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REGULARITY OF FINITE H-SPACES

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Abstract

Let X be an H-space of the homotopy type of a connected finite CW complex. Suppose the generators of the rational cohomology of X all have dimension $\leq m$. *Theorem.* If p is a prime satisfying $2p-1 \geq m$, then X is mod p equivalent to a product of odd dimensional spheres and generalized Lens spaces $L(p, 1, \ldots, 1)$ obtained as the orbit space of an action of Z_p on S^{2p-1}.

Introduction

We call an H-space *finite* if it has the homotopy type of a connected finite CW complex. In [6], Serre defined a prime p to be *regular* for a finite H-space X if the rational cohomology generators all have dimension $\leq 2p-1$. Serre proved that at regular primes p, a simply connected compact Lie group is mod p equivalent to a product of odd dimensional spheres. In [5], Kumpel extended this result to simply connected finite H-spaces.[2]

The Lie group $PSU(p)$ with fundamental group Z_p shows that additional factors are needed to obtain a non-simply connected version of the Serre-Kumpel theorem. Abbreviate the generalized Lens space $L(p, 1, \ldots, 1)$, obtained from S^{2p-1} by the usual action of Z_p, by $L(p)$ and its localization at p ($L(p)$ is simple) by L_p. We prove that L_p is an H-space. We then prove the theorem stated in the abstract. The proof contains the simply connected version. A concluding remark concerns the rank of loop spaces with p-torsion in the fundamental group.

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The results

THEOREM 1. *L_p is an H-space.*

Proof. When $p=2$, $L(2)=RP^3$. For the remainder of this proof, p is an

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[2] W. Browder remarked to the author that the restriction in [5] to spaces with p-torsion free homology is unnecessary. But Theorem 4.7 of [1], regularity and simple connectivity yield that $H_*(X; Z)$ is p-torsion free.

odd prime and cohomology is understood with Z_p coefficients. We have

$$H^*(L(p)) \cong \Lambda(x_1) \otimes Z_p[y_2]/y_2^p$$

with subscripts denoting dimension and $\beta x = y$. Note that $\dim L(p) = 2p-1$. We construct an H-space X and a map $L(p) \to X$ which is a mod p cohomology isomorphism in dimensions $\leq 6p-6$. Then, after localization at p, the obstructions to compressing

$$L(p) \times L(p) \to X \times X \to X$$

into $L(p)$ vanish and the result is established.

Construction of X. Let E be the stable two-stage Postnikov system over $K(Z_p, 1)$ with k-invariant the integral class $(\beta\iota_1)^p$,

$$\begin{array}{ccc} K(Z, 2p-1) \to E & & \\ \downarrow & & \\ K(Z_p, 1) & \to & K(Z, 2p). \end{array}$$

By standard methods we have $H^*(E) \cong H^*(L(p)) \otimes S$ where S is the free commutative algebra over Z_p generated by $\mathcal{P}(I)\iota_{2p-1}$ and $\mathcal{P}(I)$ ranges over all admissible monomials in the Steenrod operations $\mathcal{P}^k$, β subject to the conditions, excess $\mathcal{P}(I) < 2p-1$, and $\mathcal{P}(I)$ does not end in β. Thus in low dimensions S appears as

$$S \cong \Lambda[\mathcal{P}^1\iota_{2p-1}, \ldots] \otimes Z_p(\beta\mathcal{P}^1\iota_{2p-1}, \ldots).$$

Hence, for dimensional reasons, the decomposition of $H^*(E)$ above as a tensor product holds as algebras over the Steenrod algebra.

Since $(\beta\iota_1)^p$ is a primitive integral cohomology class, E is an H-space. Construct X as the 3-stage Postnikov system over E with k-invariant $\mathcal{P}^1\iota_{2p-1}$,

$$\begin{array}{ccc} K(Z_p, 4p-4) \to X & & \\ \downarrow & & \\ K(Z, 2p-1) \to E & \xrightarrow{k} & K(Z_p, 4p-3) \end{array}$$

where $k^*(\iota_{4p-3}) = \mathcal{P}^1\iota_{2p-1}$.

