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ON THE FORMALITY OF $K-1$ CONNECTED COMPACT MANIFOLDS OF DIMENSION LESS THAN OR EQUAL TO $4K-2$

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In all that follows, coefficients will be assumed to be in the field of rational numbers $\mathbf{Q}$. All spaces will be at least simply connected and of finite type. The categories of differential graded algebras, coalgebras, and Lie algebras will be denoted by DGA, DGC, and DGLA, respectively, and objects in these categories will be referred to as algebras, coalgebras, and Lie algebras. Algebras and coalgebras will be assumed to be associative, cocummutative, and homologically connected.

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The following notions are due to Sullivan [2] [3], Neisendorfer [5] [6], and Baues-Lemaire [1].

DEFINITION. A simply connected algebra M (respectively Lie algebra L) is called minimal if it is free and the differential is decomposable. A minimal model for a simply connected algebra A (respectively Lie algebra L') is a minimal algebra M (respectively minimal Lie algebra L) and a weak equivalence $\rho: M \to A$ (respectively $\rho: L \to L'$).

Minimal models exist for simply connected algebras (respectively Lie algebras) and are unique up to isomorphism [3] [5] [6].

Let $\mathscr{E}(X)$ be the algebra of rational de Rham chains on a space X. We denote the minimal model for $\mathscr{E}(X)$ by M_X.

DEFINITION. A space X is called formal if there is a weak equivalence

$$\rho: M_X \to H^*(M_X).$$

Since M_X determines the homotopy type of X [2], a formal space has its homotopy type determined by its cohomology algebra.

We can now state the major result of the paper. It is a generalization of a result found in [5].

THEOREM. *Let X be a compact, $k-1$ connected manifold of dimension less than or equal to $4k-2$, $k>1$. Then X is formal.*

Below we give a brief sketch of the theory necessary for the proof of this theorem. Readers desiring more detail are referred to [4], [5], and [6].

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Consider the dual coalgebra of M_X, C_X. Recall the functor $\mathscr{L}$ of Quillen [7] from the category DGC to the category DGLA. As a non-differential Lie algebra, $\mathscr{L}(C_X)$ is the free Lie algebra $L(s^{-1}\bar{C}_X)$ on the desuspension $s^{-1}\bar{C}_X$ of the graded vector space C_X. The differential on $\mathscr{L}(C_X)$ is the sum of two differentials, $d^*_{C_X}$ and $\Delta^*_{C_X}$, induced by the differential and the comultiplication of C_X, respectively. We note that the second differential is zero on the Lie algebra generated by $s^{-1}PC_X$, where PC_X is the subspace of primitive elements. Denote the minimal model for $\mathscr{L}(C_X)$ by L_X.

Now consider the free Lie Algebra $L(s^{-1}H_*(C_X))$. Recall that $L(s^{-1}H_*(C_X)) \cong \mathscr{L}(H_*(C_X))$. Since $H_*(C_X)$ has the zero differential, the differential on $\mathscr{L}(H_*(C_X))$ is given by $\Delta^*_{H_*(C_X)}$. The following proposition is a summary of the remarks following Proposition 8.4 of [6].

PROPOSITION. *There exists a decomposable differential d on $L(s^{-1}H_*(C_X))$ such that*

$$(L(s^{-1}H_*(C_X)), d) \cong L_X.$$

Moreover, d agrees with $\Delta^*_{H_*(C_X)}$ up to terms of length three.

Remark. If X is formal, $d = \Delta^*_{H_*(C_X)}$

Using the functor $\mathscr{C}$ of Quillen [7] which is adjoint to $\mathscr{L}$, and using the bijection established by Sullivan between the set of simply connected rational homotopy types and the set of isomorphism classes of simply connected minimal algebras [2], the following can be shown [4].

PROPOSITION. *There is a bijection of sets between the set of simply connected rational homotopy types of finite type and the set of isomorphism classes of minimal Lie algebras of finite type.*

From this proposition and the definition of formality, we have the next proposition [5].

