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A PROBLEM IN THE EXTENSION OF MEASURES

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1. Introduction

The problem referred to in the title is as follows. Suppose $(X, \mathscr{B})$ is a measurable space, $\mathfrak{A}$ a sub-σ-field of $\mathscr{B}$ and μ a probability measure on $\mathfrak{A}$. Can μ be extended to a measure on $\mathscr{B}$?

E. Marczewski has discussed various aspects of this problem in a number of articles, notably [10], [11], [12]. In [10], he constructed an example of a non-separable probability measure μ on a sub-σ-field $\mathfrak{A}$ of the Borel σ-field $\mathscr{B}$ of the unit interval [0, 1]. Plainly μ cannot be extended to a measure on $\mathscr{B}$, which shows that the answer to the problem posed above is, in general, no. Marczewski [11] then asked if a separable measure μ on a sub-σ-field $\mathfrak{A}$ of the Borel σ-field $\mathscr{B}$ of the unit interval could be extended to $\mathscr{B}$. The answer is again no as the following (unpublished) example of Marczewski shows. Take $\mathfrak{A}$ to be the σ-field of meager and comeager Borel subsets of the unit interval and let μ be the measure on $\mathfrak{A}$ which is 0 on meager Borel sets and 1 on comeager Borel sets. So defined, μ is a separable probability measure on $\mathfrak{A}$. But, as is well known, a probability measure on the Borel σ-field of the unit interval sits on a meager Borel set. Hence μ cannot be extended to a measure on the Borel σ-field.

In view of these examples, one is led to reformulate the above problem. To do so we first introduce some definitions. We shall say that a measurable space $(X, \mathscr{B})$ has the *extension property* if for every countably generated sub-σ-field $\mathfrak{A}$ of $\mathscr{B}$ and any probability measure μ on $\mathfrak{A}$, μ can be extended to a measure on $\mathscr{B}$. Of particular interest will be measurable spaces $(X, \mathscr{B})$ such that $\mathscr{B}$ is countably generated and contains singletons. It is known that, if $(X, \mathscr{B})$ is such a measurable space, then X can be metrized in such a way that it becomes a separable metric space and $\mathscr{B}$ is just the Borel σ-field of X. Conversely, if X is a separable metric space, then the Borel σ-field $\mathscr{B}$ of X is countably generated and contains singletons. This leads us to the following definition. We say that a separable metric space X has the *extension property* if $(X, \mathscr{B}_X)$ has the extension property, where, for a metric space Y, $\mathscr{B}_Y$ denotes the Borel σ-field of Y.

The question naturally arises if every separable metric space has the extension property. We do not know if a separable metric space without the extension property can be shown to exist in ZFC. However, with additional axioms it is possible to prove the existence of such separable metric spaces.

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Indeed, under CH (continuum hypothesis) Banach [1] and Rao [16] have observed that there is a countable family $\mathscr{C}$ of subsets of [0, 1] such that Lebesgue measure on the Borel σ-field of [0, 1] does not extend to a measure on the σ-field $\mathscr{B}$ on [0, 1] generated by sets in $\mathscr{C}$ and the Borel subsets of [0, 1]. Mauldin [15] proved the same result under the weaker axiom MA (Martin's axiom). Plainly the measurable space $([0, 1], \mathscr{B})$ does not have the extension property. Since the σ-field $\mathscr{B}$ is countably generated and contains singletons, it follows, by virtue of the remarks made in the preceding paragraph, that there is a separable metric space without the extension property.

In the other direction, Varadarajan ([18], p. 194) proved (in ZFC) that every analytic set has the extension property. In recent years, this result has been rediscovered by several authors; see [3], [7], [8]. The present article is the outcome of our attempts to extend Varadarajan's result to coanalytic sets.

