ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 2, June 1979

THE DIMENSIONS OF PERIODIC MODULES OVER MODULAR GROUP ALGEBRAS

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1. Introduction

Let G be a finite group and let K be a field of characteristic $p>0$. We shall assume that all KG-modules are finitely generated and hence have finite K-dimensions. A KG-module M is periodic if there exists an exact sequence

$$(1.1) \qquad 0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of KG-modules such that $P_0, \ldots, P_{n-1}$ are projective. The period of M is the least length n of any such sequence.

We prove in this paper that if G is an abelian p-group and if M is an indecomposable periodic KG-module, then there is a subgroup H of G such that G/H is cyclic and the restriction of M to a KH-module is free. This implies that the period of M is at most 2. For any finite group G, the dimension of a periodic KG-module is divisible by p^{r-1} where r is the p-rank of G. That is, the maximal elementary abelian p-subgroup of G has order p^r. These results answer some questions raised by Alperin in [1].

The author wishes to thank E. C. Dade for help with the proofs of Theorem 5.3 and Corollary 5.4. Some of the results of this paper, particularly Corollary 5.6, have also been proved, using different techniques, by Eisenbud in [8].

2. Notation and preliminaries

Throughout this paper G denotes a finite group and K is a field of characteristic $p>0$. The radical of KG is denoted $\operatorname{Rad} KG$. If H is a subgroup of G and M is a KG-module, then M_H is the restriction of M to a KH-module. The socle of M, $\operatorname{Soc}(M)$, is the sum of the minimal submodules of M. If G is a p-group, then

$$\operatorname{Soc}(M) = \{m \in M \mid xm = m \quad \text{for all} \quad x \in G\}.$$

Let $\tilde{H} = \sum_{h \in H} h \in KG$, and let $1(G) = K$ denote the trivial one-dimensional KG-module. The symbol $U(KG)$ denotes the group of units in KG.

For any KG-module M there exists a projective module F and an epimorphism $\varphi\colon F \to M$. Let $\Omega(M)$ be the direct sum of the nonprojective

Received October 28, 1977.

components of the kernel of φ. It is well known [9] that the isomorphism class of $\Omega(M)$ is independent of the choice of F and φ. Recall that a KG-module is projective if and only if it is injective (see [6, Theorem (62.3)]). So there exists a monomorphism $\theta\colon M \to F'$ where F' is projective. If $\Omega^{-1}(M)$ is the nonprojective part of the cokernel of θ, then it too is unique up to isomorphism. Inductively we define $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for all $n > 1$. If there exists an integer t such that $\Omega^n(M) \cong \Omega^{n+t}(M)$ for some n, then M is periodic and its period divides t.

The first three lemmas of this paper are well known and are included only for completeness.

LEMMA 2.1. *Let M, N be KG-modules. If $n > 0$, then*

$$\mathrm{Ext}_{KG}^{n+1}(M, N) \cong \mathrm{Ext}_{KG}^{1}(\Omega^n(M), N) \cong \mathrm{Ext}_{KG}^{1}(M, \Omega^{-n}(N)).$$

Proof. There exists an exact sequence

$$0 \to \Omega(M) \xrightarrow{\theta} F \xrightarrow{\psi} M \to 0$$

where F is projective. There is a corresponding long exact sequence [13]

$$\begin{aligned}(2.2)\qquad & 0 \to \mathrm{Hom}_{KG}(M, N) \xrightarrow{\psi^*} \mathrm{Hom}_{KG}(F, N) \xrightarrow{\theta^*} \mathrm{Hom}_{KG}(\Omega(M), N) \\ & \xrightarrow{\partial} \mathrm{Ext}_{KG}^{1}(M, N) \to \mathrm{Ext}_{KG}^{1}(F, N) \to \cdots \to \mathrm{Ext}_{KG}^{n}(F, N) \\ & \to \mathrm{Ext}_{KG}^{n}(\Omega(M), N) \to \mathrm{Ext}_{KG}^{n+1}(M, N) \to \mathrm{Ext}_{KG}^{n+1}(F, N) \to \cdots\end{aligned}$$

Since F is projective $\mathrm{Ext}_{KG}^{t}(F, N) = 0$ for all $t > 0$. Thus

$$\mathrm{Ext}_{KG}^{n+1}(M, N) \cong \mathrm{Ext}_{KG}^{n}(\Omega(M), N).$$

Now continue by induction. The second isomorphism in (2.1) follows from the similar long exact sequence for the second variable of the functor Ext.