Next we check that k is an H-map. Note that the mod p reduction of $(\beta\iota_1)^p = \mathcal{P}^1\beta\iota_1$ which is a loop class. Then applying the results of [4] to the relation $\mathcal{P}^1(\mathcal{P}^1\beta) = 2\mathcal{P}^2\beta$ yields that $\mathcal{P}^1\iota_{2p-1}$ is a primitive mod p cohomology class. It follows that k is an H-map and X is an H-space.

The first class is $\ker k^*$ is $(\beta\mathcal{P}^1 - 2\mathcal{P}^1\beta)\iota_{4p-3}$ in dimension $6p-4$. Hence $H^*(X) \cong H^*(L(p))$ in dimensions $\leq 6p-6$. There are obviously no obstructions to mapping $L(p) \to X$ to realize the cohomology isomorphism. This concludes the construction and the proof of Theorem 1. ■

THEOREM 2. *Let X be a finite H-space with all rational cohomology generators of dimension $\leq 2p-1$. Then X localized at p has the homotopy* *type of a product of p-local odd dimensional spheres of dimension* $\leq 2p-1$ *and* L_p.

Proof. We first observe that $\pi_1 X$ has no higher p-torsion, for if so, then by Corollary 4.2 [2] there exists an element in $PH_2(X)$ with 1-implication. Hence by Theorem 4.6 [1], there exists a rational generator in dimension $\geq 2p^2-1$ a contradiction.

By the Borel structure theorem and Theorem 4.7 of [1] we have $H^*(X)$ decomposed as a tensor product of exterior algebras on odd dimensional generators $\Lambda(w_i)$, $1 \leq n_i = \dim w_i \leq 2p-1$ and algebras

$$\Lambda(x_1) \otimes Z_p[y_2]/y_2^{p^k}.$$

From the Hurewicz isomorphism and the first paragraph, we have $\beta x = y$. The proof of Theorem 4.6 [1] and the regularity hypothesis yield that $k \leq 1$, for otherwise a rational cohomology generator appears via the biprimitive spectral sequence in dimension $\geq 2p^2-1$.

Now define a map $g: X \to E \times K(Z, n_1) \times \cdots \times K(Z, n_i)$, where E is as in the proof of Theorem 1, by setting $g^*(\iota_{n_i}) = w_i$ and lifting the obvious map $X \to K(Z_p, 1)$ to E, taking as many factors E, $K(Z, n_i)$ as there are odd dimensional generators in the cohomology of X. It follows that g can be constructed to be a mod p cohomology isomorphism in dimensions $\leq 2p$, with the first possible element in ker g^* of form $\mathcal{P}^1 \iota_3$ of dimension $2p+1$. Regard g as a fibration.

Now set $W = L(p) \vee S^{n_i} \vee \cdots \vee S^{n_i}$, and define a map

$$h: W \to E \times K(Z, n_1) \times \cdots \times K(Z, n_i)$$

by the obvious maps $n_i: S \to K(Z, n_i)$ and $L(p) \to E$ as in Theorem 1. For dimensional reasons, after localization at p there are no obstructions to lifting h to a map f: $W_p \to X_p$. Since X_p is an H-space, the map f extends to a map

$$L_p \times S_p^{n_1} \times \cdots \times S_p^{n_1} \xrightarrow{\bar{f}} X_p$$

which is an isomorphism of mod p cohomology algebras and hence a homotopy equivalence, concluding the proof. ∎

Remark 1. Theorem 2 can be used to prove Theorem 1. The fact that L_p is an H-space is not required in the proof of Theorem 2. Moreover Theorem 2 can be applied to the Lie group $PSU(p)$. The resulting decomposition yields a factor L_p. Thus L_p is an H-space by virtue of being a factor of an H-space.

Remark 2. The example $PSU(p)$ motivating this study has rank $p-1$. Using Clark's results [3] it is elementary to prove:

THEOREM 3. *Let X be finite H-space of the homotopy type of a loop space, $\tilde{X}$ its universal covering space.*

(a) *If p is a prime dividing the order of $\pi_1 X$ then rank $\tilde{X} \geq p-1$.*
(b) *If in addition rank $\tilde{X} = p-1$, then $\tilde{X}$ is rationally equivalent to $SU(p)$.*

We leave the proof to the reader.

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