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FUNCTIONS OF UNIT MODULUS ON BOUNDARY PORTIONS OF DOMAINS WITH A CERTAIN CIRCULAR SYMMETRY

BY
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1. Introduction

Let Δ^N denote the unit ball in the space C^N of N complex variables, and consider functions f holomorphic in Δ^N. When $N=1$, the function $\log|f|$ can be prescribed almost arbitrarily on the boundary $\partial\Delta^N$. When $N>1$, however, the behavior of $|f|$ on smaller subsets of $\partial\Delta^N$ tends to be enough to determine f completely. For instance, if $|f|=1$ on an open subset of $\partial\Delta^N$ then f is constant. Recently Forelli ([2], Theorem 1.5) has shown that if f_1 and f_2 are holomorphic in Δ^N and continuous in the closure, with $|f_1|=|f_2|$ on an open subset of $\partial\Delta^N$, then in fact f_1/f_2 reduces to a constant.

In the present paper we will find that there are subsets $U\subset\partial\Delta^N$ which are topologically thinner than open sets, such that f is completely determined by the non-tangential limits of $|f|$ on U, under certain growth restrictions on f; we obtain a result which overlaps Forelli's but does not contain it. This is a consequence of Theorem B, stated in Section 2. Our Theorem C contains a result of Rudin (unpublished, cited in [2]) which states that if f is any non-constant inner function of $\Delta^N (N>1)$ then the cluster set of f at every boundary point of $\partial\Delta^N$ consists of the full unit disc.

The results of this paper concern not only Δ^N, but a rather wide class of domains containing Δ^N; the *slice domains* defined near the end of this introductory section.

In the remainder of this section we set out the notation and definitions to be used throughout. In Section 2 we state the main theorems, and discuss

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them in a rather informal way, placing them in context and drawing some simple inferences. Section 3 is devoted to technical lemmas concerning holomorphic continuability, and may be of independent interest. The main theorems are proved in the final three sections.

The dimension N of our complex space is fixed throughout. For reasons which will presently be clear, we write points of C^N in the form (z, w), where $z \in C^1$ and $w = (w_2, \ldots, w_N) \in C^{N-1}$. On a few occasions we find it convenient to use vector notations $\mathbf{p}$, $\mathbf{q}$, etc. for points of C^N. As is customary, subscripts denote coordinates of w and superscripts are used to denote a fixed point in w-space.

We reserve the symbol $B(w^0, c)$ to denote the open ball in w-space with center w^0 and radius c. Another special notation we find convenient is the following: if A is any subset of the real interval $[0, 2\pi]$, then

$$e^{iA} = \exp\{iA\} = \{z \colon z = e^{i\theta},\ \theta \in A\}.$$

Finally, Δ^N has the meaning above; the open unit ball of C^N.

In addition, we use the following standard set-theoretic notations. If $A \subset C^M$, then ∂A is the boundary of A in C^M (which is thus a set of real dimension $2M-1$ in general) and $\operatorname{cl} A$ is the closure of A in C^M. The dimension of the space in which A "lives" will always be clear from the context. $A \times B$ is the usual Cartesian product. When $A \subset C^1$ and c is a positive real number, then cA is A expanded by the scale factor c, namely $cA = \{z \colon z/c \in A\}$.

We will say $f(z, w)$ is *holomorphic on the set* A if f is single-valued and holomorphic on some open subset of C^N containing A, and z-*analytic on* A means that when $(z, w) \in A$, f is an analytic function of z for each fixed w. If f is defined in $D \subset C^N$ and $\boldsymbol{p} \in \operatorname{cl} D$, then $C_D(f, \boldsymbol{p})$ is the full cluster set of f at p, as defined on [1, p. 1].

DEFINITION 1.1. We call $D \subset C^N$ a slice domain if it is of the form

$$D = \{(z, w) \colon |z| < R(w),\ w \in \tilde{D}\} \tag{1.1}$$

with $\tilde{D}$ some domain in C^{N-1} and $R(w)$ is continuously differentiable with respect to real coordinates and bounded away from zero on the compact subsets of $\tilde{D}$.