Given KG-modules A, B, C there is a standard isomorphism

$$\psi\colon \mathrm{Hom}_{KG}(A \otimes_K B, C) \cong \mathrm{Hom}_{KG}(A, \mathrm{Hom}_K(B, C))$$

which is natural in A. Here $A \otimes_K B$ and $\mathrm{Hom}_K(B, C)$ are G modules by the action

$$g(a \otimes b) = ga \otimes gb, \qquad (gf)(b) = gf(g^{-1}b)$$

for all $a \in A$, $b \in B$, $g \in G$, $f \in \mathrm{Hom}_K(B, C)$. The isomorphism ψ is defined by

$$[(\psi f)(a)](b) = f(a \otimes b)$$

for $f \in \mathrm{Hom}_{KG}(A \otimes_K B, C)$. Using this and the first five terms of (2.2) we get an induced isomorphism

$$\mathrm{Ext}_{KG}^{1}(A \otimes_K B, C) \cong \mathrm{Ext}_{KG}^{1}(A, \mathrm{Hom}_K(B, C)).$$

Let B^* denote the dual module $B^* = \mathrm{Hom}_K(B, K)$ where $K = 1(G)$ has the trivial G-action. We have an isomorphism $\theta: B^* \otimes C \cong \mathrm{Hom}_K(B, C)$ given by

$$\theta(f \otimes c)(b) = f(b) \cdot c \quad \text{for} \quad f \in B^*, \qquad b \in B, \qquad c \in C.$$

Now $B \otimes_K F$ is projective whenever F is projective. So for any positive integer n, $\Omega^n(A \otimes_K B) \cong \Omega^n(A) \otimes_K B$. Using these isomorphisms and Lemma 2.1 we can prove the following.

LEMMA 2.3. *If A, B, C are KG-modules and $n > 0$, then*

$$\mathrm{Ext}^n_{KG}(A \otimes_K B, C) \cong \mathrm{Ext}^n_{KG}(A, B^* \otimes_K C).$$

A KG-module M is said to be bounded if for any KG-module N there exists a number $b = b(N)$, depending only on N, such that $\mathrm{Dim}_K \mathrm{Ext}^n_{KG}(M, N) \le b$ for all $n > 0$. Lemma 2.1 implies that every periodic module is bounded.

PROPOSITION 2.4. *Let G be a p-group. A KG-module M is bounded if and only if there exists a number b such that $\mathrm{Dim}_K \Omega^n(M) \le b$ for all $n \ge 0$.*

Proof. Suppose first that there exists such a number b. The connecting homomorphism $\partial: \mathrm{Hom}_{KG}(\Omega(M), N) \to \mathrm{Ext}^1_{KG}(M, N)$ is onto. Hence

$$\mathrm{Dim}\, \mathrm{Ext}^1_{KG}(M, N) \le \mathrm{Dim}\, \mathrm{Hom}_{KG}(\Omega(M), N) \le b \cdot \mathrm{Dim}\, N.$$

Similarly by Lemma 2.1,

$$\mathrm{Dim}\, \mathrm{Ext}^n_{KG}(M, N) \le b \cdot \mathrm{Dim}\, N.$$

Therefore M is bounded.

Now suppose M is bounded. Let $N = 1(G)$ in the sequence (2.2). The homomorphism $\theta^*: \mathrm{Hom}_{KG}(F, 1(G)) \to \mathrm{Hom}_{KG}(\Omega(M), 1(G))$ is the zero map. For if $f: F \to 1(G)$ is a KG-homomorphism, then $f(\mathrm{Rad}\, KG \cdot F) = 0$. But $\theta(\Omega(M)) \subseteq \mathrm{Rad}\, KG \cdot F$ since $\Omega(M)$ has no free direct summands. Therefore

$$\mathrm{Ext}^1_{KG}(M, 1(G)) \cong \mathrm{Hom}_{KG}(\Omega(M), 1(G)).$$

Since M is bounded there exists a number b' such that

$$\mathrm{Dim}\, \mathrm{Ext}^n_{KG}(M, 1(G)) = \mathrm{Dim}\, \mathrm{Hom}_{KG}(\Omega^n(M), 1(G)) \le b'.$$

But

$$\mathrm{Dim}\, \mathrm{Hom}_{KG}(\Omega^n(M), 1(G)) = \mathrm{Dim}\, \Omega^n(M)/\mathrm{Rad}\, KG \cdot \Omega^n(M) \ge \frac{1}{|G|} \mathrm{Dim}\, \Omega^n(M).$$

Therefore $\mathrm{Dim}\, \Omega^n(M) \le |G| \cdot b'$ for all $n > 0$.

LEMMA 2.5. *Suppose H is a normal subgroup of G with G/H cyclic. If M is* *a KG-module such that M_H is a projective KH-module, then M is periodic of period at most* 2.

