arbitrary coproducts,*
(2) *mapping cylinders,*
(3) *mapping torus,*
(4) *finite products,*
(5) *infinite telescopes.*

Clearly 2.3 holds for the category $FC\updownarrow B$.

The next proposition is part of the folklore of homotopy theory but we have supplied the proof since it is not available in the literature to my knowledge.

2.4. Lemma. *If $\alpha \in F\downarrow B$ and $\bar{H}$ is a lifting of H*

$$\begin{array}{ccc} (X, A) & \xrightarrow{f} & (X_\alpha, A_\alpha) \\ {\scriptstyle i}\downarrow & \overset{\bar{H}}{\nearrow} & \downarrow{\scriptstyle p_\alpha} \\ (X, A)\times I & \xrightarrow[H]{} & (B, B) \end{array}$$

then $\bar{H}$ is unique up to fiber homotopy over H.

Proof. Just as in [26 Corollary 11, page 101]. ■

2.5. PROPOSITION. *If $f: \alpha \to \beta$ is a closed inclusion in $\downarrow B$ and ordinary deformation retract then $\gamma \in F\downarrow B$ implies $f^*[\beta, \gamma] \to [\alpha, \gamma]$ is a bijection where $[\ ,\ ]$ denotes fiber homotopy classes of maps.*

Proof. By hypothesis there is an ordinary retraction $g: (X_\beta, A_\beta) \to (X_\alpha, A_\alpha)$ and an ordinary homotopy $S: fg \simeq 1_{(X_\alpha, A_\alpha)}$ with $S_0 = fg$, $S_1 = 1_{X_\beta}$ and $gf = 1_{(X_\alpha, A_\alpha)}$. We construct the proof in three steps:

Step (1). Define an ordinary homotopy $H: (X_\beta, A_\beta) \times I \to (B, B)$ by $H = p_\beta S$. We then have $H_0 = p_\alpha g$ and $H_1 = p_\beta$. For $h: \alpha \to \gamma$ define $\hat{h}: \beta \to \gamma$ by setting $\hat{h} = \bar{H}_1$ where $\bar{H}$ is a lifting of H as shown below:

$$\begin{array}{ccc} (X_\beta, A_\beta) \times 0 & \xrightarrow{hg} & (X_\gamma, A_\gamma) \\ \downarrow & \overset{\bar{H}}{\nearrow} & \downarrow p_\gamma \\ (X_\beta, A_\beta) \times I & \xrightarrow[H]{} & (B, B). \end{array}$$

Now by 2.4 $\hat{h}$ is unique up to fiber homotopy.

Step (2). Suppose $h_1, h_2: \alpha \to \gamma$ and T is a fiber homotopy with $T_0 = h_1$ and $T_1 = h_2$. Define ordinary homotopies

$$R: (X_\beta, A_\beta) \times I \to (X_\gamma, A_\gamma) \quad \text{and} \quad H': (X_\beta, A_\beta) \times I \times I \to (B, B)$$

by $R(x, t) = T(g(x), t)$ and $H'(x, t, t') = H(x, t')$ thus $H'(x, t, 0) = H(x, 0) = p_\alpha g(x) = p_\gamma R(x, t)$. Let D' be a lifting of H' as shown below:

$$\begin{array}{ccc} (X_\beta, A_\beta) \times I \times 0 & \xrightarrow{R} & (X_\gamma, A_\gamma) \\ \downarrow & \overset{D'}{\nearrow} & \downarrow p_\gamma \\ (X_\beta, A_\beta) \times I \times I & \xrightarrow[H']{} & (B, B) \end{array}$$

Defining $D(x, t) = D'(x, t, 1)$ we have $D(x, 0) = D'(x, 0, 1) = \hat{h}_1$, $D(x, 1) = D'(x, 1, 1) = \hat{h}_2$ and $p_\gamma D(x, t) = p_\gamma D'(x, t, 1) = H(x, 1) = p_\gamma$, i.e., D is a fiber homotopy from $\hat{h}_1$ to $\hat{h}_2$.