The main result of the paper is the following:

THEOREM. *The following conditions are equivalent* (*in ZFC*):
(i) *Every PCA set has the extension property.*
(ii) *Every coanalytic set has the extension property.*
(iii) *Every PCA set is universally measurable.*

Now if $V=L$ then there is a PCA set of reals which is not Lebesgue measurable [4]. So it follows from our theorem that under $V=L$ there is a coanalytic set which does not have the extension property. On the other hand, if we assume Martin's Axiom (to be abbreviated hereafter by MA) and $2^{\aleph_0}>\aleph_1$, then every PCA set is universally measurable [13, p. 169], and, consequently, every coanalytic set has the extension property. We conclude that the proposition "every coanalytic set has the extension property" is undecidable in ZFC.

The proof of our main result is presented in the next section. In the final section, we give examples to show that the extension property of separable metric spaces neither implies nor is implied by universal measurability.

2. Proofs

We shall need below the notion of a perfect measure. Recall that a probability measure μ on a measurable space $(X, \mathfrak{A})$ is said to be *perfect* if for every real-valued, $\mathfrak{A}$-measurable function f on X, there is a Borel subset B of the real line such that $B \subset f(X)$ and $\mu(f^{-1}(B))=1$. It is known that if X is a coanalytic set, then any probability measure on $(X, \mathscr{B}_X)$ is perfect. This follows easily from the fact that every coanalytic set is universally measurable.

LEMMA 1. *Suppose* $(X, \mathfrak{A})$, $(Y, \mathscr{B})$ *are measurable spaces and* f *is a* $(\mathfrak{A},$

$\mathcal{B}$)-measurable function of X onto Y. If $(X, \mathfrak{A})$ has the extension property, then so does $(Y, \mathcal{B})$.

The easy proof is omitted. The next lemma is known; the proof is included here for completeness.

LEMMA 2. *Assume that all PCA sets are universally measurable. Let X be a coanalytic set and f a real-valued Borel measurable function on X. Then there is a function g: $f(X) \to X$ such that g is universally measurable and $f \circ g(y) = y$ for every $y \in f(X)$.*

Proof. Let F be the graph of f, that is,

$$F = \{(x, y) \in X \times \mathbf{R} : f(x) = y\}.$$

It is well known that F is a Borel subset of $X \times \mathbf{R}$, so F is coanalytic. According to a result of Kondô [5], F admits a coanalytic uniformization G (parallel to the second axis), that is, $G \subseteq F$ and $G \cap (X \times \{y\})$ is a singleton for each $y \in f(X)$. Plainly G defines a function g on $f(X)$ to X such that $f \circ g(y) = y$. Finally, if B is a (relatively) Borel subset of X, then $g^{-1}(B)$ is the projection to the second coordinate of the set $G \cap (B \times \mathbf{R})$. Since $G \cap (B \times \mathbf{R})$ is coanalytic, it follows that $g^{-1}(B)$ is PCA and so universally measurable.

Proof of the theorem. The implication (i)$\Rightarrow$(ii) in the theorem is obvious and (ii)$\Rightarrow$(i) is an easy consequence of Lemma 1.

We now prove (ii)$\Rightarrow$(iii). Let Y be a PCA subset of a Polish space Z. Let λ be a probability measure on the Borel subsets of Z. Suppose that the outer measure $\lambda^*(Y) > 0$. Find a Borel subset Y' of Z such that $Y \subseteq Y'$ and $\lambda(Y') = \lambda^*(Y)$. Define ν on $\mathcal{B}_Y$, the Borel σ-field on Y by

$$\nu(E \cap Y) = \lambda(E)/\lambda(Y') \quad \text{for} \quad E \in \mathcal{B}_{Y'}.$$

It is easy to check that ν is a probability measure on $\mathcal{B}_Y$.

Since Y is a PCA set, there is a coanalytic set X and a continuous function f on X onto Y. Let $\mathcal{B}$ be the Borel σ-field on X and put $\mathfrak{A} = f^{-1}(\mathcal{B}_Y)$. Since $\mathfrak{A}$ and $\mathcal{B}_Y$ are isomorphic, there is a measure μ on $\mathfrak{A}$ such that $\nu = \mu f^{-1}$. By (ii), μ extends to a measure μ' on $\mathcal{B}$. Since X is coanalytic it follows that μ' is perfect. Consequently, there is a Borel subset B of Z such that $B \subseteq f(X) = Y$ and $\mu'(f^{-1}(B)) = 1$. But $\mu'(f^{-1}(B)) = \mu(f^{-1}(B))$ and hence $\nu(B) = 1$. This implies that $\lambda(B) = \lambda(Y')$, so that Y is λ-measurable. It now follows that Y is universally measurable.