Proof. Let $1(H)^G = KG \otimes_{KH} 1(H)$. By Frobenius reciprocity

$$M \otimes_K 1(H)^G \cong (M_H)^G$$

is projective (see [6]). Suppose $G = \langle x, H\rangle$ where $x^t \in H$, $x^s \notin H$ for $0 < s < t$. We have an exact sequence

$$0 \to 1(G) \to 1(H)^G \xrightarrow{\tau} 1(H)^G \xrightarrow{\sigma} 1(G) \to 0$$

where $\sigma(\sum_{i=0}^{t-1} x^i \otimes k_i) = \sum k_i$ and τ is multiplication by $x-1$. If we tensor this sequence over K with M we get

$$0 \to M \to (M_H)^G \to (M_H)^G \to M \to 0.$$

But this says M is periodic.

3. Groups of order p^2

Let $G = \langle x, y\rangle$ be an elementary abelian group of order p^2. Assume throughout this section that K is an algebraically closed field of characteristic p. Ths main result of this section is the following.

THEOREM 3.1. *Let M be an indecomposable bounded KG-module. Either $M_{\langle x\rangle}$ is a free $K\langle x\rangle$-module or $M_{\langle y\rangle}$ is a free $K\langle y\rangle$-module.*

Before beginning the proof let us mention some consequences of this theorem.

COROLLARY 3.2. *A KG-module is bounded if and only if it is periodic, and every periodic module has period at most* 2.

As previously noted any periodic module is bounded. If M is bounded, then Lemma 2.5 and the theorem imply that M is periodic.

COROLLARY 3.3. *Suppose M is a nonprojective indecomposable periodic KG-module. Let $\omega_\infty = x$ and $\omega_\alpha = y + \alpha(x-1)$ for $\alpha \in K$. Let $W = \langle \omega_\alpha \rangle$. Then ω_α is a unit in KG and W is a cyclic group of order p. There is exactly one α, $\alpha \in K$ or $\alpha = \infty$, such that M_W is not a free KW-module.*

Proof. By a lemma of Dade [7, Lemma 11.1] there exists one such α. Suppose there exist two such. Call them α, β. Let G' be the subgroup of $U(KG)$, the units of KG, generated by ω_α, ω_β. Then G' is elementary abelian of order p^2. The inclusion of G' into KG induces a homomorphism $\psi\colon KG' \to KG$. An easy calculation shows that

$$\psi(\tilde{G}') = \psi((\omega_\alpha - 1)^{p-1}(\omega_\beta - 1)^{p-1}) = \tilde{G} \quad \text{if} \quad \alpha = \infty \quad \text{or} \quad \beta = \infty,$$

and $\psi(\tilde{G}')=(\alpha-\beta)^{p-1}\tilde{G}$ otherwise. Since $K\tilde{G}'$ is the unique minimal ideal, ψ is an isomorphism. Thus M is an indecomposable KG'-module. But this contradicts Theorem 3.1.

We now proceed with the proof of the theorem. First note that if $p=2$ the theorem follows from the classification of all KG-modules given by Basev in [2] and Heller and Reiner in [10]. It is not difficult to show in this case that all odd dimensional KG-modules are unbounded (compare [11] with Lemma 2.5), while all even dimensional indecomposables are periodic of period 1 and satisfy the theorem. Therefore we shall assume for the remainder of this section that $p\neq 2$.

The proof of the following is straightforward and is left to the reader as an exercise (see also [12]).

PROPOSITION 3.4. *Let $G=\langle x, y \mid x^q=y^r=1, xy=yx\rangle$ where $q=p^a$, $r=p^b$. For all $n>0$ there is an exact sequence*

$$0\to\Omega^n(1(G))\xrightarrow{\theta_n}F_n\xrightarrow{\psi_n}\Omega^{n-1}(1(G))\to 0 \tag{3.5}$$

where F_n is a free module with KG-basis $a_1,\ldots,a_n$ and where $\theta(\Omega^n(1(G)))\subseteq F_n$ is generated as follows.

(i) *If $n=2m+1$, then $\theta_n(\Omega^n(1(G)))$ is generated by*

$$\begin{aligned}
l_1&=(x-1)a_1;\\
l_{2j}&=(y-1)a_{2j-1}+(x-1)^{q-1}a_{2j},\quad j=1,\ldots,m;\\
l_{2j+1}&=(y-1)^{r-1}a_{2j}+(x-1)a_{2j+1},\quad j=1,\ldots,m;\\
l_{2m+2}&=(y-1)a_{2m+1}.
\end{aligned}$$

(ii) *If $n=2m$, then $\theta_n(\Omega^n(1(G)))$ is generated by*

$$\begin{aligned}
l_1&=(x-1)^{q-1}a_1;\\
l_{2j}&=(y-1)a_{2j-1}-(x-1)a_{2j},\quad j=1,\ldots,m;\\
l_{2j+1}&=(y-1)^{r-1}a_{2j}-(x-1)^{q-1}a_{2j+1},\quad j=1,\ldots,m-1;\\
l_{2m+1}&=(y-1)^{r-1}a_{2m}.
\end{aligned}$$

We shall use this to prove the following.

