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MEASURABLE SUBBUNDLES IN LINEAR SKEW-PRODUCT FLOWS

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Introduction

The main purpose of this paper is to construct and discuss two ODEs and one real flow. Our work is based on techniques of topological dynamics developed by Furstenberg in [5]. These techniques also allow a discussion of certain other topics, namely Bohr's theorem, and a type of quasi-periodic function considered by Sell in [16] (see also Brezin, Ellis, and Shapiro [2]).

To avoid confusion due to the number of topics considered, we give here an outline of the paper, as well as an indication of why one might be interested in the ODEs and the flow. In §1, we prove Bohr's theorem using Furstenberg's techniques. In §2, we consider a certain irrational (Kronecker) flow $(K^2, \mathbf{R})$ on the 2-torus K^2. Let $\omega \cdot t$ represent the position of $\omega \in K^2$ after time t under this flow. We construct a non-continuous function $R \in L^2(K^2, m)$ (m is Lebesgue measure on K^2) and an analytic function b on K^2 such that "R is an antiderivative of b along orbits"; i.e.,

$$\int_0^t b(\omega \cdot s)\, ds = R(\omega \cdot \omega t) - R(\omega) \quad (\omega \in K^2, t \in \mathbf{R}).$$

The function b has mean value zero, but $\int_0^t b(\omega \cdot s)\, ds$ is not almost-periodic (a.p.). the functions R and b are of fundamental importance in constructing our three examples. In §2, we also consider Sell's results.

In §§ 3, 4, and 5, we treat the examples.

(§3) Consider the analytic differential equations E_ω:

$$\dot{x} = \frac{1}{2}\begin{bmatrix} 0 & -b(\omega \cdot t) \\ b(\omega \cdot t) & 0 \end{bmatrix} x \quad (\omega \in K^2, x \in \mathbf{R}^2).$$

We view this collection of ODEs as "generated" by some one quasiperiodic ODE E_{ω_0} (ω_0 a fixed element of K^2). The equations E_ω induce a "linear skew-product flow", or LSPF [13], [14], [15] on $K^2 \times \mathbf{R}^2$. This LSPF has interesting structure: the vector bundle $K^2 \times \mathbf{R}^2$ foliates into measurable, non-continuous, invariant, one-dimensional subbundles (3.3). Now, by Floquet theory, non-continuous subbundles cannot occur for periodic ODE's (3.4). The point we wish to make with this example is that, if one is to

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classify linear a.p. ODEs (i.e., "extend Floquet theory" to this case), then the possibility of measurable subbundles must be taken into account.

(§4) We construct a real flow $(K^3, \mathbf{R})$ (K^3 is a 3-torus) which extends the flow $(K^2, \mathbf{R})$ mentioned in §2. The flow $(K^3, \mathbf{R})$ is strictly ergodic (4.1, 4.7). However, there are times $t_k \to 0$ such that, if T_k is the time t_k map, then (K^3, T_k) has uncountably many ergodic measures. In fact there is an analytic function $f\colon K^3 \to \mathbf{C}$ such that the Cesaro means

$$\frac{1}{l}\sum_{i=0}^{l} f\circ(T_k)^i(\omega, \rho)$$

diverge for a residual set of $(\omega, p) \in K^3$, for all $k \geq 1$ (4.8). Compare this with the fact that, by strict ergodicity and [11, Chapter 6, proof of Theorem 9.05],

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f((\omega, \rho)\cdot s)$$

exists (uniformly in (ω, ρ) in fact). Thus "existence of integral means does not imply existence of Cesaro means".

(§5) This example is closely related to the flow of §4. Consider the ODEs E_ω given by

$$\dot{x} = \frac{1}{2}\begin{bmatrix} 0 & -\lambda_1 - b(\omega\cdot t) \\ \lambda_1 + b(\omega\cdot t) & 0 \end{bmatrix} x \quad (\omega \in K^2, x \in \mathbf{R}^2),$$

where λ_1 is a certain real number (see 4.2). The equations induce an LSPF on $K^2 \times \mathbf{R}^2$. It turns out that the LSPF $(K^2 \times \mathbf{R}^2, \mathbf{R})$ admits no invariant, one-dimensional subbundles. However, the *integer* flows $(K^2 \times \mathbf{R}^2, T_k)$ (T_k is a time-t_k map, t_k as in §4) *do* have measurable, T_k-invariant, non-continuous, one-dimensional subbundles, and these foliate $K^2 \times \mathbf{R}^2$. Thus, if one studies only the *real* flow $(K^2 \times \mathbf{R}^2, \mathbf{R})$, and none of the corresponding *integer* flows, some of the complexity of the flow is not observed.

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1. A proof of Bohr's theorem

1.1. Definition. Let Ω be a compact Hausdorff space, and let T be a topological group (we will consider only cases where $T = \mathbf{R}$ or $T = Z$). A flow (Ω, T) is defined by a continuous map $\Phi\colon \Omega \times T \to \Omega\colon (\omega, t) \to \omega \cdot t$ satisfying (i) $\omega \cdot idy = \omega$; (ii) $\omega \cdot (t_1 t_2) = (\omega \cdot t_1) \cdot t_2$ ($\omega \in \Omega$; $t, t_1, t_2 \in T$). A set $S \subset \Omega$ is *invariant* if $S \supset S \cdot t = \{\omega \cdot t \mid \omega \in S\}$ for all $t \in T$. The flow (Ω, T) is *minimal* if the only nonempty closed invariant subset of Ω if Ω itself.

Let $\tilde{b}\colon \mathbf{R} \to \mathbf{R}$ be an almost-periodic (a.p.) function with mean value b_0;

i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde{b}(s)\,ds=b_0.$$

Bohr's theorem states that $\int_0^t \tilde{b}(s)\,ds-b_0t$ is bounded iff $\int_0^t \tilde{b}(s)\,ds$ is a.p. [11].