PROPOSITION. *X is formal if and only if $L_X \cong \mathscr{L}(H_*(C_X))$, i.e. if and only if the differential on L_X is $\Delta^*_{H_*(C_X)}$.*

Let A be an algebra and C a coalgebra, both simply connected with zero differential, and let $X \otimes \mathbf{Q}$ denote the rational homotopy type of X. Define the following two sets:

$\mathrm{RHT}(A) = \{$simply connected rational homotopy types $X \otimes Q$ with $H^*(X; Q) \cong A\}$.

Coalg Diff $(L(s^{-1}C)) = \{$automorphism classes of differential structures on $L(s^{-1}C)$ which agree with $\Delta^*_{H_*(C_X)}$ up to terms of length three$\}$.

The above bijection restricts in the following way [4].

PROPOSITION. *There exists a bijection of sets between* $\mathrm{RHT}(H^*(M_X))$ *and the set*

$$\text{Coalg Diff } (L(s^{-1}H_*(C_X))).$$

COROLLARY. *If* Coalg Diff $(L(s^{-1}H_*(C_X)))$ *consists of only one element, then* $X\otimes\mathbf{Q}$ *is the only simply connected rational homotopy type having cohomology algebra* $H^*(M_X)$, *and* X *is formal.*

Proof of the theorem. We show that the hypothesis of the corollary holds. We will assume the dimension of the manifold to be $4k-2$, and simple adjustments of the proof in this case will yield the result in lower dimensions.

Choose the following generators for $L(s^{-1}H_*(C_X))$:

$x_1^1,\dots,x_{n_1}^1$ in dimension $k-1$,
$x_1^2,\dots,x_{n_2}^2$ in dimension k,

$\vdots$

$x_1^{k-1},\dots,x_{n_{k-1}}^{k-1}$ in dimension $2k-3$,
$y_1,\dots,y_{2n}$ in dimension $2k-2$
$\bar{x}_1^{k-1},\dots,\bar{x}_{n_{k-1}}^{k-1}$ in dimension $2k-1$,

$\vdots$

$\bar{x}_1^1,\dots,\bar{x}_{n_1}^1$ in dimension $3k-3$,
μ in dimension $4k-3$,

where

$$(sx_j^i)^*\cdot(s\bar{x}_j^i)^*=(s\mu)^*,\quad 1\le i\le k-1,\quad 1\le j\le n_i,$$
$$(sy_i)^*\cdot(sy_{i+1})^*=(s\mu)^*,\quad i<2n,\quad i \text{ odd},$$

in $H^*(X;\mathbf{Q})$. Hence

$$\Delta^*_{H_*(C_X)}(\mu)=\sum_{i=1}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}[x_j^i,\bar{x}_j^i]-\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}[y_i,y_{i+1}].$$

Let $d_1\in\operatorname{Coalg\,Diff}L(s^{-1}H_*(C_X))$ be the formal differential, i.e. $d_1=\Delta^*_{H^*(C_X)}$, and let d_2 be an arbitrary element of the same set. It is clear that d_1 and d_2 agree on a generator x different from μ, since $d_2(x)$ is of dimension no higher than $3k-4$, and there are no terms of length three in dimensions lower than $3k-3$. Therefore we need only check that d_1 and d_2 agree on μ.

First we obtain a general form for $d_2(\mu)$. Since this term lies in dimension $4k-4$, we seek a basis for terms of length three and four in that dimension (by dimensional reasons there are no terms of length five or greater). Recall that Serre has given a formula for the basis of a free Lie algebra [8], which easily generalizes to the graded case by inclusion of elements of the form $[x,x]$, where x is an element of the Hall basis and the degree of x is odd. Consider the set of all basis elements of length three in dimension $4k-4$

whose first term is x_j^i, and let n_{ij} be the number of elements in this set. Let $\{a_{ij}^l\}_{l=1}^{n_{ij}}$ be the "tails" of these basis elements. Hence a typical basis element whose first term is x_j^i has the form $[x_j^i, a_{ij}^l]$, $1 \le i \le k-1$, $1 \le j \le n_i$, and $1 \le l \le n_{ij}$. Denote the set of all such (i, j, l) by A.