PROPOSITION 3.6. *Let G be as above. Suppose M^* is a bounded KG-module. If $m\in\mathrm{Soc}(M)$, then there exist elements $m', m''\in M$ such that*

$$m=(y-1)^{r-1}m'+(x-1)^{q-1}m''. \tag{3.7}$$

Moreover $(y-1)^{r-1}m'$, $(x-1)^{q-1}m''$ are in $\mathrm{Soc}(M)$.

Proof. Let $m\in\mathrm{Soc}(M)$ with $m\neq 0$. Let $n=2t-1$ and choose

$\alpha_1, \ldots, \alpha_t \in K$. Now define the KG-homomorphism $f = f(\alpha_1, \ldots, \alpha_t)$: $\Omega^{2t-2}(1(G)) \to M$ by the rule

$$f(l_{2j-1}) = \alpha_j m, \quad j = 1, \ldots, t,$$
$$f(l_{2j}) = 0, \quad j = 1, \ldots, t-1.$$

Here $l_1, \ldots, l_{2t-1}$ are as in Proposition 3.4. Now note that

$$f(\alpha_1, \ldots, \alpha_t) + f(\beta_1, \ldots, \beta_t) = f(\alpha_1 + \beta_1, \ldots, \alpha_t + \beta_t),$$

and

$$\alpha f(\alpha_1, \ldots, \alpha_t) = f(\alpha\alpha_1, \ldots, \alpha\alpha_t).$$

Therefore the collection V_t of all such homomorphisms is a K-subspace of dimension t in $\mathrm{Hom}_{KG}(\Omega^{2t-2}(1(G)), M)$.

In the sequence (2.2) replace M by $\Omega^{2t-3}(1(G))$ and replace N by our module M. Now $\mathrm{Ext}^1_{KG}(F_n, M) = 0$ since F_n is projective. Thus

$$\mathrm{Ext}^1_{KG}(\Omega^{2t-3}(1(G)), M) \cong \mathrm{Hom}_{KG}(\Omega^{2t-2}(1(G)), M)/\theta_n^*(\mathrm{Hom}_{KG}(F_n, M)).$$

Now by (2.1) and (2.3),

$$\mathrm{Ext}^1_{KG}(\Omega^{2t-3}(1(G)), M) \cong \mathrm{Ext}^{2t-2}_{KG}(1(G), M) \cong \mathrm{Ext}^{2t-2}_{KG}(M^*, 1(G)).$$

Suppose that $V_t \cap \mathrm{Im}\,\theta_n^* = \{0\}$ for all t. Then $\mathrm{Ext}^{2t-2}_{KG}(M^*, 1(G))$ has dimension at least t and M^* is not bounded. Since we are assuming M^* is bounded, there exists a non-zero element $f = f(\alpha_1, \ldots, \alpha_t) \in V_t \cap \mathrm{Im}\,\theta_n^*$, for some t. Suppose $\alpha_j \neq 0$. There exists a KG-homomorphism $g\colon F_n \to M$ such that $f = g\theta_n = \theta_n^*(g)$. Then

$$m = \alpha_j^{-1} f(l_{2j-1}) = \alpha_j^{-1}(y-1)^{r-1} g(a_{2j-2}) - \alpha_j^{-1}(x-1)^{q-1} g(a_{2j-1}).$$

If we let $m' = \alpha_j^{-1} g(a_{2j-2})$, $m'' = -\alpha_j^{-1} g(a_{2j-1})$ we are done.

The final statement of the proposition follows from the fact that

$$(x-1)m = 0 = (x-1)(y-1)^{r-1} m'.$$

So $\mathrm{Rad}\,KG \cdot (y-1)^{r-1} m' = 0$. The same holds for $(x-1)^{q-1} m''$.