We will state and prove a slight generalization of this result. Let Ω be the *hull* of b; i.e., the uniform closure in $C(\mathbf{R})$ of the set of all translates $\tilde{b}_\tau(t)=b(t+\tau)$). Define a flow on Ω by translation; i.e., $\omega\cdot\tau=\omega_\tau$ ($\omega\in\Omega$, $\tau\in\mathbf{R}$). Consider the function $b:\Omega\to\mathbf{R}$: $b(\omega)=\omega(0)$ (evaluate ω at $0\in\mathbf{R}$). Let ω_0 denote the function $\tilde{b}$. Then $b(\omega_0\cdot t)=\tilde{b}(t)$. Thus, if $\mathbf{R}$ is mapped into Ω via $t\to\omega_0\cdot t$, then b may be thought of as an extension of $\tilde{b}$ to Ω. Moreover, it turns out that Ω may be given the structure of a compact, metric abelian topological group such that (i) $t\to\omega_1\cdot t$ maps $\mathbf{R}$ onto a dense subgroup of Ω; (ii) the map $(\omega,t)\to\omega\cdot t$ is group multiplication [11, p. 394]. Let m be normalized Haar measure on Ω. It is well known that the mean value b_0 of $\tilde{b}$ equals $\int_\Omega b(\omega)\,dm(\omega)$ [11, p. 510].

The following result implies Bohr's theorem.

1.2. Theorem. *The following are equivalent.*

(a) *There exists $\omega_0\in\Omega$ such that $\int_0^t b(\omega_0\cdot s)\,ds-b_0t$ is bounded.*

(b) *There is a continuous function $S:\Omega\to\mathbf{R}$ such that*

$$S(\omega\cdot t)-S(\omega)=\int_0^t b(\omega\cdot s)\,ds-b_0t\quad(\omega\in\Omega).$$

We need only prove (a) $\Rightarrow$ (b). The proof will use 1.3 and 1.4. We can and will assume that $b_0=0$. Choose a real $\lambda_0>0$ so that

$$\text{(A)}\qquad \left|\lambda_0\int_0^t b(\omega_0\cdot d)\,ds\right|<2\pi.$$

1.3. Let $\Sigma=\Omega\times S^1$; we denote points of Σ by (ω,ρ). Fix (ω,ρ), with $\rho=e^{i\varphi}$. Define a flow $(\Sigma,\mathbf{R})$ (which depends on λ_0) by

$$(\omega,\rho)\cdot t=\exp i\left(\varphi+\lambda_0\int_0^t b(\omega\cdot s)\,ds\right).$$

Then $(\omega,\rho)\cdot t=(\omega\cdot t, g_t(\omega)\cdot\rho)$, where $g_t(\omega)=\exp(i\lambda_0\int_0^t b(\omega\cdot s)\,ds)$. The flow $(\Sigma,\mathbf{R})$ is an example of a real *cocycle flow* (e.g., [12]).

The next lemma applies to any cocycle flow. See also [3, Lemma 1.9].

1.4. Lemma. *The flow $(\Sigma,\mathbf{R})$ is minimal if and only if the equation*

$$[g_t(\omega)]^n=\frac{\sigma(\omega\cdot t)}{\sigma(\omega)}\quad(t\in\mathbf{R},\ \omega\in\Omega)$$

has no continuous solution σ, $|\sigma(\omega)|$ a non-zero constant, for any $n\neq0$.

Proof. If $[g_t(\omega)]^n = \sigma(\omega \cdot t)/\sigma(\omega)$ for all t, ω and some $n \neq 0$, then it is easily seen that $f(\omega, p) \equiv \sigma(\omega)\rho^n$ is a continuous, non-constant function which is invariant: $f((\omega, \rho) \cdot t) = f(\omega, \rho)$. Hence $(\Sigma, \mathbf{R})$ is not minimal. Suppose $(\Sigma, \mathbf{R})$ is not minimal, and let $F \neq K$ be a non-empty closed invariant set. Let X_F be its characteristic function. Expand X_F in a partial Fourier series: $X_F \sim \sum_{n \neq 0} a_n(\omega)\rho^{-n}$. Since X_F is not constant, there exists $n \neq 0$ such that $a_n(\omega) \neq 0$ γ-a.e. Since $X_F((\omega, \rho) \cdot t) = X_F(\omega, \rho)$, we obtain $a_n(\omega \cdot t)[g_t(\omega)]^{-n} = a_n(\omega)$ γ-a.e. for each fixed t. The proof that $|a_n(\omega)|$ is a non-zero constant γ-a.e. and that a_n is equal to a continuous function γ-a.e. mimics the argument in [3, pp. 17–18], and is omitted. Let σ be continuous with $\sigma = a_n$ γ-a.e. ∎

1.5. *Proof of* 1.2. The flow $(\Sigma, \mathbf{R})$ is minimal iff every orbit $\{(\omega, \rho) \cdot t \mid t \in \mathbf{R}\}$ is dense in Σ. By (A), no orbit of the form $\{(\omega_0, \rho) \cdot t \mid t \in \mathbf{R}\}$ can be dense in Σ. By 1.4, then, there is a continuous function σ on Ω and an integer $k \neq 0$ such that

$$(*) \qquad \exp\left(i\lambda_0 k \int_0^t b(\omega \cdot s)\, ds\right) = \frac{\sigma(\omega \cdot \omega t)}{\sigma(\omega)} \quad (\omega \in \Omega,\ t \in \mathbf{R}).$$

Fix $\tilde{\omega} \in \Omega$, and let $Q = \{\tilde{\omega} \cdot t \mid t \in \mathbf{R}\} \subset \Omega$. Define $f: \mathbf{R} \to \mathbf{R}$ and $s: Q \to \mathbf{R}$ by

$$f(t) = \sigma(\tilde{\omega} \cdot t)/\sigma(\tilde{\omega}), \qquad s(\tilde{\omega} \cdot t) = \lambda_0 \int_0^t b(\tilde{\omega} \cdot s)\, ds.$$

We need only show that s is uniformly continuous on Q, since then the continuous extension S of s to Ω satisfies 1.2(b).