For example, Δ^N is a slice domain, with $R(w) = \sqrt{1 - \|w\|^2}$, and $\tilde{D}$ the open unit ball of C^{N-1}.

The function $R(w)$ is defined on $\tilde{D}$, and the points $(R(w)e^{i\theta}, w)$ comprise all of ∂D except for the negligible set where $w \in \partial \tilde{D}$. If f is defined in the slice domain D, we introduce the following special limit at $(R(w)e^{i\theta}, w) \subset \partial D$;

$$L_f(\theta, w) = \lim_{t \to 1-0} \log |f(tR(w)e^{i\theta}, w)| \tag{1.2}$$

provided this limit exists in the extended real numbers.

DEFINITION 1.2. For slice domains D, we define function classes as follows:

(i) $f \in \mathfrak{J}(D)$ if f is a non-constant holomorphic function in D, $f| \leq 1$ in D, and $L_f(\theta, w) = 0$ almost everywhere on $[0, 2\pi] \times \tilde{D}$.
(ii) $f \in \mathfrak{S}(D)$ if $f \in \mathfrak{J}(D)$ and has no zeros in D.

DEFINITION 1.3. A subset S of the w-space $C^N - 1$ will be called a *determining set relative to the ball* $B(w^0, \delta)$ if S is dense in $B(w^0, \delta)$, and S meets $\partial B(w^0, \delta_n)$ in a set of positive measure for some sequence $\delta_n \to 0$.

We conclude this section by pointing out a simple fact concerning the special limit L_f.

LEMMA 1.1. *If* $\log|f|$ *has a non-tangential limit at*

$$(R(w)e^{i\theta}, w), \quad w \notin \partial\tilde{D},$$

then this non-tangential limit is equal to $L_f(\theta, w)$.

Proof. Put $\mathbf{p} = (R(w)e^{i\theta}, w)$. The normal to ∂D at $\mathbf{p}$ is the gradient there of the function

$$x^2 + y^2 - R^2(u_2, v_2, u_3, v_3, \ldots, u_N, v_N)$$

($z = x + iy$, $w_j = u_j + iv_j$), and since $R \neq 0$ on $\tilde{D}$ this gradient is co-directional with

$$\mathbf{N} = \left(\cos\theta, \sin\theta, -\frac{\partial R}{\partial u_2}, -\frac{\partial R}{\partial v_2}, \ldots, -\frac{\partial R}{\partial u_N}, -\frac{\partial R}{\partial v_N}\right).$$

The path $(tR(w)e^{i\theta}, w)$, $0 \leq t < 1$, has tangent vector

$$\mathbf{T} = (R(w)\cos\theta, R(w)\sin\theta, 0, \ldots, 0).$$

We see that $\mathbf{T}$ and $\mathbf{N}$ are not orthogonal.

2. Statement and discussion of the main theorems

THEOREM A. *Let D be the slice domain* (1.1), *and let Ω be an open ball about some point of ∂D. Suppose f is holomorphic in $D \cap \Omega$, continuous in* $\mathrm{cl}\, D \cap \Omega$, *and real valued on $\partial D \cap \Omega$.*
Then either f is constant, or $\log R(w)$ is pluriharmonic in some open set. ■

We remark that Theorem A generalizes a result which is well-known (and alluded to above) for Δ^N to all slice domains.

THEOREM B. *Let $f = f_1/f_2$, where f_1 and f_2 are holomorphic in the slice domain* (1.1). *Suppose:*

(i) *f_1 and f_2 are free of zeros in a set*

$$\{(z, w) \colon z = R(w)E, w \in B(w^0, \delta)\}$$

with E *a simply connected subdomain of the unit disc such that* $\mathrm{cl}\, E$ *contains the arc* e^{iI}, I *an interval;*

(ii) $L_f(\theta, w)=0$, $(\theta, w)\in I\times S$, *where* $S\subset \tilde{D}$ *is a determining set relative to* $B(w^0, \delta)$;