Proof of Theorem 3.1. Suppose M is a bounded KG-module. As noted in Section 2, $\mathrm{Hom}_K(M, M) \cong M \otimes M^*$ is also a bounded KG-module. Recall that the action of G on $\mathrm{Hom}_K(M, M)$ is given by $(zf)(m) = zf(z^{-1}m)$ for $z \in G$, $f \in \mathrm{Hom}_K(M, M)$, and $m \in M$. Hence

$$\mathrm{End}\,(M) = \mathrm{Hom}_{KG}(M, M) = \mathrm{Soc}\,(\mathrm{Hom}_K(M, M)).$$

Let $I\colon M \to M$ be the identity homomorphism. Now $(M \otimes M^*)^* \cong M \otimes M^*$ is bounded. By the last proposition there exist $f, g \in \mathrm{HOm}_K(M, M)$ such that

$$I = (y-1)^{p-1} f + (x-1)^{p-1} g.$$

Whereas $(y-1)^{p-1} f$, $(x-1)^{p-1} g \in \mathrm{End}\,M$ and $\mathrm{End}\,M$ is a local ring, then one of these two homomorphisms is not in Rad (End M). For the rest of the proof we assume that $(y-1)^{p-1}f \notin$ Rad (End M).

Since K is algebraically closed End M/Rad (End M) $\cong K$. Therefore there exists a nonzero element $k \in K$ such that $(y-1)^{p-1}f = kI + r$ where $r \in$ Rad (End M). Let t be a positive integer such that $(r)^{p^t} = 0$. If

$$h = k^{-p^t} f \circ [(y-1)^{p-1}f]^{p^t-1},$$

then $I = I^{p^t} = (y-1)^{p-1}h$. The proof of the theorem will be complete when we have proved the following lemma. For we can let $\langle y \rangle = H$ and $M_{\langle y \rangle} = L$ in the lemma.

LEMMA 3.7 (Gaschütz, see [6, (62.3]). *Let H be a p-group and let K be a field of characteristic p. Suppose L is a KH-module such that the identity map $I: L \to L$ is in $\tilde{H} \cdot \mathrm{Hom}_K(L, L)$. Then L is a free KH-module.*

Proof. There exists $g \in \mathrm{Hom}_K(L, L)$ such that $I = \tilde{H}g$. Suppose $\varphi \in \mathrm{Hom}_{KG}(L, L)$. Then for any $m \in L$

$$\varphi(m) = (I \circ \varphi)(m) = \sum_{h \in H} hg(h^{-1}\varphi(m) = \sum h(g \circ \varphi)(h^{-1}m).$$

Therefore $\varphi = \tilde{H}(g \circ \varphi) \subseteq \tilde{H}\,\mathrm{Hom}_K(L, L)$.

There is an exact sequence

$$0 \to \Omega(L) \to F \xrightarrow{\psi} L \to 0 \tag{3.8}$$

where F is a free KH-module. The long exact sequence for Ext implies that

$$\mathrm{Hom}_{KH}(L, F) \xrightarrow{\psi_*} \mathrm{Hom}_{KH}(L, L) \to \mathrm{Ext}^1_{KH}(L, \Omega(L)) \to 0$$

is exact. Note that $\mathrm{Ext}^1_{KH}(L, F) = 0$ since F is also injective. Given any $\varphi \in \mathrm{Hom}_{KH}(L, L)$ there exists $\sigma \in \mathrm{Hom}_{KH}(L, L)$ with $\varphi = \tilde{H}\sigma$. But (3.8) splits as a sequence of K-modules. Hence there exists $\tau \in \mathrm{Hom}_K(L, F)$ with $\sigma = \psi\tau$. So $\varphi = \tilde{H}(\psi\tau) = \psi(\tilde{H}\tau)$. Since $\tilde{H}\tau \in \mathrm{Hom}_{KH}(L, F)$, we conclude that ψ_* is onto and $\mathrm{Ext}^1_{KH}(L, \Omega(L)) = 0$. In particular (3.8) must split and L must be a projective KH-module.

4. Elementary abelian p-groups

We begin this section with a general result which will be used several times later.

THEOREM 4.1. *Let K be a field of characteristic p and let H be a normal subgroup of a p-group G. Suppose M is a bounded (respectively periodic) KG-module such that M_H is a free KH-module. Then $\tilde{H} \cdot M$ is a bounded (periodic) $K(G/H)$-module.*

Proof. There is an exact sequence $0 \to \Omega(M) \to F \to M \to 0$ where F is a free KG-module. The restriction of this must split as a sequence of KH-modules. Therefore the sequence $0 \to \tilde{H}\Omega(M) \to \tilde{H}F \to \tilde{H}M \to 0$ is exact. Since H acts trivially on these modules they may be regarded as $K(G/H)$-modules. Also $\tilde{H}F$ is free as a $K(G/H)$-module. If Ω_1 denotes the syzygy operator for $K(G/H)$-modules, then $\Omega_1(\tilde{H}M) \cong \tilde{H} \cdot \Omega(M)$, and inductively $\Omega_1^n(\tilde{H}M) \cong \tilde{H} \cdot \Omega^n(M)$. If M is bounded, then Proposition 2.4 implies $\tilde{H}M$ is a bounded $K(G/H)$-module. If M is periodic with period n, then $\Omega_1^n(\tilde{H}M) \cong \tilde{H}\Omega^n(M) \cong \tilde{H}M$.