So, let $0 < \varepsilon < \pi$ be given. Choose $\varepsilon_0 > 0$ so that $|e^{ix} - e^{iy}| < \varepsilon_0$ and $|x - y| < \pi$ imply that $|x - y| < \varepsilon$. Let d be a metric on Ω. Since σ is continuous, there exists $\delta > 0$ such that

$$d(\tilde{\omega} \cdot t_2, \tilde{\omega} \cdot t_1) < \delta \Rightarrow |f(t_2 + t) - f(t_1 + t)| < \varepsilon_0 \quad \text{for all } t \in \mathbf{R}.$$

Then $s(\tilde{\omega} \cdot (t_2 + t)) - s(\tilde{\omega} \cdot (t_1 + t)) = 2\pi L + \eta_1(t)$, where L is an integer and $|\eta_1(t)| < \varepsilon < \pi$ for all $t \in \mathbf{R}$. Let $t_{n+1} = n(t_2 - t_1) + t_1$ $(n = 2, 3, \ldots)$. By induction,

$$|s(\tilde{\omega} \cdot t_{n+1}) - s(\tilde{\omega} \cdot t_1)| \geq n(2\pi |L| - \varepsilon) \quad \text{if} \quad L \neq 0.$$

But then s is unbounded, a contradiction. So $L = 0$, and s is uniformly continuous. ∎

2. The function R

The construction is a simple modification of one due to Furstenberg [5, p. 585]. Let $\alpha = \sum_{k=1}^{\infty} 2^{-v_k}$, where $v_1 = 1$, $v_{k+1} = 2^{v_k} + v_k + 1$. Then α is irrational (its binary expansion is non-repeating). Let $n_k = 2^{v_k}$; then

$$n_k\alpha - [n_k\alpha] = \sum_{l=k+1}^{\infty} 2^{v_k - v_l} < 2 \cdot 2^{v_k - v_{k+1}} = 2^{-n_k}.$$

Let $m_k = -[n_k\alpha]$, and let $n_{-k} = n_k$, $m_{-k} = m_k$. Then

$$\text{(B)} \qquad |n_k\alpha + m_k| < (\sqrt{2})^{-|n_k|-|m_k|} \quad (k \in Z - \{0\}).$$

For $0 \le \theta_1$, $\theta_2 < 1$, define a square-integrable function R by

$$R(\theta_1, \theta_2) \sim \sum_{k \neq 0} \frac{1}{|k|} \exp\left(2\pi i(n_k\theta_1 + m_k\theta_2)\right).$$

Then R is real; extend R to the plane so as to be 1-periodic in θ_1 and θ_2.

2.1. Proposition. *There is no continuous function* $\bar{R}: [0, 1] \times [0, 1] \to \mathbf{R}$ *such that* $\bar{R} = R$ a.e.

Here "a.e." refers to Lebesgue measure on $[0, 1] \times [0, 1]$. We do *not* require that $\bar{R}$ have a doubly-1-periodic, continuous extension to the plane.

Proof. Suppose $R = \bar{R}$ a.e. where $\bar{R}$ is continuous on $[0, 1] \times [0, 1]$. For fixed θ_2, consider the function $\bar{R}_{\theta_2}(\theta_1) \equiv \bar{R}(\theta_1, \theta_2)$. We claim that the Fourier series of $\bar{R}_{\theta_2}$ is

$$\bar{R}_{\theta_2} \sim \sum_{k \neq 0} \frac{e^{2\pi i m\theta_2}}{|k|} \cdot e^{2\pi i n_k\theta_1}.$$

To see this, let

$$a_j(\theta_2) = \int_0^1 \bar{R}(\theta_1, \theta_2) e^{-2\pi i j\theta_1}\, d\theta_1.$$

Note that

$$\int_0^1 a_j(\theta_2) e^{-2\pi i l\theta_2}\, d\theta_2 = \int_0^1 \int_0^1 \bar{R}(\theta_1, \theta_2) e^{-2\pi i(j\theta_1 + l\theta_2)}\, d\theta_1\, d\theta_2$$

$$= \begin{cases} \dfrac{1}{|k|} & \text{if } j = n_k \text{ and } l = m_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$a_j(\theta_2) = \begin{cases} \dfrac{1}{|k|} e^{2\pi i m_k\theta_2} & \text{if} \quad j = n_k \quad \text{for some } k \\ 0 & \text{otherwise} \end{cases}$$

$$= \left(\frac{1}{|k|} e^{2\pi i m_k\theta_2}\right) \cdot \delta_{j,n_k}.$$

Here $\delta_{j,n_k} = 0$ if $j \neq n_k$, and $\delta_{j,n_k} = 1$ if $j = n_k$. Now $\bar{R}_{\theta_2} \sim \sum c_j e^{2\pi i j\theta_1}$, where

$$c_j = \int_0^1 \bar{R}_{\theta_2} e^{-2\pi i j\theta_1}\, d\theta_1 = a_j(\theta_2) = \frac{1}{|k|} e^{2\pi i m_k\theta_2} \cdot \delta_{j,n_k}.$$

This proves the claim.

Now let $\theta_2=0$: the function $\bar{R}_0$ is continuous on [0, 1] with Fourier series

$$\sum \frac{1}{|k|} e^{2\pi i n_k \theta_1}.$$

This is a contradiction because the Cesaro sums of this series diverge at $\theta_1=0$ (see [17, Chapter III, Equation 2.22]). ■

2.2. DEFINITION. For each $t \in \mathbf{R}$, let

$$G_t(\theta_1, \theta_2)=R(\theta_1+\alpha t, \theta_2+t)-R(\theta_1, \theta_2) \quad (-\infty<\theta_1, \theta_2<\infty).$$

2.3. PROPOSITION. *For each $t \in \mathbf{R}$, the function $G_t(\theta_1, \theta_2)$ is equal* a.e. *to an analytic function which is* 1*-periodic in θ_1 and θ_2.*

Proof. Consider the Fourier series

$$\sum_{k\neq 0} \frac{1}{|k|} [e^{2\pi i(n_k\alpha+m_k)t}-1]e^{2\pi i(n_k\theta_1+m_k\theta_2)}$$

of G_T. By (B) and the uniform continuity of $x \to e^{2\pi i t x}$, there exists $M=M(t)$ such that, for large $|k|$,

$$\frac{1}{|k|}[e^{2\pi i(n_k\alpha+m_k)}-1]\leq M(\sqrt{2})^{-|n_k|-|m_k|}.$$

Hence [1, Chapter I, §25] the Fourier series is that of an analytic function. Clearly G_t is 1-periodic in θ_1 and θ_2.