Next, consider the set of all basis elements of length three (in dimension $4k-4$) whose first term is y_i, $1 \le i \le 2n$. Such an element has the form $[y_i, [x_j^1, x_l^1]]$, where $j < l$, and the set of generators is ordered in the natural way. Denote the set of all such (j, l) by B', and the set of triples (i, j, l) by B, where $1 \le i \le 2n$, $j < l$.

Finally, the set $[x_j^1, [x_k^1, [x_l^1, x_m^1]]]$ spans the space of terms of length four. Choose a basis from this set, and index it by C, i.e. $(j, k, l, m) \in C$ if and only if

$$[x_j^1, [x_k^1, [x_l^1, x_m^1]]]$$

is in the basis. Let C_j index the set of basis elements whose first term is x_j^1, i.e. $(k, l, m) \in C_j$ if and only if $(j, k, l, m) \in C$.

Using the notation developed above, we can write $d_2(\mu)$ as follows:

$$d_2(\mu) = \Delta_{H*}^*(\mu) + \sum_{(i,j,l)\in A} q_{ijl}^A[x_j^i, a_{ij}^l] + \sum_{(i,j,l)\in B} q_{ijl}^B[y_i, [x_j^1, x_l^1]]$$
$$+ \sum_{(j,k,l,m)\in C} q_{jklm}[x_k^1, [x_k^1, [x_l^1, x_m^1]]],$$

where the coefficients are, of course, in $\mathbf{Q}$.

We now define $T: (L(s^{-1}H_*(C_X)), d_1) \to (L(s^{-1}H_*(C_X)), d_2)$, a map of Lie algebras, as follows:

$$T(x_j^i) = x_j^i, \quad 1 \le i \le k-1, \quad 1 \le j \le n_i,$$

$$T(\bar{x}_j^1) = \bar{x}_j^1 + (-1)^{|x_j^1|+1} \sum_{l=1}^{n_{ij}} q_{1jl}^A(a_{1j}^l)$$
$$+ (-1)^{|x_j^1|+1} \sum_{(k,l,m)\in C_j} p_{jklm}[x_k^1, [x_l^1, x_m^1]], \quad 1 \le j \le n_1,$$

where we do not specify p_{jklm} yet,

$$T(\bar{x}_j^i) = \bar{x}_j^i + (-1)^{|x_j^i|+1} \sum_{l=1}^{n_{ij}} q_{ij}^A(a_{ij}^l), \quad 2 \le i \le k-1, \quad 1 \le j \le n_i,$$

$$T(y_i) = y_i + \sum_{(j,l)\in B'} q_{i+1j}^B[x_j^1, x_l^1],$$

$$T(y_{i+1}) = y_{i+1} - \sum_{(j,l)\in B'} q_{ijl}^B[x_j^1, x_l^1], \quad \text{where} \quad 1 \le i \le 2n-1, \quad i \text{ odd}.$$

(Note how the coefficients are "flip-flopped".)

$$T(\mu) = \mu.$$

It is clear that T is an isomorphism, since the matrix for T in each dimension is lower triangular. We check that T commutes with the differentials on generators other than μ. In the dimensions of these generators, d_1 and d_2 agree. Since sx_j^i is primitive, $d_2(x_j^1)$ and $d_2(a_{ij}^l)$ are both zero. Commutativity follows by noting that T is the identity plus terms involving x_j^1 and a_{ij}^l.