For the implication (iii)$\Rightarrow$(ii), let X be a coanalytic set, $\mathfrak{A}$ a countably generated sub-σ-field of the Borel σ-field $\mathcal{B}$ on X and μ a probability measure on $\mathfrak{A}$. Choose sets A_n, $n \geq 1$, such that $\langle A_n, n \geq 1\}$ generates $\mathfrak{A}$. Let

$$f = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{A_n}$$

be the characteristic function of the sequence $\{A_n\}$. According to Lemma 2, there is a universally measurable function $g: f(X) \to X$ such that $f \circ g(y) = y$ for each $y \in f(X)$.

Now let λ be the measure on the Borel σ-field of the unit interval which is defined by setting $\lambda = \mu f^{-1}$. Let ν denote the restriction of the completion of λ to the σ-field of universally measurable subsets of $f(X)$. Note that ν is a probability measure. Define μ' on $\mathcal{B}$ by setting $\mu' = \nu g^{-1}$. Using a familiar property of the characteristic function f and the fact that $f \circ g$ is the identity on $f(X)$, one checks that $\mu' = \mu$ on $\mathfrak{A}$. This completes the proof.

3. Remarks

We conclude with some remarks which will elucidate the relationship for separable metric spaces between universal measurability and the extension property.

(A) We shall exhibit a subset of the unit interval which has the extension property but is not universally measurable. Towards this consider a subset Z of $[0, 1]$ such that neither Z nor $[0, 1] - Z$ contains a perfect set. Such a set can be shown to exist in ZFC [6, p. 514]. Plainly Z is not universally measurable. Indeed, Z is not measurable with respect to any continuous probability measure. We claim that Z has the extension property.

Let $\mathfrak{A}$ be a countably generated sub-σ-field of $\mathcal{B}_Z$ and let μ be a probability measure on $\mathfrak{A}$. To show that μ can be extended to $\mathcal{B}_Z$ it suffices to consider the case where μ is a continuous measure on $\mathfrak{A}$. Let $\mathfrak{A}'$ be a countably generated sub-σ-field of $\mathcal{B}_{[0,1]}$ such that $\mathfrak{A} = \mathfrak{A}' \cap Z$. Define μ' on $\mathfrak{A}'$ by $\mu'(A') = \mu(A' \cap Z)$. By Varadarajan's theorem (quoted in Section 1), μ' can be extended to a measure ν on $\mathcal{B}_{[0,1]}$. Since μ is continuous we note that ν is continuous. It follows that the outer measure $\nu^*(Z) = 1$. This enables us to define a measure λ on $\mathcal{B}_Z$ by the formula

$$\lambda(B \cap Z) = \nu(B) \quad \text{for} \quad B \in \mathcal{B}_{[0,1]}$$

An easy calculation now shows that $\lambda = \mu$ on $\mathfrak{A}$.

(B) Next we show in ZFC+CH that there is a separable metric space which is universally measurable but does not possess the extension property. Our construction is based on a method of Luzin and Sierpiński [9].

Let A be an analytic non-Borel set in $[0, 1]$. There is then a system $\{A(p)\}$, indexed by finite sequences of positive integers, of Borel sets in $[0, 1]$ such that A is the result of operation (A) on the system $\{A(p)\}$. Let A_α, $\alpha < \omega_1$, be the constituents of A induced by $\{A(p)\}$ (see [17]). It is known that the sets A_α are Borel subsets of $[0, 1]$, disjoint and $A = U_{\alpha < \omega_1} A_\alpha$. The analytic non-Borel set A and the defining system $\{A(p)\}$ can be so chosen that the sets A_α, $\alpha < \omega_1$, are all non-empty. For each $\alpha < \omega_1$, pick a point $y_\alpha \in A_\alpha$. By CH, enumerate the elements of $[0, 1]$ as x_α, $\alpha < \omega_1$. Put $S = \{(x_\alpha, y_\alpha): \alpha < \omega_1\}$. We have thus obtained a subset of the unit square.