We write G_t for the analytic function in 2.3, as well as the function of 2.2. Let

$$b(\theta_1, \theta_2)=\lim_{t\to 0}\frac{G_t(\theta_1, \theta_2)}{t}$$
$$=\sum_{k\neq 0}\frac{1}{|k|}\cdot 2\pi i(n_k\alpha+m_k)e^{2\pi i(n_k\theta_1+m_k\theta_2)}.$$

An argument like that of 2.3 proves

2.4. PROPOSITION. *b is analytic and* 1*-periodic in θ_1 and θ_2.*

Observe now that R, G_t, and b all define functions on K^2, e.g. if

$$\omega=(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}),$$

the mapping $\omega \to R(\theta_1, \theta_2)$ is well defined. We denote the functions induced on K^2 by R, G_t, and b, also. Define an irrational flow $(K^2, \mathbf{R})$ by

$$\omega \cdot t=(e^{2\pi i(\theta_1+\alpha t)}, e^{2\pi i(\theta_2+t)}),$$

where ω is as above. Let m be Lebesgue measure on K^2. Observe that, if we fix t and compute the Fourier series of the function $\omega \to \int_0^t b(\omega\cdot s)\, ds$

(using-term-by-term integration), we obtain the Fourier series of $R(\omega \cdot t) - R(\omega)$. Hence, for each t,

$$R(\omega \cdot t) - R(\omega) = \int_0^t b(\omega \cdot s)\, ds \ (= G_t(\omega)) \quad m\text{-a.e.}$$

2.5. PROPOSITION. *The function b is analytic on K^2 and has mean value zero, but $\int_0^t b(\omega \cdot s)\, ds$ is unbounded (i.e., is not a.p.) for all $\omega \in K^2$.*

Proof. Write $(t \cdot R)(\omega) = R(\omega \cdot t)$ $(\omega \in K^2, t \in \mathbf{R})$. Then $t \cdot R - R$ is equal to the function

$$G_t : \omega \to \int_0^t b(\omega \cdot s)\, ds$$

in $L^2(K^2, m)$. Let b_0 be the mean value of b. Then

$$\lim_{|t| \to \infty} \frac{1}{t} \int_0^t b(\omega \cdot s)\, ds = b_0 \quad \text{uniformly in } \omega.$$

Hence $b_0 = \lim_{|t|\to\infty} G_t/t = \lim_{|t|\to\infty} (t \cdot R - R)/t$ in $L^2(K^2, m)$. But then, letting $\| \ \|_2$ refer to the norm in $L^2(K^2, m)$, we have $\|R\|_2 = \|t \cdot R\|_2 \geq |b_0 t|/2 - \|R\|_2$ for large $|t|$, a contradiction unless $b_0 = 0$. So b has mean value zero.

Next, suppose $\int_0^t b(\omega_0 \cdot s)\, ds$ *is* bounded for some ω_0. Then (1.2) there is a continuous S such that $(t \cdot S)(\omega) - S(\omega) = \int_0^t b(\omega \cdot s)\, ds$. It follows that $[t \cdot (R - S)] - [R - S] = 0$ m-a.e. Hence $R - S$ is a measurable function on K^2 which is invariant with respect to the irrational flow $(K^2, \mathbf{R})$. So $R - S$ is constant m-a.e. [11, p. 468], [6, p. 25]. This contradicts 2.1. ■

We now give another proof that $\int_0^t b(\omega \cdot s)\, ds$ is unbounded for all $\omega \in K^2$. This proof yields a little additional information (2.10(a)) about R. The proof will use 2.6–2.8. assume for contradiction that, for some $\omega_0 \in \Omega$, $\int_0^t b(\omega_0 \cdot s)\, ds$ is bounded.

2.6. PROPOSITION. *Let $\Lambda = \{\lambda \in \mathbf{R} \mid$ the function $r(\omega) = e^{i\lambda R(\omega)}$ is not equal m-a.e. to a continuous function$\}$. Then Λ is residual in* $\mathbf{R}$.

Proof. By 2.1, there is no continuous function $\bar{R}$ on K^2 such that $\bar{R} = R$ m-a.e. Hence we may apply word for word the proof of Proposition A1, p. 83 of [3], except that "the interval [0, 1]" must be replaced by "the square $[0, 1] \times [0, 1]$". ■

2.7. Let $\Lambda_n = \{\lambda \in \mathbf{R} \mid e^{in\lambda R(\omega)}$ is not equal m-a.e. to a continuous function$\}$ (n an integer). Then, by 2.6, Λ_n is residual in $\mathbf{R}$ for all $n \neq 0$. Therefore, we can choose $\lambda_0 \in \bigcap_{n \neq 0} \Lambda_n$ such that $|\lambda_0 \int_0^t b(\omega_0 \cdot s)\, ds\,| < 2\pi$ for all $t \in \mathbf{R}$.

Define $g_t : K^2 \to S^1$ by

$$g_t(\omega) = \exp\left(i\lambda_0 \int_0^t b(\omega \cdot s)\, ds\right) \quad (\omega \in K^2, t \in \mathbf{R}).$$

Let $r(\omega)=e^{i\lambda_0 R(\omega)}$. Since $R(\omega\cdot t)-R(\omega)=\int_0^t b(\omega\cdot s)\,ds$ m-a.e., one has

$$g_t(\omega)=r(\omega\cdot t)/r(\omega)\quad m\text{-a.e. } (t\in\mathbf{R}).$$

Let $K^3=K^2\times S^1$. Define a flow $(K^3,\mathbf{R})$ by

$$(\omega,\rho)\cdot t=(\omega\cdot t, g_t(\omega)\cdot\rho)\quad (\omega\in K^2, \rho\in S^1, t\in\mathbf{R}).$$

2.8. PROPOSITION. *The flow* $(K^3,\mathbf{R})$ *is minimal.*

Proof. We use Lemma 1.4. We must show that the equation

$$[g_t(\omega)]^n=\sigma(\omega\cdot t)/\sigma(\omega)$$

has no continuous solution σ for any $n\neq 0$. Suppose σ *is* a continuous solution for some $n\neq 0$. Then $r^n(\omega)/\sigma(\omega)=r^n(\omega\cdot t)/\sigma(\omega\cdot t)$ m-a.e. for each $t\in\mathbf{R}$. But then r^n/σ is a measurable invariant function for the irrational flow $(K^2,\mathbf{R})$; hence r^n/σ is constant m-a.e. This contradicts the fact that $\lambda_0\in\bigcap_{n\neq 0}\Lambda_n$ (see 2.7). ■

2.9. *Second proof that* $\int_0^t b(\omega\cdot s)\,ds$ *is unbounded* $(\omega\in K^2)$. We chose λ_0 so that

$$\left|\lambda_0\int_0^t b(\omega_0\cdot s)\,ds\right|<2\pi\quad\text{for all } t\in\mathbf{R}.$$

But then no orbit $\{(\omega_0,\rho)\cdot t\mid t\in R\}$ of the flow $(K^3,\mathbf{R})$ could be dense in K^3. This contradicts 2.8. ■