(iii) *there exists some finite valued function* $\rho(w)$ *on* $\tilde{D}$ *such that*

$$\sup_{|\zeta|<1} |f_1(\zeta R(w), w)| + \sup_{|\zeta|<1} |f_2(\zeta R(w), w)| < \rho(w), \quad w\in \tilde{D}.$$

Then either f *is constant, or* $\log R(w)$ *is pluriharmonic on some open set (which as a matter of fact can be taken to lie in* $B(w^0, \delta)$*).*

THEOREM C. *Let* $f\in\mathfrak{F}(D)$, D *the slice domain* (1.1). *Then either* $C_D(f, \mathbf{p})$ *is the full unit disc for every* $\mathbf{p}\in\partial D$, *or* $\log R(w)$ *is pluriharmonic on some open set.*

Let us see what Theorem B tells us when $D=\Delta^N$. A simple computation shows that the corresponding function $\log R(w)$ is nowhere even separately harmonic in the coordinates w_j, thus our conclusion is that f_1/f_2 is constant if $|f_1(z, w)|=|f_2(z, w)|$ on $e^{iI}\times S$ in the sense of non-tangential limits. We compare this with the result of Forelli [2, Theorem 1.5] alluded to in our introduction. Forelli does not place any restrictions on the zeros of f_1 and f_2. Our conclusion though is stronger than the Forelli result in two directions; we do not require f_1 and f_2 to be continuous in $\mathrm{cl}\,\Delta^N$ and, more significantly we feel, $e^{iI}\times S$ can be topologically much thinner than the open subsets of $\partial\Delta^N$.

COROLLARY 2.1. *Let* g_1, g_2 *be holomorphic with bounded real parts in the slice domain* D. *Suppose that, in the sense of non-tangential limits,*

$$\mathrm{Re}\, g_1(R(w)e^{i\theta}, w) = \mathrm{Re}\, g_2(R(w)e^{i\theta}, w), \quad (\theta, w)\in I\times S,$$

where I *is an interval and* S *a determining set.*

Then either $g_1(z, w)=g_2(z, w)+ic$, c *a real constant, or* $\log R(w)$ *is pluriharmonic in some open set.*

Proof. Let $f_n=\exp g_n$, $n=1, 2$. Then f_1 and f_2 satisfy the conditions of Theorem B, condition (i) being met vacuously. ■

We say f is an *inner function* for the domain Ω if f is holomorphic in Ω, $|f|\le 1$ in Ω, and f has a radial limit of unit modulus almost everywhere on $\partial\Omega$. (We note that if Ω is a polydisc it is customary to use the term "inner function" with a different meaning.) The existence of non-constant inner functions, even for the ball, is still open. A recent result of Rudin (unpublished, cited in [2]) is that if f is a non-constant inner function for Δ^N then the cluster set at every boundary point contains the unit disc. In view of Lemma 2.1 below, Theorem C contains Rudin's result, and generalizes it to any slice domain for which the notion of radial limit makes sense; furthermore radial limits can be replaced by any kind of non-tangential limit.

LEMMA 2.1. *Let f be a non-constant holomorphic function in the slice domain D, with $|f| \leq 1$ in D. Let $\mathbf{p} = (R(w)e^{i\theta}, w)$, $w \in \tilde{D}$. Suppose*

$$\lim_{(z,w)\to\mathbf{p}} f(z, w) = e^{i\tau}, \quad \tau \text{ real},$$

along some non-tangential path Γ out to $\mathbf{p}$. Then $L_f(\theta, w) = 0$.

Proof. Because of Lemma 1.1, it suffices to show $f(z, w) \to e^{i\tau}$ uniformly in any Stolz cone with vertex at $\mathbf{p}$.

Let $u = \operatorname{Re}(1 - e^{-i\tau} f)$. Then u is harmonic in D, and the maximum modulus property of f shows $u > 0$ in D. Furthermore u tends to zero along Γ.