We now check that $T(d_1(\mu)=d_2(T(\mu))$. The proof is tedious but straightforward.

$$
\begin{aligned}
T(d_1(\mu)) &= T\Bigg(\sum_{i=1}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}[x_j^i,\bar{x}_j^i]-\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}[y_i,y_{i+1}]\Bigg)\\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}[x_j^i,T(\bar{x}_j^i)]-\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}[T(y_i),T(y_{i+1})]\\
&= \sum_{j=1}^{n_i}(-1)^{|x_j^1|+1}\Bigg[x_j^1,x_j^1+(-1)^{|x_j^1|+1}\sum_{l=1}^{n_{ij}}q_{1jl}^{A}(a_{1j}^l)\\
&\qquad +(-1)^{|x_j^1|+1}\sum_{(k,l,m)\in C_j}p_{jklm}[x_k^1,[x^1,x_m^1]]\Bigg]\\
&\qquad +\sum_{i=2}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}\Bigg[x_j^i,\bar{x}_j^i+(-1)^{|x_j^i|+1}\sum_{l=1}^{n_{ij}}q_{ijl}^{A}a_{ij}^l\Bigg]\\
&\qquad -\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}\Bigg[y_i+\sum_{(j,l)\in B'}q_{i+1jl}^{B}[x_j^1,x_l^1],\,y_{i+1}-\sum_{(j,l)\in B'}q_{ijl}^{B}[x_j^1,x_l^1]\Bigg]\\
&= \sum_{j=1}^{n_1}(-1)^{|x_j^1|+1}[x_j^1,\bar{x}_j^1]+\sum_{j=1}^{n_1}(-1)^{|x_j^1|+1}\sum_{l=1}^{n_{ij}}(-1)^{|x_j^1|+1}q_{ijl}^{A}\\
&\qquad \times[x_j^1,a_{1j}^l]+\sum_{j=1}^{n_1}(-1)^{|x_j^1|+1}\sum_{(k,l,m)\in C_j}(-1)^{|x_j^1|+1}p_{jklm}\\
&\qquad \times[x_j^1,[x_k^1,[x^1,x_m^1]]]+\sum_{i=2}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}[x_j^i,\bar{x}_j^i]\\
&\qquad +\sum_{i=2}^{k-1}\sum_{j=1}^{n_i}(-1)^{|x_j^i|+1}\sum_{l=1}^{n_{ij}}(-1)^{|x_j^i|+1}q_{ijl}^{A}[x_j^i,a_{ij}^l]\\
&\qquad -\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}[y_i,y_{i+1}]+\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}\sum_{(j,l)\in B'}q_{ijl}^{B}[y_i,[x_j^1,x_l^1]]\\
&\qquad +\sum_{\substack{i=1\\ i\text{ odd}}}^{2n-1}\sum_{(j,l)\in B}q_{i+1jl}^{B}[x_j^1,x_l^1]]\\
&\qquad +\sum_{i=1}^{2n-1}\Bigg[\sum_{(j,l)\in B'}q_{i+1jl}^{B}[x_j^1,x_l^1],\sum_{(j,l)\in B'}q_{ijl}^{B}[x_j^1,x_l^1]\Bigg]
\end{aligned}
$$

$$= \sum_{i=1}^{k-1} \sum_{j=1}^{n_i} (-1)^{|x_j^i|+1}[x_j^i, \bar{x}_j^i] + \sum_{(i,j,l)\in A} q_{ijl}^A [x_j^i, a_{ij}^l]$$

$$+ \sum_{(i,j,l)\in B} q_{ijl}^B [y_i, [x_j^1, x_l^1]] - \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-1} [y_i, y_{i+1}]$$

$$+ \sum_{(j,k,l,m)\in C} p_{jklm} [x_j^1, [x_k^1, [x_l^1, x_m^1]]]$$

$$+ \sum_{i=1}^{2n-1} \left[\sum_{(j,l)\in B'} q_{i+1jl} [x_j^1, x_l^1], \sum_{(j,l)B'} q_{ijl} [x_j^1, x_l^1] \right].$$

Express this last in terms of a basis, and then choose p_{jklm} so that the coefficients turn out correctly. Q.E.D.

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