Step (3). Suppose $\bar{h}: \beta \to \gamma$ and let $h = \bar{h}f$. Define an ordinary homotopy

$$T: (X_\beta, A_\beta) \times I \to (X_\gamma, A_\gamma)$$

by $T(x, t) = \bar{h}S(x, t)$. Then T is a lifting of H since

$$p_\gamma T(x, t) = p_\gamma \bar{h} S(x, t) = p_\beta S(x, t) = H(x, t) \quad \text{and} \quad T(x, 0) = \bar{H}fg(x) = hg(x).$$

Now $T(x, 1) = \bar{h}$ and thus, by step (1), $\bar{h}$ is fiber homotopic to $\hat{h}$. The above steps have accomplished the construction of a set function $f^{-1}: [\alpha, \gamma] \to [\beta, \gamma]$ which is inverse to f^*. ∎

2.6. COROLLARY. *If $f: \alpha \to \beta$ is an ordinary homotopy equivalence in $\downarrow B$ and $\gamma \in F\downarrow B$ then $f^*: [\beta, \gamma] \to [\alpha, \gamma]$ is a bijection.*

If $\alpha \in \downarrow B$ we let $\tilde{\alpha} \in F\downarrow B$ be the path fibration of α as described in [26]. Clearly $\sim$ is a functor $\downarrow B \to F\downarrow B$ and there is a natural transformation $t_\alpha: \alpha \to \tilde{\alpha}$ which is an ordinary homotopy equivalence.

2.7. PROPOSITION. *If $\alpha \in \downarrow B$ and $A_\alpha \subseteq X_\alpha$ is an ordinary cofibration then $\tilde{\alpha} \in FC\downarrow B$.*

Proof. By Lemma 3.2 of [3], $A_{\tilde{\alpha}} \subseteq X_{\tilde{\alpha}}$ is an ordinary cofibration and thus by Lemma 2.2 of [4], $\alpha \in FC\downarrow B$.

If α is an ex-space with $s_\alpha(B) \subseteq X_\alpha$ an ordinary cofibration we let $\bar{\alpha}$ be the ex-space formed by collapsing $A_{\tilde{\alpha}}$ to a section, i.e., we have a quotient map $q: \tilde{\alpha} \to \bar{\alpha}$. From our previous results it follows that $\alpha \in FC\updownarrow B$ and $qt_\alpha: \alpha \to \bar{\alpha}$ is an ordinary homotopy equivalence.

Let $b \in B$ be a fixed point in B and define $\gamma_n = (S^n, *, p_n)$ where $(S^n, *)$ is the pointed n-sphere and $p_n(S^n) = \{b\}$. We let σ_n be the ex-space formed from $\tilde{\gamma}_n$ by collapsing $A_{\tilde{\gamma}_n}$ to a section. By our previous results $\sigma_n \in FC\updownarrow B$. ■

2.8. PROPOSITION. *Let $f: \alpha \to \beta$ be a map in $F\updownarrow B$ and suppose α and β have connected fibers. If $f_*: [\sigma_n, \alpha] \to [\sigma_n, \beta]$ is a bijection for all $n > 0$ then f is a fiber homotopy equivalence.*

Proof. We first observe that if $\tau \in F \updownarrow B$ then, by 2.6,

$$\pi_n(\tau^b) = [\gamma_n, \tau] \simeq [\tilde{\gamma}_n, \tau] \simeq [\sigma_n, \tau]$$

where τ^b is the pointed fiber of τ over b. The result now follows by the Whitehead theorem (in view of our blanket hypothesis) and 2.1. ■

As we see from the above result the σ_n play the role of n-spheres in the category $F\updownarrow B$.

III. Ex-spaces and homotopy functors

We let $\mathscr{C}$ be the fiber homotopy category determined by those $\alpha \in FC\updownarrow B$ with connected fibers and $\mathscr{C}_0$ the full subcategory of $\mathscr{C}$ determined by the σ_n, $n > 0$.

Referring the reader to [13] for terminology we have:

3.1. PROPOSITION. *The pair $(\mathscr{C}, \mathscr{C}_0)$ is a homotopy category in the sense of Brown.*

Proof. That $\mathscr{C}_0$ is a small Whitehead subcategory is 2.8. The other conditions follow from 2.3. ■

We thus obtain the following theorem as in [13]:

3.2. THEOREM. *Each homotopy functor on $(\mathscr{C}, \mathscr{C}_0)$ is representable.*

We may extend the above result in the following way. A contravariant functor from some category of ex-spaces to sets satisfies the strong homotopy axiom if it sends each ordinary homotopy equivalence to an isomorphism. Let $\mathscr{D}$ be the full subcategory of the fiber homotopy category of $\updownarrow B$ determined by those α for which there is an ordinary homotopy equivalence $f\colon \alpha \to \beta$ with $\beta \in \mathscr{C}$.