Choose V and V' to be Stolz cones in D with vertex at $\mathbf{p}$, V' properly including V and V wide enough so that Γ eventually lies in V. Let

$$V_\varepsilon = \{\mathbf{q} : \mathbf{q} \in V, \varepsilon/2 \leq \|\mathbf{p} - \mathbf{q}\| \leq \varepsilon\}$$

and

$$V'_\varepsilon = \{\mathbf{q} : \mathbf{q} \in V', \varepsilon/4 \leq \|\mathbf{p} - \mathbf{q}\| \leq 2\varepsilon\}.$$

By Harnack's Principle [4, p. 263] (note that the method of proof is independent of the number of variables) there exists a constant c determined only by V_ε and V'_ε such that

$$u(\mathbf{q}_2) \leq c u(\mathbf{q}_1), \quad \mathbf{q}_1 \in V_\varepsilon, \mathbf{q}_2 \in V'_\varepsilon. \tag{2.1}$$

Because the geometry is homogeneous, c is actually independent of ε.

Now in (2.1) let $\mathbf{q}_1 = \mathbf{q}_1(\varepsilon)$ be the point which maximizes u on $\Gamma \cap V_\varepsilon$. Thus for any point $\mathbf{q}_2$ in cl V_ε we have $u(\mathbf{q}_2) \leq c u(\mathbf{q}_1(\varepsilon))$, and since $u(\mathbf{q}_1(\varepsilon))$ tends to zero with ε the conclusion follows. ■

We conclude this section by pointing out a couple of generalizations which follow from inspection.

The condition on S (Definition 1.3) is used only in Lemma 3.3, where it is necessary to have S meet $\partial B(w^0, c)$ in a set of positive measure for some c sufficiently small (so that certain sets overlap properly). Thus:

COROLLARY 2.2. *The conclusion of Theorem B holds if the condition that S be a determining set is replaced by the condition that S be dense in $B(w^0, \delta)$ and meet $\partial B(w^0, \delta)$ in a set of positive measure, for some positive δ sufficiently small depending on I, K, w^0 and the function R.*

Finally, our theorems are subject to a kind of "localization". Rather than require that D be a slice domain, our conclusions follow if only D contains a set of the form

$$\{(z, w) : z < R(w)e^{i\theta}, w \in \tilde{D}, 0 \leq \theta \leq 2\pi\}, \tag{2.2}$$

with $\tilde{D}$ any open set, provided in the case of Theorem B that S lies in $\tilde{D}$ or, in the case of Theorem C, that $\mathbf{p}$ lies in the closure of the set (2.2).

3. Technical lemmas

These lemmas are all concerned with holomorphic continuation. Throughout, D, $\tilde{D}$ and the function R are as in Definition 1.1.

LEMMA 3.1. *Put $B = B(w^0, c)$. Let the real-valued functions $\varphi_n(w)$ be plurisubharmonic in* cl B. *Suppose there are numbers α and β, and a subset S of ∂B of positive measure, such that*

$$\varphi_n(w) \leq \alpha, \qquad w \in \text{cl}\, B,$$
$$\lim_{n\to\infty} \varphi_n(w) \leq \beta, \qquad w \in S.$$

Then for any positive ε there exists an open subset of B on which

$$\overline{\lim_{n\to\infty}}\, \varphi_n(w) < \beta + \varepsilon.$$

Proof. A plurisubharmonic function is *subharmonic*, in the usual sense of dominance by harmonic functions, thus

$$\varphi_n(w) \leq \int_{\partial B} P(w, \omega)\varphi_n(\omega)\, d\sigma(\omega) \leq \int_{\partial B} P(w, \omega)v_n(\omega)\, d\sigma(\omega), \quad \omega \in B$$

where $v_n(\omega) = \sup_{m \geq n} \varphi_m(\omega)$, $d\sigma(\omega)$ is normalized Lebesgue measure on ∂B, and P is the Poisson kernal. It follows from Fatou's Lemma that

$$\varphi_n(w) \leq \int_{\partial B} P(w, \omega)v(\omega)\, d\sigma(\omega), \quad w \in B \tag{3.1}$$

where $v = \overline{\lim}\, \phi_n$ $(= \lim v_n)$.