2.10. *Remarks.* (a) We can now improve 2.7 by showing that there is *no* $\lambda\neq 0$ for which $e^{i\lambda R(\omega)}$ is equal m-a.e. to a continuous function. For, suppose $\lambda\neq 0$, and suppose $e^{i\lambda R(\omega)}$ is equal m-a.e. to a continuous function $r_\lambda(\omega)$. Then

$$r_\lambda(\omega\cdot t)/r_\lambda(\omega)=\exp\left(i\lambda\int_0^t b(\omega\cdot s)\,ds\right)\quad\text{for all }\omega, t.$$

Fix $\bar\omega\in K^2$. By [4, Lemma 6.7], we see that $\lambda\int_0^t b(\bar\omega\cdot s)\,ds=ct+B(t)$, where $B(t)$ is a.p. However, by 2.7, the mean value of b is zero. Hence $c=0$, and $\int_0^t b(\bar\omega\cdot s)\,ds$ is bounded. This contradicts 2.7.

(b) The function R cannot be essentially bounded. To see this, let ρ be a *strong lifting* of $L^\infty(K^2,m)$ which commutes with translations [7]. Let $\bar R=\rho(R)$; then $\bar R(\omega\cdot t)-\bar R(\omega)=\int_0^t b(\omega\cdot s)\,ds$ for *all* ω, t (the proof of this uses properties of ρ listed on p. 64 of [7]). From the definition of strong lifting, $\bar R$ is *bounded*, hence $\int_0^t b(\omega\cdot s)\,ds$ is bounded. This contradicts 2.5.

Let us now consider how our functions R and b are related to Sell's examples [16]. He constructs functions $\bar R$, $\bar b$ on K^2 and an irrational flow $(K^2,\mathbf{R})$ such that

(i) $\bar R$ is in $C^2(K^2)$ but not $C^3(K^2)$;
(ii) $\bar b$ is in $C^4(K^2)$;
(iii) $\bar R(\omega\cdot t)-\bar R(\omega)=\int_0^t \bar b(\omega\cdot s)\,ds$.

In other words, "integration leads to *loss* of derivatives". In 2.11 below, we will show that our R may be altered on a set of measure zero so that $R(\omega \cdot t) - R(\omega) = \int_0^t b(\omega \cdot s)\, ds$ for *all* ω, t. Our functions, then, represent an extreme case of loss of derivatives: b is analytic, while R is not even C^0.

2.11. PROPOSITION. *We may assume that the equation*

$$R(\omega \cdot t) - R(\omega) = \int_0^t b(\omega \cdot s)\, ds$$

holds for all $\omega \in K$ and $t \in \mathbf{R}$.

Proof. We appeal to [8]. First, [8, 3.8(b)] and the argument in the "(a) $\Rightarrow$ (b)" part of [8, 3.9] may be used to show that there is some m-measurable $\tilde{R}$ satisfying $\tilde{R}(\omega \cdot t) - \tilde{R}(\omega) = \int_0^t b(\omega \cdot s)\, ds$ for all ω, t. Hence

$$[t \cdot (\tilde{R} - R)] - [\tilde{R} - R] = 0 \quad m\text{-a.e.},$$

so $\tilde{R} - R$ is constant m-a.e. [11, p. 468] and [6, p. 25]. Clearly, we may take the constant to be zero. ■

2.12. *Remark.* Observe that the vector $\langle \alpha, 1 \rangle$ satisfies no inequality of the form

$$|\langle \alpha, 1 \rangle \cdot \langle n, m \rangle| \equiv |\alpha n + m| \geq \gamma(|n| + |m|)^{-\tau} \quad \text{for} \quad \gamma, \tau > 0.$$

For, if it did, $\int_0^t b(\omega \cdot s)\, ds$ would be a.p. [10, pp. 148–149]. Hence α is in none of the sets B_k of [16].

3. An analytic ODE

We construct a linear skew-product flow which foliates into measurable, non-continuous, invariant subbundles. Let b, G_t, R be as in §2. We may and will suppose that $R(\omega \cdot t) - R(\omega) = \int_0^t b(\omega \cdot s)\, ds$ for all $\omega \in K^2$, $t \in \mathbf{R}$ (2.11).

3.1. DEFINITIONS. Let V be a vector bundle with base Ω, projection π, and fibers of constant finite dimension. (See [13]; we will consider only the case $V = K^2 \times \mathbf{R}^2$, $\Omega = K^2$, and $\pi: K^2 \times \mathbf{R}^2 \to K^2$: $(\omega, x) \to \omega$.) A (real) *linear skew-product flow* (LSPF) on V is a pair of flows $(V, \mathbf{R})$ and $(\Omega, \mathbf{R})$ such that

(i) $\pi(v \cdot t) = \pi(v) \cdot t (v \in V, t \in \mathbf{R})$;
(ii) each map $t_\omega: \pi^{-1}(\omega) \to \pi^{-1}(\omega \cdot t)$ is linear $(t \in \mathbf{R})$.

A *continuous k-dimensional subbundle* of V is a closed subset of V which intersects each fiber $\pi^{-1}(\omega)$ in a k-dimensional subspace. See [13], [14], [15].

3.2. DEFINITION. Let $V = K^2 \times \mathbf{R}^2$. Let S^1 be the unit circle in $\mathbf{R}^2$. Let P^1 be projective one-space $= \{l: l$ is a line through the origin in $\mathbf{R}^2\}$. Note that $P^1 \cong S^1$ and that $K^2 \times P^1 \cong K^3$, a 3-torus. Let $\eta: K^2 \times (\mathbf{R}^2 - \{0\}) \to K^2 \times P^1$: $(\omega, x) \to (\omega, l)$, be the natural map (i.e., l is the line through the origin containing x). A *measurable one-dimensional subbundle of* V is a subset Q of V such that (i) Q intersects each fiber $\{\omega\}\times\mathbf{R}^2$ in a line through the origin and (ii) $\eta(Q)$ is μ-measurable, where μ is Lebesgue measure on $K^2\times P^1$.