We claim that S is universally measurable but does not possess the extension property.

Indeed, S is a universal null set. To see this, consider the system $\{B(p)\}$ defined by $B(p)=[0,1]\times A(p)$. If B is the result of operation (A) on $B(p)$ and B_α, $\alpha<\omega_1$, are the constituents of B induced by $\{B(p)\}$, then $B=[0,1]\times A$ and $B_\alpha=[0,1]\times A_\alpha$, $\alpha<\omega_1$. Note that the set S meets each B_α in exactly one point. Let λ be a continuous probability measure on the Borel sets of the unit square. According to a known property of constituents [17], there is $\alpha<\omega_1$ such that the outer measure $\lambda^*(\bigcup_{\beta>\alpha} B_\beta)=0$. Now, the set $S\cap(\bigcup_{\beta\le\alpha} B_\beta)$ being countable, we have $\lambda(S\cap(\bigcup_{\beta\le\alpha} B_\beta))=0$. Since $S\subseteq B$, it follows that $\lambda^*(S)=0$.

To see that S does not have the extension property, denote by π the projection on S to the first coordinate. Then π is a one-one continuous and hence Borel measurable function on S onto $[0,1]$. Let $\mathfrak{A}=f^{-1}(\mathcal{B}_{[0,1]})$ and μ the measure on $\mathfrak{A}$ such that μf^{-1} is Lebesgue measure on $\mathcal{B}_{[0,1]}$. Since f is one-one, $\mathfrak{A}$ contains all singletons and μ is continuous on $\mathfrak{A}$. Since S is a universal null set, it follows that μ cannot be extended to the Borel σ-field $\mathcal{B}_S$.

We remark that it is possible to prove in ZFC+MA that there is a separable metric space which is universally measurable but which does not have the extension property. Indeed, a modification due to Mauldin [14] of a construction of Darst [2] yields in ZFC+MA a subset S' of the unit square which is a universal null set and whose projection to the first coordinate is the unit interval. Just as above one shows that S' does not have the extension property.

(C) We do not know if a separable metric space without the extension property can be shown to exist in ZFC. However we now show that it is impossible to prove in ZFC that there is a separable metric space of cardinality $\aleph_1$ which does not possess the extension property.

Indeed, we show in ZFC+MA+$2^{\aleph_0}>\aleph_1$ that if X is any set of cardinality $\aleph_1$ then $(X, P(X))$ has the extension property, where $P(X)$ is the power-set of X. To see this let $\mathfrak{A}$ be a countably generated σ-field on X and let μ be a probability measure on $\mathfrak{A}$. Let f be the characteristic function of a sequence $\{A_n\}$ which generates $\mathfrak{A}$. Put $Y=f(X)$. Then Y has cardinality at most $\aleph_1$, so that, by MA and $2^{\aleph_0}>\aleph_1$, Y is a universal null set of reals [13, p. 167]. Consequently $\mathcal{B}_Y$ supports no continuous probability measure. So μf^{-1} is discrete on $\mathcal{B}_Y$. As f is an isomorphism of $\mathfrak{A}$ and $\mathcal{B}_Y$, it follows that μ is discrete on $\mathfrak{A}$. Let $B_1, B_2, \ldots$ be the atoms of $\mathfrak{A}$ such that $\sum_{n=1}^\infty \mu(B_n)=1$. Pick $x_n\in B_n$, $n\ge 1$. Then $\mu=\sum_{n=1}^\infty \mu(B_n)\partial(x_n)$ is a measure on $P(X)$ which extends μ.

The preceding should be compared with the last paragraph in [7] where the authors purportedly prove in ZFC that on any set X of cardinality $\aleph_1$ there is a countably generated σ-field $\mathcal{B}$ such that $(X, \mathcal{B})$ does not have the extension property! This is of course incorrect.

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