If v is not integrable over ∂B then (3.1) is true with both sides equal $-\infty$, and we are done. Otherwise, almost every point ω^0 of ∂B is a *regular point* for v, meaning a point where the right-hand side of (3.1) tends uniformly to $v(\omega^0)$ in any Stolz cone at ω^0 [8, pp. 197–8]. (In [8] this principle is proved for a half-space, but the proof adapts to the context of a ball in view of an inequality in [9, p. 10].) If we take ω^0 a regular point such that $v(\omega^0) \leq \beta$, we have $v(w) < \beta + \varepsilon$ in an open subset of a Stolz cone at ω^0. ∎

LEMMA 3.2. *Let U be an open subset of $\tilde{D}$, and K a simply-connected plane domain containing an arc of the unit circle. Let*

$$T = \{(z, w) \colon z \in R(w)K, w \in U\}.$$

Then if f is z-analytic in T and holomorphic in $T \cap D$, f is actually holomorphic in T.

Proof. We fix attention on a point $(z^0, w^0) \in T$, and construct a neighborhood of this point in which f is holomorphic.

Because R is continuous, we can find a sufficiently small polydisc P about the origin of C^{N-1}, and a simply-connected relatively compact subdomain

K' of K, such that the set

$$T' = \{(z, w): z \in R(w^0)K', \quad w \in w^0 + P\}$$

will fit inside T and will cover (z^0, w^0). Furthermore, we can arrange that for some point z' common to K' and the open unit disc, the set

$$T'' = \{(z, w): z = R(w^0)z', \quad w \in w^0 + P\}$$

lies in $T \cap D$. We will show that f extends to be holomorphic on T'.

Let $z = \phi(\zeta)$ be the conformal mapping of $|\zeta| < 1$ onto K' such that $\phi(0) = z'$. The biholomorphism $z = R(w^0)\phi(\zeta)$, $w = w^0 + \omega$ transforms $f(z, w)$ into

$$\tilde{f}(\zeta, \omega) = f(R(w^0)\phi(\zeta), \omega + w^0).$$

Then $\tilde{f}$ is ζ-analytic on the N-dimensional polydisc

$$P^N = \{(\zeta, \omega): |\zeta| < 1, \quad \omega \in P\}$$

since f is z-analytic on T', and $\tilde{f}$ is holomorphic on a set of the form

$$\{(\zeta, \omega): |\zeta| < c, \quad \omega \in P\}$$

since f is holomorphic on T''. From a theorem of Rothstein [7, p. 8], $\tilde{f}$ is holomorphic on P^N and hence f is holomorphic on T'. ■

LEMMA 3.3. *Let K be a simply connected plane domain containing an arc e^{iI} of the unit circle. Let S be a determining set* (*cf. Definition* 1.3) *relative to the ball $B(w^0, \delta) \subset \tilde{D}$. Put K_0 for the intersection of K with the open unit disc. Let*

$$T = \{(z, w): z \in R(w)K, \quad w \in S\}.$$

Suppose f is holomorphic on the set

$$\{(z, w): w \in R(w)K_0, \quad w \in B(w^0, \delta)\}$$

and z-analytic on T.

Then f extends holomorphically to a set containing an open patch of ∂D of the form $\{(z, w): z \in R(w)e^{iI}, w \in U\}$ with U an open subset of $B(w^0, \delta)$.

Proof. Let I' be an arbitrary closed subinterval of I. Because of Lemma 3.2, we are done if we can produce K^*, a relatively compact simply-connected subdomain of K containing $\exp\{iI'\}$, and an open subset U of $B(w^0, \delta)$, such that f is z-analytic in the set

$$\{(z, w): z \in R(w)K^*, \quad w \in U\}. \tag{3.2}$$

First, choose K' any relatively compact simply connected subdomain of K which contains $\exp\{iI'\}$. Because $R(w)$ is continuous at w^0, there exists $\eta > 0$ such that when $\|w - w^0\| < \eta$ the plane set $R(w)K'$ meets both $R(w^0)K_0$ and $R(w^0)(K - K_0)$ in non-empty open sets, and is contained in $R(w^0)K$.