3.3. COROLLARY. *If H is a contravariant functor from $\mathscr{D}$ to sets satisfying the strong homotopy axiom and having its restriction to $\mathscr{C}$ isomorphic to a homotopy functor then H is representable on $\mathscr{D}$.*

Proof. This follows easily from 3.2 and 2.6. ■

IV. Cohomology theories

A cohomology theory over B is a graded contravariant functor $K = \{K^q\}$, $q \in Z$, with domain $\downarrow B$ and range the category of abelian groups satisfying the following axioms:

A1. For each $\alpha \in \downarrow B$ there is a functorial long exact sequence of the pair

$$(X_\alpha, p_\alpha)(A_\alpha, p_\alpha \mid A_\alpha).$$

A2. If $f, g\colon \alpha \to \gamma$ are fiber homotopic then $f^* = g^*$.
A3. If $e\colon \alpha \to \gamma$ is an exision then e^* is an isomorphism.
A4. K sends each coproduct to a product (wedge axiom).

We let $T(B)$ be the category of theories over B with natural transformations of theories as morphisms.

A theory satisfies the strong homotopy axiom if it sends each ordinary homotopy equivalence over B to an isomorphism. We let $T'(b)$ denote the full sub-category of $T(B)$ determined by the strong homotopy axiom.

There is a functor $M\colon T(B) \to T(B)$ and natural transformation $l\colon M \to 1_{T(B)}$ defined as follows: $M(K)(\alpha) = K(\tilde{\alpha})$, $M(K)(f) = K(\tilde{f})$ and $l_\alpha = K(t_\alpha)$ where $t_\alpha\colon \alpha \to \tilde{\alpha}$ is the natural inclusion of α into its path fibration. A routine verification yields:

4.1. PROPOSITION. *$T'(B)$ is a coreflective subcategory of $T(B)$ with M the coreflector.* (See [20] for this terminology.)

We mention two examples of theories.

4.2. *Example.* Sheaf cohomology. Fixing a sheaf of abelian groups on B yields a theory over B by pulling back this sheaf and computing cohomology by injective resolutions. That one obtains a theory in this way is a consequence of results found in [6].

4.3. *Example.* Prestack cohomology. The references for this example are [8], [9], and [18]. Let Sin be the singular complex functor. We may view Sin as a functor from $\downarrow B$ to pairs of simplicial sets over $\mathrm{Sin}(B)$ that preserves the fiber homotopy relation. If G is a fixed prestack of abelian groups on $\mathrm{Sin}(B)$ we may define a theory $K \in T(B)$ by setting $K(\alpha; G) = H(\mathrm{Sin}(\alpha); G)$ where the right side is the prestack cohomology defined in the references. That we obtain a theory is a consequence of results found in the references.

We conclude this section with a remark on the results of Dold in [11]. In this paper Dold treats only theories satisfying the strong homotopy axiom and so his results do not apply to theories such as 2.2 and 2.3. However, by using the functor M or what amounts to the same thing restricting the theories to $F\downarrow B$ we obtain analogues of all of Dold's results provided we restrict their scope to $F\downarrow B$. In particular one obtains a weak form of uniqueness for cohomology of local systems provided we modify the dimension axiom to the vanishing of the cohomology of the object $\tilde{b}$ where $b \to B \in \downarrow B$ is the inclusion of a point in B.

V. Reduced theories and spectra

In dealing with reduced theories we find it convenient to restrict our attention to the category $C'\updownarrow B$, i.e. the full subcategory of $\updownarrow B$ determined by $s_\alpha: B \to X_\alpha$ is an ordinary cofibration. A reduced cohomology theory over B is a graded contravariant functor K from $C'\updownarrow B$ to the category of abelian groups satisfying the following axioms:

R1. If f and g are ex-homotopic then $f^* = g^*$.

R2. For each sequence

$$\alpha \xrightarrow{f} \gamma \xrightarrow{i} \mathrm{con}(f)$$

where $\mathrm{con}(f)$ is the ex-mapping cone of f the sequence of groups

$$K(\mathrm{con}(f)) \xrightarrow{f^*} K(\gamma) \xrightarrow{i^*} K(\alpha)$$

is exact.