THEOREM 4.2. *Let $G = \langle x_1, \ldots, x_n \rangle$ be an elementary abelian p-group of order p^n where $n \geq 2$. Let K be an algebraically closed field of characteristic p. Suppose M is a bounded KG-module. There exist units $y_1, \ldots, y_n \in KG$ which satisfy the following conditions.*

(i) *Each y_i is of the form $y_i = 1 + \sum_{j=1}^{n} a_{ij}(x_j - 1)$ for some $a_{ij} \in K$.*

(ii) *The group $G' = \langle y_1, \ldots, y_n \rangle \subseteq U(KG)$ is elementary abelian of order p^n.*

(iii) *The inclusion map of G' into KG induces an algebra isomorphism of KG' onto KG.*

(iv) *Let $H = \langle y_2, \ldots, y_n \rangle \subseteq G'$. Then M_H is a free KH-module.*

Of course the action of KG' on M is induced by the inclusion of G' into KG.

Proof. Suppose first that $n = 2$. By Corollary 2.3, for each component of M there is at most one element $a \in K$ such that this component is not free as a $K\langle y_2 \rangle$-module where $y_2 = x_2 + a(x_1 - 1)$. Since K is infinite we can choose a so that every component is free as a $K\langle y_2 \rangle$-module. Let $y_1 = x_1$. As in the proof of (3.3) we can show that $KG' \cong KG$.

Now assume $n > 2$. Let $V = \langle x_2, \ldots, x_n \rangle$. If M_V is a free KV-module there is nothing left ot prove. So assume M_V is not free. By induction on n there exist units $z_2, \ldots, z_n \in KV$ which satisfy all of the conditions of the theorem. Let

$$V' = \langle z_2, \ldots, z_n \rangle \quad \text{and} \quad U = \langle z_3, \ldots, z_n \rangle.$$

Then $KV' \cong KV$ and M_U is a free KU-module. Let $W = \langle x_1, z_2, \ldots, z_n \rangle$. Now $\tilde{V}' = k\tilde{V}$ for some $k \in K$, $k \neq 0$. Hence $\tilde{W} = (x_1 - 1)^{p-1}\tilde{V}' = k(x_1 - 1)^{p-1}\tilde{V} = k\tilde{G} \neq 0$. So the inclusion of W into KG induces an isomorphism of KW onto KG. Therefore M is a bounded KW-module, and by Theorem 4.1, $\tilde{U}M$ is a bounded $K(W/U)$-module.

The action of W/U on $\tilde{U}M$ is the same as that of $\langle x_1, z_2 \rangle$. According to the previous case there exists a unit $y_2 = z_2 + a(x_1 - 1)$ such that $\tilde{U}M$ is free as a $K\langle y_2 \rangle$-module. Now set $y_1 = x_1$, $y_i = z_i$ for $i = 3, \ldots, n$. Then $\tilde{H} = (y_2 - 1)^{p-1}\tilde{U}$ and

$$\operatorname{Dim} \tilde{H}M = \frac{1}{p} \operatorname{Dim} \tilde{U}M = \frac{1}{|H|} \operatorname{Dim} M.$$

This implies that M_H is a free KH-module (see [4, Lemma 2.1]). It is easy to check that this particular collection of units satisfies the remaining conditions of the theorem.

5. Abelian p-groups

The main result of this section is the following generalization of Theorem 4.2.

THEOREM 5.1. *Let G be an abelian p-group and let K be an algebraically closed field of characteristic p. Suppose M is a bounded KG-module. There exists a group $G' \subseteq U(KG)$ satisfying the following conditions.*

(i) *$G \cong G'$ and the inclusion of G' into KG induces an isomorphism of KG' onto KG.*

(ii) *G' has a subgroup H such that G'/H is cyclic and the restriction M_H is a free KH-module.*

We shall need the following lemma whose proof can be found in [3] or [5].