Let z' be any point common to $R(w^0)K'$ and $R(w^0)K_0$, and let $z=\varphi(\zeta)$ be the conformal mapping of $|\zeta|<1$ onto $R(w^0)K'$ such that $z'=\varphi(0)$. There exists a number $c<1$ such that the pre-image of $R(w)K'$ covers $|\zeta|<c$ and the image of $|\zeta|<c$ contains an open neighborhood of $\exp\{iI'\}$ (so long as $\|w-w^0\|<\eta$). Consider now the function $F(\zeta, w)=f(\varphi(\zeta), w)$. Choose n so large that $\delta_n<\eta$. Then the function F is holomorphic in a neighborhood of $(0, w^0)$, and is ζ-analytic on the set

$$\{(\zeta, w): |\zeta|<c, w\in S\cap\partial B(w^0, \delta_n)\}.$$

Our task is to show that for any $\varepsilon>0$ there exists an open $U\subset B(w^0, \delta_n)$ such that F is ζ-analytic on

$$\{(\zeta, w): |\zeta|<c-\varepsilon, \quad w\in U\}. \tag{3.3}$$

This will actually complete the proof of the lemma, for if ε is small enough the image of the set (3.3) under the biholomorphism $z=\varphi(\zeta)$, $w=w$, will contain a set of the form (3.2), with K^* meeting the appropriate conditions, and f will be z-analytic on this set.

F has a Taylor expansion about $(0, w^0)$ which, by normal convergence, can locally be arranged to read

$$F(\zeta, w)=\sum_n a_n(w)\zeta^n. \tag{3.4}$$

For fixed w this series converges in any disc $|z|<r$ in which F is ζ-analytic. Let $r(w)$ be the radius of convergence of (3.4). If $w\in S\cap\partial B(w^0, \delta_n)$ then $r(w)\geq c$, and also

$$\log 1/r(w)=\overline{\lim_{n\to\infty}}\,(1/n)\log|a_n(w)|\,.$$

The usual Cauchy estimate for coefficients shows there is a finite upper bound for the functions $(1/n)\log|a_n(w)|$ in $\operatorname{cl} B(w^0, \delta_n)$, and they are plurisubharmonic in $\operatorname{cl} B(w^0, \delta_n)$ (cf. [3, p. 44]). Identifying $(1/n)\log|a_n(w)|$ with the functions $\varphi_n(w)$ of Lemma 3.1 and taking $B=B(w^0, \delta_n)$, we infer that for any $\varepsilon>0$ there is an open subset U_ε of $B(w^0, \delta_n)$ such that $r(w)>c-\varepsilon$, $w\in U_\varepsilon$. Thus F is ζ-analytic on a set (3.3). ■

4. Proof of Theorem A

We may suppose $\partial D\cap\Omega$ is of the form

$$S=\{(z, w): z=R(w)e^{i\theta}, \theta\in I, \quad w\in U\}$$

where I is an interval and U is an open subset of $\tilde{D}$.

By the reflection principle f is z-analytic in Ω, and the continuation across

∂D is given by the formula

$$f(z, w) = \bar{f}(R^2(w)/\bar{z}, w), \quad (z, w) \in \Omega - D. \tag{4.1}$$

By Lemma 3.2, f is holomorphic in Ω. We will find, however, that the right-hand side of (4.1) cannot in general be analytic in the coordinates of w unless R is subject to special conditions.