R3. There is a natural isomorphism for all q and α called suspension:

$$\Sigma^q : K^q(\alpha) \to K^{q+1}(S\alpha)$$

where $S\alpha$ is the fiberwise suspension of α.

R4. K sends each ex-coproduct to a product (wedge axiom).

A reduced theory satisfies the strong homotopy axiom if it sends each ordinary homotopy equivalence to an isomorphism. Given a cohomology theory satisfying the strong homotopy axiom one can construct from it a reduced theory satisfying the strong homotopy axiom. Clearly one can also prove a version of 4.1 for reduced theories. We leave this for the reader.

We now turn to the notion of an Ω-spectrum over B.

An Ω-spectrum $X=\{\chi_n, t_n\}$ in $\updownarrow B$ is a sequence of ex-homotopy equivalences

$$t_n: \chi_n \to L\chi_{n+1}$$

where L is the vertical loop space functor defined on $\updownarrow B$.

An Ω-spectrum X gives rise in the usual manner (see [15]) to a reduced cohomology theory. Furthermore, if the spectrum X has each $\chi_n \in F\updownarrow B$ then the corresponding reduced theory satisfies the strong homotopy axiom. This follows from 2.6.

We now give an important example of a reduced theory.

5.1. *Example.* Group bundle cohomology. The references for this example are [21], [22], and [23]. Let γ be an abelian compactly generated NDR group bundle over B. For example, every numerable locally trivial compactly generated group bundle is such a γ provided the identity of G, the fiber, is a nondegenerate basepoint. This follows from the fact that every numerable LNDR group bundle is NDR. (See [10, example 3.7]). Now by results of Mielke there is an Ω-spectrum $X_\gamma=\{\chi_n, t_n\}$ with each χ_n, $n>0$ a compactly generated abelian group bundle such that χ_n classifies the functor $H^n(\alpha, \overline{p_\alpha^{-1}(\gamma)})$ for $n>0$ where $\overline{p_\alpha^{-1}(\gamma)}$ is the sheaf of germs of continuous sections of the group bundle $p_\alpha^{-1}(\gamma)$. Here we assume $\alpha=(X_\alpha, \phi, p_\alpha)$ is a paracompact K-space. This Ω-spectrum defines a reduced theory that we call group bundle cohomology with coefficients in γ. Furthermore, if γ is locally trivial the χ_n's in the representing spectrum are locally trivial by Mielke's construction and thus the corresponding theory satisfies the strong homotopy axiom.

We note some consequences of Mielke's results.

5.2. Proposition. *If $f: X\to Y$ is an ordinary homotopy equivalence of paracompact K-spaces and $\mathscr{L}$ is a locally trivial sheaf of abelian groups on Y then*

$$f^*: H^n(Y;\mathscr{L})\to H^n(X;f^{-1}(\mathscr{L}))$$

is an isomorphism for all n.

The above result generalizes Theorem 1, page 601 of [24].

5.3. Corollary. *If $f: X\to Y$ is an ordinary homotopy equivalence of compact spaces and*

$$\mathscr{L}=\varinjlim_j \mathscr{L}_j$$

where $\mathscr{L}_j$ is a direct system of locally trivial sheaves on Y then

$$f^*: H^n(Y;\mathscr{L})\to H^n(X;\varinjlim_j f^{-1}(\mathscr{L}_j))=\varinjlim_j H^n(X;f^{-1}(\mathscr{L}_j))$$

is an isomorphism for all n.

Proof. By 5.2, $f_i^*: H^n(Y; \mathscr{L}_j) \to H^n(X; f^{-1}(\mathscr{L}_j))$ is an isomorphism for all j. The result follows from Corollary 14.5 of [6]. ■

We now turn to the main result of this paper.

5.4. THEOREM. *If K is a reduced cohomology theory then there is an Ω-spectrum*

$$X = \{\chi_n, t_n\} \quad \text{with} \quad \chi_n \in FC\updownarrow B \quad \text{for all } n$$

that represents K on $FC\updownarrow B$.