Consider the set of analytic, 2-dimensional ODE's E_ω:

$$\dot{x}=\frac{1}{2}\begin{bmatrix} 0 & -b(\omega\cdot t) \\ b(\omega\cdot t) & 0 \end{bmatrix}x \quad (\omega\in K^2,\ x\in\mathbf{R}^2).$$

In polar coordinates (r,θ), E_ω is given by

$$\dot{r}=0,\quad \dot{\theta}=\tfrac{1}{2}b(\omega\cdot t).$$

These equations define an LSPF on $K^2\times\mathbf{R}^2$ in the following way:

$$(\omega, x_0)\cdot t=(\omega\cdot t, x(t)),$$

where $x(t)$ satisfies E_ω and the initial condition $x(0)=x_0$. We may parametrize P^1 with $\rho=e^{2i\theta}$ (view P^1 as S^1 with antipodal points identified). By (ii) of 3.1, the LSPF $(K^2\times\mathbf{R}^2,\mathbf{R})$ induces a flow on $K^3=K^2\times P^1$. This flow is given by

$$(\omega,\rho)\cdot t=(\omega\cdot t, g_t(\omega)\cdot\rho),$$

where $g_t\colon K^2\to P^1$, $\omega\to\exp(i\int_0^t b(\omega\cdot s)\,ds)$. Here $g_t(\omega)\cdot\rho$ is the product, in the topological group P^1, of the elements $g_t(\omega)$ and ρ.

3.3. We show that $K^2\times\mathbf{R}^2$ "foliates" into a collection $\{\mathcal{S}_\beta \mid \beta\in P^1\}$ of measurable, one-dimensional, invariant subbundles (i.e., $K^2\times\mathbf{R}^2=\bigcup_{\beta\in P^1}\mathcal{S}_\beta$). Let $r(\omega)=e^{iR(\omega)}$; then $r(\omega\cdot t)/r(\omega)=g_t(\omega)$. Hence, if $\beta\in P^1$, then each set

$$S_\beta=\{(\omega,\rho)\mid \omega\in K^2,\quad \rho=\beta\cdot r(\omega)\}$$

is invariant. Here $\beta\cdot r(\omega)$ is the product of the elements β and $r(\omega)$ of P^1. For each $\omega\in K^2$, consider the line in $\{\omega\}\times\mathbf{R}^2$ which is defined by $\beta\cdot r(\omega)\in P^1$. The union $\mathcal{S}_\beta$ of all such lines is a measurable, invariant subbundle of $K^2\times\mathbf{R}^2$. Also, $\mathcal{S}_\beta$ is not a continuous subbundle. For, if it were, then the set S_β would be closed, which would imply that r is continuous. That would contradict 2.10(a). The collection $\{\mathcal{S}_\beta \mid \beta\in P^1\}$ of subbundles clearly foliates $K^2\times\mathbf{R}^2$.

3.4. We elaborate briefly on the remark in the Introduction concerning nonexistence of measurable, invariant subbundles for periodic ODEs. For simplicity, we consider only the two-dimensional case, though the discussion can be generalized to dimension n.

Let

$$(*)\qquad \dot{x}=a_{ij}(t))x$$

be a 2-dimensional ODE such that each $a_{ij}(t)$ has period p. Let $\Phi(t)=P(t)e$ be the Floquet representation of a fundamental matrix $\Phi(t)$ for $(*)$. Assume

$\Phi(0)=I$. The ODE (*) generates an LSPF on $K\times\mathbf{R}^2$, where in this case the base K is a circle [14], [15]. The LSPF $(K\times\mathbf{R}^2,\mathbf{R})$ induces a flow on $K\times P^1\cong K^2$. It may be shown that this flow $(K\times P^1,\mathbf{R})$ is independent of

$$\operatorname{trace}(a_{ij}(t))=a_{11}(t)+a_{22}(t);$$

we assume $a_{11}(t)+a_{22}(t)=0$.

Consider the period matrix $\Phi(p)$. There is a real matrix A such that $A\Phi(p)A^{-1}$ take one of the following four forms:

(1) $$\begin{pmatrix}\cos 2\pi\theta & \sin 2\pi\theta\\ -\sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}\quad \text{where } \theta \text{ is irrational;}$$

(2) $$\begin{pmatrix}\cos 2\pi\theta & \sin 2\pi\theta\\ -\sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}\quad \text{where } \theta \text{ is rational;}$$

(3) $$\begin{pmatrix}\lambda & 0\\ 0 & \dfrac{1}{\lambda}\end{pmatrix}\quad \text{where}\quad \lambda\neq 0;$$

(4) $$\begin{pmatrix}\lambda & 0\\ 1 & \lambda\end{pmatrix}\quad \text{where } \lambda=\pm 1.$$

Now, invariant one-dimensional subbundles of $K\times\mathbf{R}^2$ correspond naturally to subsets of $K\times P^1$ which are invariant with respect to $(K\times P^1,\mathbf{R})$, and which intersect each fiber $\{\omega\}\times P^1$ in exactly one point (call such a subset an "invariant section"). In case (1), the flow $(K\times P^1,\mathbf{R})$ is conjugate to an irrational (Kronecker) flow on $K^2\cong K\times P^1$, and the conjugacy preserves Lebesgue measure. But an irrational flow admits no measurable invariant section (if there were such a section, one could use it to define a measure on K^2 invariant under the irrational flow, but unequal to Lebesgue measure). So there are no measurable, invariant, one-dimensional subbundles in case (1). In cases (3) and (4), there may be shown to be, respectively, 2 and 1 measurable invariant sections; these are all homeomorphs of S^1, hence correspond to continuous subbundles. Finally, consider case (2). If $\theta=0$, there are infinitely many measurable invariant sections, all of which are continuous. If $\theta\neq 0$, there are no measurable invariant sections; instead, $K\times P^1$ foliates into a union of torus knots.

4. A strictly ergodic flow $(K^3,\mathbf{R})$

4.1. DEFINITIONS. Let $(\Omega,\mathbf{R})$ be a flow, with Ω compact Hausdorff. A measure μ on Ω is *invariant* if $\mu(\Omega)=1$ and $\mu(B\cdot t)=\mu(B)$ for all Borel sets $B\subset\Omega$ $(t\in\mathbf{R})$. It is *ergodic* if, in addition, $\mu(Bt\,\Delta\,B)=0$ $(t\in\mathbf{R})$ implies $\mu(B)=0$ or $\mu(B)=1$. The flow $(\Omega,\mathbf{R})$ is *uniquely ergodic* (u.e.) if it has a unique invariant measure (which is then necessarily ergodic; see [11, Chapter 6, 9.05 and 9.20]. If $(\Omega,\mathbf{R})$ is u.e. and minimal, then it is *strictly ergodic* (s.e.).