We use the Wirtinger operators $\partial/\partial w_j$, $\partial/\partial \bar{w}_j$ (cf. [3, p. 1]). The Wirtinger operators are not differentiations in the usual sense, but they satisfy the usual chain rule and it is easily checked that

$$\partial G/\partial \bar{w}_j = \overline{(\partial \bar{G}/\partial w_j)}. \tag{4.2}$$

The condition that f be w_j-analytic is $\partial f/\partial \bar{w}_j = 0$. If $(z, w) \in \Omega - D$ then $|z| > R(w)$, and we can write $\zeta = R^2(w)/\bar{z}$ with $|\zeta| < R(w)$. Using (4.2) to apply $\partial/\partial \bar{w}_j$ to (4.1), we find that the condition for f to be w_j-analytic in $\Omega - D$ is

$$(2\zeta/R(w)) f_\zeta(\zeta, w) \frac{\partial R}{\partial w_j} + f_{w_j}(\zeta, w) = 0, \quad j = 2, \ldots, N, \tag{4.3}$$

where f_ζ and f_{w_j} are partial derivatives with ζ regarded as an independent variable, and $\partial/\partial w_j$ is still the Wirtinger operator, not necessarily a differentiation.

If f_ζ vanishes identically in $\Omega \cap D$, then equations (4.3) show that f is constant. Otherwise (4.3) is equivalent off the zero-set of f_ζ to the equations

$$\partial(\log R)/\partial w_j = -f_{w_j}(\zeta, w)/\zeta f_\zeta(\zeta, w) \tag{4.4}$$

a set of equations valid in some open set Ω'. Now since the right-hand side of (4.4) is holomorphic, it is sent to zero by $\partial/\partial \bar{w}_k$, and also the left-hand side of (4.4) is twice-continuously differentiable on Ω'. We have

$$\partial^2(\log R)/\partial w_j\, \partial \bar{w}_k = 0 \quad \text{for all } j, k,$$

which means $\log R(w)$ is pluriharmonic in the projection of Ω' onto w-space. ■

5. Proof of Theorem B

First, *fix w in S*. Because of hypothesis (iii) and the fact that $f_2(z, w) \neq 0$ for fixed $w \in S$, the function $f(z, w)$ is a *function of bounded type*, as a function of z, relative to the disc $|z| < R(w)$, hence by well known theory (cf. [6, pp. 185 ff.]) has a representation

$$f(z, w) = (B_1(z, w)/B_2(z, w)) \exp \int_0^{2\pi} \frac{R(w)e^{it} + z}{R(w)e^{it} - z}\, d\nu(w, t) \tag{5.1}$$

where, w being fixed, B_1 and B_2 are Blashke products in z relative to the disc $|z|<R(w)$ and $\nu(w, t)$ is a function of bounded variation in t. The zeros of B_1 and B_2 are bounded away from the arc

$$R(w)e^{iI} \tag{5.2}$$

Thus, as is not hard to show, B_1 and B_2 converge uniformly out to (5.2) (for each fixed $w \in S$) to continuous functions of unit modulus. Thus, for each fixed $w \in S$, the function

$$\operatorname{Re} \int_0^{2\pi} \frac{R(w)e^{it}+z}{R(w)e^{it}-z}\, d\nu(w, t)$$

has vanishing radial limit on the interval I. It follows trivially from a uniqueness theorem of Lohwater [5], that $d\nu(w, t)$ vanishes identically on I for each $w \in S$. We infer that for $w \in S$, $f(z, w)$ is z-continuous out to and on the arc (5.2), and being of unit modulus on the arc can be continued analytically across the arc by reflection.

Thus f is z-analytic on a set $T=\{(z, w)\colon z \in R(w)K, w \in S\}$ where K meets the conditions imposed in Lemma 3.3. Because of hypothesis (i), we can apply Lemma 3.3 to infer that f and $\log f$ are holomorphic on an open patch of ∂D of the form

$$T'=\{(z, w)\colon z \in R(w)e^{iI}, \quad w \in U\}$$

with U an open subset of $B(w^0, \delta)$. In particular f and $\log f$ are continuous on the set T'; thus, since S is dense in U and $L_f(\theta, w)=0$ on $I \times S$ we have in fact

$$L_f(\theta, w)=0, \quad (\theta, w) \in I \times U. \tag{5.3}$$

We can now apply Theorem A to the function $i \log f$,, which is z-analytic on T' and, by (5.3), real valued on $T' \cap \partial D$. We conclude that either f is constant or $\log R(w)$ is pluriharmonic on some open subset of U. ■