Proof. This follows from 3.2 in the usual way. See [15] or [27]. ■

5.5. COROLLARY. *If K is a reduced cohomology theory satisfying the strong homotopy axiom then there is an Ω-spectrum $X = \{\chi_n, t_n\}$ with $\chi_n \in FC\updownarrow B$ that represents K on $C'\updownarrow B$.*

Proof. By 5.4 there is an Ω-spectrum $X = \{\chi_n, t_n\}$ that represents K on $FC\updownarrow B$ i.e., there are isomorphism, natural in α, $K^n(\alpha) \simeq [\alpha, \chi_n]$ for all n and $\alpha \in FC\updownarrow B$. If $\beta \in C'\updownarrow B$ then by 2.7 and the subsequent comment there is an ordinary homotopy equivalence $qt_\beta: \beta \to \bar{\beta}$ with $\bar{\beta} \in FC\updownarrow B$. That X represents K on $C'\updownarrow B$ now follows from the strong homotopy axiom and 2.6. ■

REFERENCES

1. J. F. ADAMS, *Stable homotopy and generalized homology*, University of Chicago Press, Chicago, Illinois, 1974.
2. ———, *A variant of E. H. Brown's representability theorem*, Topology, vol. 10 (1971), pp. 185–198.
3. J. E. ARNOLD, *Local to global theorems*, Trans. Amer. Math. Soc., vol. 164 (1972), pp. 179–188.
4. ———, *Attaching Hurewicz fibrations with fiber preserving maps*, Pacific J. Math., vol. 46 (1973), pp. 325–355.
5. J. C. BECKER, *Extensions of cohomology theories*, Illinois J. Math., vol. 14 (1970), pp. 551–584.
6. G. BREDON, *Sheaf theory*, McGraw Hill, Hightstown, N.J., 1966.
7. E. H. BROWN, *Abstract homotopy theory*, Trans. Amer. Math. Soc., vol. 119 (1965), pp. 79–85.
8. Y. C. CHEN, *Stacks, costacks and axiomatic homology*, Trans. Amer. Math. Soc., vol. 145 (1969), pp. 105–116.
9. ———, *On Spaniers higher order operations*, Proc. Amer. Math. Soc., vol. 31 (1972), pp. 601–604.
10. A. DOLD, *Local extension properties in topology*, Proceedings of the Advanced Institute on Algebraic topology, vol. I, Aarus Lecture Notes, 1970, pp. 76–87.
11. ———, *Chern classes in general cohomology*, Symposia Mathematics, vol. V (1970), pp. 385–410.
12. M. H. EGGER, *The piecing comparison theorem*, Indag. Math., vol. 76 (1973), pp. 320–330.
13. A. HELLER, *Completions in abstract homotopy theory*, Trans. Amer. Math. Soc., vol. 147 (1970), pp. 573–602.
14. ———, *Abstract homotopy in categories of fibrations*, Illinois J. Math., vol. 16 (1972), pp. 454–474.
15. P. HILTON, *General cohomology theory and K-theory*, London Math. Soc. Lecture Notes, 1968.

16. I. M. JAMES, *Ex-homotopy theory I.* Illinois J. Math., vol. 15 (1971), pp. 324–337.
17. L. LARMORE, *Twisted cohomology theories and the single obstruction to lifting,* Pacific J. Math., vol. 41, 1972, pp. 755–769.
18. D. M. LATCH, *The uniqueness of homology for the category of small categories,* J. Pure Appl. Algebra, vol. 9 (1977), pp. 221–237.
19. A. LUNDELL and S. WEINGRAM, *Topology of C.W. complexes,* Van Nostrand, 1969.
20. S. MACLANE, *Categories for the working mathematician,* Springer Verlag, New York, 1971.
21. M. MIELKE, *The algebraic structure of certain Ω-spectra,* Illinois J. Math., vol. 18 (1974), pp. 602–607.
22. ———, *Cohomology of fiber spaces with group bundle coefficients,* Proc. Amer. Math. Soc., vol. 39 (1973), pp. 629–632.
23. ———, *Cohomology with module bundle coefficients,* Duke Math. J. vol. 40 (1973), pp. 449–454.
24. M. PARKS, *Applications of homotopy in sheaf theory,* Proc. Amer. Math. Soc., vol. 34 (1972), pp. 601–604.
25. R. SCHÖN, *Cohomology theories over B are representable,* to appear.
26. E. SPANIER, *Algebraic topology,* McGraw Hill, Hightstown, N.J., 1966.
27. R. M. SWITZER, *Algebraic topology, homotopy and homology,* Springer Verlag, New York, 1975.

UNIVERSITY OF ALASKA
FAIRBANKS, ALASKA