LEMMA 5.2. *Let G be an abelian p-group and let H be a subgroup of G which contains all elements of order p in G. Let K be a field of characteristic p. If M is a KG-module such that M_H is a free KH-module, then M is a free KG-module.*

Proof of Theorem 5.1. Since G is abelian we can write

$$G = H_1 \times \cdots \times H_t$$

where $H_i = \langle x_i \rangle$ and $|H_i| = p^{b_i} = q_i$. Assume $b_1 \le b_2 \le \cdots \le b_t$. The subgroup

$$J = \langle z_i = x_i^{q/p} \mid i = 1, \ldots, t \rangle$$

is elementary abelian of order p^t. We can assume M_J is not a free KJ-module since otherwise Lemma 5.2 implies M is a free module. But M_J is bounded. There exist units $y_1, \ldots, y_t \in KJ$ satisfying the conditions of Theorem 4.2. In particular, there exist $a_{ij} \in K$ such that

$$y_i = 1 + \sum a_{ij}(z_j - 1)$$

for each $i = 1, \ldots, t$. If $P = \langle y_1, \ldots, y_t \rangle$ and $Q = \langle y_2, \ldots, y_t \rangle$, then $KP \cong KJ$ and M_Q is a free KQ-module. We have two cases to consider.

Case 1. Suppose $a_{i1} = 0$ for all $i = 2, \ldots, t$. Let $A = \langle z_2, \ldots, z_t \rangle$. Then $y_i \in KA$ for all $i = 2, \ldots, t$. Now

$$\tilde{Q} = \prod_{i=2}^{t} (y_i - 1)^{p-1} \in (\text{Rad } KA)^{(t-1)(p-1)} = K \cdot \tilde{A}$$

(see [4, Lemma 4.2]). Whereas $\tilde{Q}$ is not zero in KJ, we must have $\tilde{Q} = k\tilde{A}$

for some nonzero $k \in K$. Hence

$$\operatorname{Dim} \tilde{A}M = \operatorname{Dim} \tilde{Q}M = \frac{1}{|A|} \operatorname{Dim} M.$$

This implies that M_A is a free KA-module (see [4, Lemma 2.1]). Let $H = \langle x_2, \ldots, x_t \rangle$. By Lemma 5.2, M_H is a free KH-module. Since G/H is cyclic we are done.

Case 2. Suppose $a_{i1} \neq 0$ for some fixed i with $2 \leq i \leq t$. Let

$$\omega = 1 + \sum_{j=1}^{t} c_{ij}(x_j - 1)^{q_j/q_1},$$

where $c_{ij}^{q_1/p} = a_{ij}$. Then $\omega^{q_1/p} = y_i$. Now let $B = \langle \omega \rangle$, $C = \langle x_2, \ldots, x_t \rangle$ and $A = B \times C$. Since $H_1 \cong B$, $A \cong G$. Let ψ: $KA \to KG$ be the homomorphism induced by the inclusion of A into KG. The image of $\tilde{A}$ is $\psi(\tilde{A}) = (\omega - 1)^{q_1 - 1}\tilde{C}$. Now $(x_j - 1)\tilde{C} = 0$ whenever $j = 2, \ldots, t$. So

$$\psi(\tilde{A}) = c_{i1}^{q_1-1}(x_1 - 1)^{q_1-1}\tilde{C} = c_{i1}^{q_1-1}\tilde{G} \neq 0.$$

Therefore ψ is an isomorphism because $\tilde{A}$ generates the unique minimal ideal in KA.

By Lemma 5.2, M_B is a free KB-module because it is a free $K\langle y_i \rangle$-module. By Theorem 4.1, $\tilde{B}M$ is a bounded KC-module. By induction on $|G|$ there is a subgroup C' of $U(KC)$ and a subgroup H' of C' such that $KC' \cong KC$ by the inclusion homomorphism, C'/H' is cyclic and $(\tilde{B}M)_{H'}$ is a free KH'-module. Let $G' = B \times C'$ and $H = B \times H'$. It is easy to check that these satisfy the conclusion of the theorem.

THEOREM 5.3. *Let G be as in Theorem* 5.1 *and let K be any field of characteristic p. For each $i = 1, \ldots, n$, let $J_i = \langle x_j \mid j \neq i \rangle$. If M is an indecomposable bounded KG-module then M_{J_i} is a free KJ_i-module for some i.*

Proof. Let $N = \operatorname{Hom}_K(M, M) \cong M \otimes M^*$. As before N is a bounded KG-module. Let $S = \{i_1, \ldots, i_s\}$ be a maximal subset of $\{1, \ldots, n\}$ such that the restriction of N to a $K\langle x_{i_1}, \ldots, x_{i_s} \rangle$-module is free. Let

$$U = \langle x_{i_1}, \ldots, x_{i_s} \rangle \quad \text{and} \quad L = \tilde{U}N.$$

We shall show that $s = n - 1$, and hence $U = J_i$ for some i. Assume $s < n - 1$. By renumbering we get $U = \langle x_{n-s+1}, \ldots, x_n \rangle$. Let $V = \langle x_1, \ldots, x_t \rangle$, $t = n - s$, and number the elements so that

$$|\langle x_1 \rangle| = q_1 \leq \cdots \leq |\langle x_t \rangle| = q_t.$$

Note that L is a bounded KV-module and $\operatorname{Soc}(N) = \operatorname{Hom}_{KG}(M, M) \subseteq L$.