6. Proof of Theorem C

We show first that it is sufficient to prove Theorem C for the sub-class $\mathfrak{S}(D)$. If $f \in \mathfrak{J}(D)$ we can form a new function $g=\Psi_\theta \circ f$ where

$$\Psi_\theta(\zeta)=\exp\{(e^{i\theta}+\zeta)/(e^{i\theta}-\zeta)\}.$$

The relevant properties of Ψ_θ are well known; Ψ_θ is a singular inner function of the unit disc, and $|\Psi_\theta(\zeta)| \to 1$ uniformly as ζ tends, in any manner, to a point of the circumference other than $e^{i\theta}$. It is easy to check that g is holomorphic in D, non-constant, and $|g|<1$ in D. We have

$$\lim_{t \to 1-0} g(tR(w)e^{i\sigma}, w)=\Psi_\theta\Big(\lim_{t \to 1-0} f(tR(w)e^{i\sigma}, w)\Big)$$

unless

$$\lim_{t\to 1-0} f(tR(w)e^{i\sigma}, w) = e^{i\theta}. \tag{6.1}$$

Therefore $L_g(\sigma, w) = 0$ unless (6.1) is true. Thus $g \in \mathfrak{S}(D)$ unless (6.1) holds on a set of positive measure on $[0, 2\pi] \times \tilde{D}$. This latter possibility, however, can happen for at most countably many θ, which we may assume have been avoided in our choice of Ψ_θ.

Suppose now that $C_D(f, \mathbf{p})$ is not the full disc. Then $C_D(f, \mathbf{p})$ omits some open subset of the disc, and by looking at the (multiple-valued) function Ψ_θ^{-1} we see that $C_D(g, \mathbf{p})$ is not the full disc.

We next prove a simple lemma.

Lemma 6.1. *Let $f \in \mathfrak{S}(D)$, D the slice domain (1.1). Then, for almost every $w \in \tilde{D}$, $f(z, w)$ is a singular inner function of z for the disc $|z| < R(w)$.*

Proof. Lebesgue measure on $I \times \tilde{D}$ can be decomposed as $dm = d\theta\, dw$. By dominated convergence

$$0 = \int_{I\times\tilde{D}} L_f(\theta, w)\, dm = \int_{\tilde{D}} \int_0^{2\pi} L_f(\theta, w)\, d\theta\, dw$$

and since L_f is non-positive we must have

$$\int_0^{2\pi} L_f(\theta, w)\, d\theta = 0 \tag{6.2}$$

for almost all $w \in \tilde{D}$.

Let $V \subset \tilde{D}$ be the set where (6.2) holds. For $w \in V$, f is either a constant function or a singular inner function in z. Unless $f_z \equiv 0$, the zero set of f_z does not have positive measure, so either $f(z, w)$ is a singular inner function of z for almost every $w \in V$, or f is independent of z throughout D. In the latter case, $|f| = 1$ for almost all w in $\tilde{D}$, and as a holomorphic function of w alone f must be constant, contradicting the definition of $\mathfrak{S}(D)$. ■

Now let $\mathbf{p}_0 \in \partial D$. If $C_D(f, \mathbf{p}_0)$ is not the full disc, then we infer by a simple diagonal argument that there is a neighborhood N of $\mathbf{p}_0$ on ∂D such that $C_D(f, \mathbf{p})$ is not the full disc for any $\mathbf{p} \in N$. We may assume

$$N = \{(z, w) : z = R(w)e^{i\theta}, \quad \theta \in I, \quad w \in U\}$$

where I and U are open. Let

$$S = \{w : w \in U, f(z, w) \text{ is a singular inner function of } z \quad \text{for} \quad |z| < R(w)\}.$$

From cluster set theory of functions of one variable [1, p. 95] we find that f is z-analytic across $\exp(iI)$ when $w \in S$, and $|f| \equiv 1$ on that set. Because of Lemma 6.1, S satisfies the hypothesis of Theorem B. Applying Theorem B with $f_1 = f$, $f_2 = 1$, and E the unit disc, we have completed the proof.

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