Let α, R, G_t, and b be as in §2. Let

$$r(\omega)=e^{iR(\omega)}, \quad g_t(\omega)=\exp\left(i\int_0^t b(\omega\cdot s)\,ds\right).$$

Choose a real number λ_1 such that $n\lambda_1\neq 2\pi(\alpha p+q)$ for all $n, p, q\in\mathbf{Z}$ $(n\neq 0)$. Define a flow $(K^3, \mathbf{R})$ as follows: $(\omega, \rho)\cdot t=(\omega\cdot t, e^{\lambda_1 it}g_t(\omega)\cdot\rho)$ (here $\omega\in K^2$, $\rho\in S^1$). If $t\in\mathbf{R}$, let T_t denote the homeomorphism $(\omega, \rho)\to(\omega, \rho)\cdot t$. We will first show that some integer flow (K^3, T_t) is s.e. This implies that $(K^3, \mathbf{R})$ is s.e. We then find times $t_k\to 0$ such that (K^3, T_k) is not u.e. Finally, we find an analytic function f such that the Cesaro sums of f with respect to T_k diverge on a residual set for all $k\geq 1$.

4.2. Proposition. *Let $t\in\mathbf{R}$, and consider the integer flow (K^3, T_t) where*

$$T_t(\omega, \rho)=(\omega\cdot t, h(\omega)\rho)$$

with $h(\omega)=e^{\lambda_1 it}g_t(\omega)$. Then (K^3, T_t) is u.e. if and only if the equation

(C) $\quad h^n(\omega)=s(\omega\cdot t)/s(\omega)\quad m$-a.e. $\quad(t\in\mathbf{R})$; $\quad|s(\omega)|=$ *non-zero constant,*

has no m-measurable solution s for any integer $n\neq 0$. In any case, normalized Haar measure μ on K^3 is invariant with respect to (K^3, T_t).

For the proof of a more general result, see [5, Lemma 2.1] (Furstenberg's "s.e." is the same as our "u.e."). Note that the flow $(K^2, \tilde{T}_t)$, $\tilde{T}_t(\omega)=\omega\cdot t$, is u.e. (and s.e.); m is the unique invariant measure.

4.3. Proposition. *The flow (K^3, T_t) of 4.2 is minimal if and only if* (C) *has no continuous solution s for any $n\neq 0$.*

For the proof of a more general statement, see [5, Theorem 1]. See also the proof of Lemma 1.4.

4.4. Proposition. *There exists a time t such that (K^3, T_t) is s.e.*

Proof. We must show that some (K^3, T_t) is u.e. and minimal. We first seek to apply 4.2. Fix $t\neq 0$, and let $\beta=e^{\lambda_1 it}$. Then

$$T_t(\omega, \rho)=(\omega\cdot t, \beta g_t(\omega)\rho).$$

Suppose

$$\beta^n[g_t(\omega)]^n=s(\omega\cdot t)/s(\omega)\quad(n\neq 0)$$

for some measurable s. Since $[g_t(\omega)]^n=r^n(\omega\cdot t)/r^n(\omega)$ (2.11), we have

$$\beta^n=[s(\omega\cdot t)/r^n(\omega\cdot t)]\cdot[r^n(\omega)/s(\omega)].$$

Let $u(\omega)=s(\omega)/r^n(\omega)$; then $u(\omega\cdot t)=\beta^n u(\omega)$. Comparing coefficients in the Fourier expansions of $u(\omega\cdot t)$ and $\beta^n u(\omega)$, we see that there are integers p,

q such that $e^{in\lambda_1 t} = \beta^n = e^{2\pi i t(\alpha p+q)}$. Hence

$$\text{(D)} \qquad \frac{n\lambda_1}{2\pi} = (\alpha p + q)t \pmod 1.$$

By choice of λ_1, we can find a t for which there are no n, p, q such that (D) holds. By 4.2, (K^3, T_t) is u.e. To see that (K^3, T_t) is minimal, observe that equation (C) has no measurable solution, hence no continuous solution, and apply 4.3. ■

4.5. PROPOSITION. *If*

$$t = l\Big/\left(\frac{n\lambda_1}{2\pi} - \alpha p - q\right) \quad (l, n, p, q \in \mathbf{Z};\ n \neq 0, l \neq 0),$$

then (K^3, T_t) *is not u.e.*

Proof. Observe that we obtain the above equalities by solving (D) for t. Let

$$\tau_{pq}(\omega) = e^{2\pi i(p\theta_1 + q\theta_2)} \quad \text{where} \quad \omega = (e^{2\pi i\theta_1}, e^{2\pi i\theta_2}).$$

Let $c(\omega, \rho) = \tau_{pq}(\omega) \cdot r^n(\omega) \cdot \rho^{-n}$. Then c is T_t-invariant, i.e.,

$$c(T_t(\omega, \rho)) = c(\omega, \rho)(\omega, \rho) \in K^3).$$

To see this, note that $r(\omega \cdot t) = r(\omega) g_t(\omega)$. So

$$\begin{aligned} c(T_t(\omega, \rho)) &= e^{2\pi i t(\alpha p+q)} \tau_{pq}(\omega) r^n(\omega)[g_t(\omega)]^n e^{-in\lambda_1 t}[g_t(\omega)]^{-n}\rho^{-n} \\ &= e^{2\pi i t(\alpha p+q)} e^{2\pi i l} \tau_{pq}(\omega) r^n(\omega) e^{-in\lambda_1 t}\rho^{-n} \\ &= c(\omega, \rho) \quad \text{(use the formula for } t). \end{aligned}$$

Now, by 4.2, Haar measure μ on K^3 is invariant for (K^3, T_t). But we have just found a function c which is μ-measurable, T_t-invariant, and not constant μ-a.e. Hence [11, p. 468] μ is not the only invariant measure for (K^3, T_t). ■

4.6. *Remarks.* (a) Let t be as in 4.5. Then (K^3, T_t) is minimal. We will prove this by using 4.3. First, let $s(\omega) = \tau_{pq}(\omega) \cdot r^n(\omega)$. Then (C) holds; i.e.,

$$h^n(\omega) = s(\omega \cdot t)/s(\omega).$$

Note that s^j is not equal m-a.e. to a continuous function for any integer $j \neq 0$, because r^{nj} is not (2.10(a)). Now, suppose $h^j(\omega) = q(\omega \cdot t)/q(\omega)$ for some continuous function q and some $j \neq 0$. Then

$$q^n(\omega \cdot t)/s^j(\omega \cdot t) = q^n(\omega)/s^j(\omega) \quad m\text{-a.e.}$$

This implies that $q^n(\omega)/s^j(\omega)$ is constant m-a.e. [11, p. 468] and [6, p. 25]. This means that $s^j(\omega)$ is a continuous function m-a.e., which contradicts the statement about s^j made above.