Let K' be the algebraic closure of K. Write

$$M' = K' \otimes_K M, \quad N' = K' \otimes_K N \cong \operatorname{Hom}_{K'}(M', M') \quad \text{and} \quad L' = K' \otimes_K L = \tilde{U}N'.$$

Now reduce L' as a $K'V$-module by the method in the proof of Theorem 5.1. Note that Case 1 can not occur. In the notation of Case 2, let $L_1 = (\omega - 1)^{q_1 - 1}L'$. Then L_1 is a bounded $K'\langle x_2, \ldots, x_t\rangle$-module which contains $\mathrm{Soc}(N')$. Continuing we get $L'' = L_{t-2} \subseteq L'$ which is a bounded $K'\langle x_{t-1}, x_t\rangle$-module containing $\mathrm{Hom}_{K'G}(M', M')$. If I is the identity homomorphism on M, then $1 \otimes I$ is that of M' and $1 \otimes I \in \mathrm{Soc}(L'')$. Note that L''^* is bounded by Corollary 5.5 (which in this case does not depend on the present theorem). Applying Proposition 3.6 we get

$$1 \otimes I = f_{t-1} + f_t \quad \text{where} \quad f_i \in (x_i - 1)^{q_i - 1}L'' \cap \mathrm{Soc}(L'') \quad \text{for} \quad i = t-1, t.$$

Let $V_i = \langle x_i, U\rangle$. Now $(x_i - 1)^{q_i - 1}L'' \subseteq \tilde{V}_i N'$. An easy investigation reveals that

$$\tilde{V}_i N' \cap \mathrm{Soc}(N') = \mathrm{Soc}(\tilde{V}_i N') = K' \otimes_K (\tilde{V}_i N \cap \mathrm{Soc}(N)),$$

where $\mathrm{Soc}(N) = \mathrm{Hom}_{KG}(M, M)$. Therefore

$$f_i \in K' \otimes_K (\tilde{V}_i N \cap \mathrm{Hom}_{KG}(M, M)).$$

Now f_{t-1} and f_t can not both be nilpotent. For convenience assume f_t is not nilpotent. We can write f_t as a finite sum: $f_t = \sum \alpha_j \otimes g_j$ for $\alpha_j \in K'$ and $g_j \in \tilde{V}_i N \cap \mathrm{Hom}_{KG}(M, M)$. At least one of the g_j's is not nilpotent, and since $\mathrm{Hom}_{KG}(M, M)$ is an Artinian local ring, this g_j has an inverse h in $\mathrm{Hom}_{KG}(M, M)$. Therefore $I = g_j \circ h$ is in $\tilde{V}_i\, \mathrm{Hom}_K(M, M)$, and by Lemma 3.7, M_{V_i} is a free KV_i-module. This contradicts the maximality of S.

Corollary 3.3 now generalizes to the following.

COROLLARY 5.4. *Let $G = \langle x_1, \ldots, x_n\rangle$ be an elementary abelian p-group and let K be algebraically closed field of characteristic p. Let M be an indecomposable bounded KG-module. Let V be the K-subspace of KG with basis*

$$\{x_1 - 1, \ldots, x_n - 1\}.$$

Let U be the subset of V consisting of 0 and of all $v \in V$ such that M is not a free $K\langle 1 + v\rangle$-module. Then U is a subspace of V of dimension 1. Moreover if $v_1, \ldots, v_t$ are linearly independent elements of V such that the subspace which they generate has trivial intersection with U, then M is free as a $K\langle 1 + v_1, \ldots, 1 + v_t\rangle$-module.

It was noted in Section 2 that any periodic module is bounded. Thus Lemma 2.5 and Theorem 5.3 (or Theorem 5.1 if K is algebraically closed) imply the following.

COROLLARY 5.5. *Let G be an abelian p-group and let K be a field of characteristic p. A KG-module is bounded if and only if it is periodic. Any periodic KG-module has period at most 2.*

Recall that that the exponent of a p-group G is the maximum of the orders of the elements of G. Since the restriction of a bounded module is bounded we have the following.

COROLLARY 5.6. *Let G be a finite group and let K be a field of characteristic p. Let H be an abelian p-subgroup of G whose order is p^n and whose exponent is p^e. If M is a bounded (or periodic) KG-module then p^{n-e} divides $\mathrm{Dim}_K(M)$. In particular if r is the p-rank of G then p^{r-1} divides $\mathrm{Dim}_K(M)$.*

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