(b) Let t be as in 4.5. We show that (K^3, T_t) has uncountably many ergodic measures. For each $\beta \in S^1$, let $\lambda_\beta(\omega) = \beta \cdot s(\omega)(s(\omega) = \tau_{pq}(\omega) \cdot r^n(\omega))$. Define

$$\mu_\beta(f) = \int_{K^2} f(\omega, \lambda_\beta(\omega))\, dm(\omega) \quad (f \in C(K^3)).$$

It is easy to see that each μ_β is (K^3, T_t)-ergodic.

4.7. PROPOSITION. *$(K^3, \mathbf{R})$ is s.e.; Haar measure μ is the unique invariant measure.*

Proof. The first statement follows from 4.4, the second from 4.2. ∎

Thus $(K^3, \mathbf{R})$ is s.e., but there exist times t such that (K^3, T_t) is minimal and has uncountably many ergodic measures.

4.8. PROPOSITION. *There exist times $t_k \to 0$ and an analytic function $f(\omega, \rho)$ such that:*

(a) *If T_k is the time-t_k map, then (K^3, T_k) is not u.e.*
(b) *For each k, the Cesaro sums*

$$\frac{1}{l} \sum_{i=1}^{l} f \circ (T_k)^i(\omega, \rho)$$

diverge for a residual set V of $(\omega, \rho) \in K^3$.

Proof. Referring to 4.5, let

$$t_k = 1 \Big/ \left(\frac{\lambda_1}{2\pi} - \alpha + k\right) \quad (k \geq 1).$$

By 4.5, (K^3, T_k) is not u.e. It may be verified that, if $\beta_k = e^{i\lambda_1 t_k}$, then $\beta_k g_t(\omega) = s_k(\omega)$, where $s_k(\omega) = \tau_{1k}(\omega)$. Let $\varepsilon\colon K^2 \to \mathbf{R}$ be an analytic function such that

$$\int_{K^2} \varepsilon(\omega) s_k(\omega)\, dm(\omega) \neq 0$$

for all $k \geq 1$ (such a function clearly exists). Let $f(\omega, \rho) = \varepsilon(\omega)\rho$. Then for fixed k, the sums

$$\frac{1}{l} \sum_{i=1}^{l} f \circ (T_k)^i(\omega, \rho)$$

do not all converge; the proof is line for line that given on p. 584 of [5], and is hence omitted. By [9, Remark 4.4(2)], the sums diverge for residual set V_k of $(\omega, \rho) \in K^3$. Let $V = \bigcap_{k=1}^{\infty} V_k$. ∎

We have shown that $(K^3, \mathbf{R})$ has all the properties stated in the Introduction.

5. The second ODE

5.1. Consider the collection of ODEs (λ_1 as in §4) E_ω given by

$$\dot{x}=\frac{1}{2}\begin{bmatrix} 0 & -\lambda_1-b(\omega\cdot t) \\ \lambda_1+b(\omega\cdot t) & 0 \end{bmatrix}x \quad (\omega\in K^2, x\in \mathbf{R}^2).$$

In polar coordinates (r, θ), E_ω is given by

$$\dot{r}=0, \quad \dot{\theta}=\tfrac{1}{2}(\lambda_1+b(\omega\cdot t)).$$

As in §3, equations E_ω define an LSPF on $K^2\times\mathbf{R}^2$, which then induces a flow on $K^2\times P^1\cong K^3$. As in §3, we coordinatize P^1 with $\rho=e^{2i\theta}$. It then turns out that the flow $(K^2\times P^1, \mathbf{R})$ is exactly the flow $(K^3, \mathbf{R})$ considered in §4.

5.2. PROPOSITION. *The flow $(K^2\times\mathbf{R}^2, \mathbf{R})$ has no measurable, invariant, one-dimensional subbundles.*

Proof. Suppose $\mathcal{S}_0$ is measurable, invariant, one-dimensional subbundle. Let S_0 be the subset of $K^2\times P^1$ induced by $\mathcal{S}_0$ (more precisely,

$$S_0=\eta(\mathcal{S}_0\sim(K^2\times\{0\})),$$

where η is the map of 3.3). Let $\lambda(\omega)$ be the point of intersection of S_0 and $\{\omega\}\times P^1\subset K^2\times P^1$. We can then define a $(K^2\times P^1, \mathbf{R})$-invariant measure μ_0 by

$$\mu_0(f)=\int_{K^2} f(\omega, \lambda(\omega))\,dm(\omega)(f\in C(K^2\times P^1)).$$

Since μ_0 is not equal to Lebesgue measure, 4.7 is contradicted.

5.3. PROPOSITION. *Let k be any number of the form*

$$1\Big/\left(\frac{\lambda_1}{2\pi}-\alpha p+q\right)(p, q\in Z).$$

Let T_k be the time-t_k map on $K^2\times P^1$. Then $K^2\times\mathbf{R}^2$ foliates into measurable, non-continuous, T_k-invariant, one-dimensional subbundles.

Proof. Use the reasoning of 3.3, with the function $r(\omega)$ in 3.3 replaced by

$$s(\omega)=\tau_{pq}(\omega)\cdot r(\omega).$$

One obtains subsets $S_\beta=\{(\omega, \beta\cdot s(\omega))\mid\omega\in K^2\}\subset K^2\times P^1$ $(\beta\in P^1)$ which foliate $K^2\times P^2$. The subbundles $\mathcal{S}_\beta=\eta^{-1}(S_\beta)$ foliate $K^2\times\mathbf{R}^2$. ■

4.4. *Remark.* Fix p and q in 4.3. Write $\beta\cdot r(\omega)=e^{2i\varphi_\beta(\omega)}$. Then the intersection of $\mathcal{S}_\beta$ with the fiber $\{\omega\}\times\mathbf{R}^2$ is a straight line which makes angle

$$\varphi_\beta(\omega)+p\theta_1+q\theta_2$$

with the positive x-axis, where $\omega=(e^{2\pi i\theta_1}, e^{2\pi i\theta_2})$.

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