ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 3, September, 1979

SMOOTH FUNCTIONS AND CONVERGENCE OF SINGULAR INTEGRALS

To the memory of N. M. Rivière

BY
CALIXTO P. CALDERÓN[1]

1. Introduction and statement of the main result

Let $x=(x_1, x_2, \ldots, x_n)$, $y=(y_1, y_2, \ldots, y_n)$ be points of the real n-dimensional Euclidean Space R^n and let $x'=|x|^{-1}x$ be a point in the unit sphere of R^n, $|x|=(\sum_1^n x_i^2)^{1/2}$. Let $K(x)$ be a positively homogeneous kernel of degree $-n$, that is

$$K(x)=|x|^{-n}K(x'), \quad x\neq 0. \tag{1.1}$$

The L^1-modulus of continuity of the kernel K is defined by

$$\omega_K(s)=\sup_{h;|h|\leq s}\int_{2<|x|<4}|K(x+h)-K(x)|\,dx, \quad |s|<1. \tag{1.2}$$

The L^1-modulus of continuity of $f\in L^1(R^n)$ is defined by

$$\omega(f,s)=\omega(s)=\sup_{h;|h|\leq s}\int_{R^n}|f(x+h)-f(x)|\,dx. \tag{1.3}$$

We are going to assume that the kernel K satisfies the following properties:

$$\text{(1.4)} \qquad \begin{aligned} &\text{(i)} \quad \int_\Sigma K(x')\,d\sigma=0 \\ &\text{(ii)} \quad \int_\Sigma |K(x')|\log^+|K(x')|\,d\sigma<\infty. \end{aligned}$$

If the kernel K is odd we assume the weaker condition

$$\text{(iii)} \quad \int_\Sigma |K(x')|\,d\sigma<\infty, K \text{ odd} \tag{1.5}$$

where Σ denotes the unit sphere and $d\sigma$ its "area" element. Throughout this paper we shall be concerned with operators defined by

$$\text{p.v.}\int_{R^n}K(x-y)f(y)\,dy \tag{1.6}$$

where K satisfies properties (i) and (ii) in (1.4), or (iii) in (1.5).

Received February 8, 1978.
[1] Research partially supported by the National Science Foundation.

THEOREM A. *Suppose that K satisfies properties* (i) *and* (ii) *of* (1.4), *or property* (iii) *of* (1.5). *Let $f(x)$ be a function in $L^1(R^n)$. Suppose that the L^1-moduli of continuity of f and K satisfy the Dini condition*

$$\int_0^1 \omega_K(s)\omega(s)\frac{ds}{s}<\infty. \tag{1.7}$$

Then

$$\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} K(x-y)f(y)\,dy \quad \textit{exists} \quad \text{a.e.,} \tag{1.8}$$

and moreover, the maximal operator $\sup_{\epsilon>0}\left|\int_{|x-y|>\epsilon} K(x-y)f(y)\,dy\right| = K^*(f)$ *satisfies*

$$|E(K^*(f)>\lambda)\cap Q_0|<\frac{C_1}{\lambda}\|f\|_1+\frac{C_2}{\lambda}\int_0^1 \omega_K(s)\omega(s)\frac{ds}{s}. \tag{1.9}$$

where Q_0 denotes an n-dimensional cube and the constants C_1 and C_2 depend on n, Q_0 and K but not on λ or f.

2. Auxiliary lemmas

2.1. LEMMA. *Let $T(r)\geq 0$ be a nonincreasing radial function belonging to $L^1(R^n)$. Let f be a nonnegative measurable function, locally integrable in R^n. Define the following operators:*

$$m(f)(x)=\inf_{S(x)}\frac{1}{|S(x)|}\int_{S(x)} f(t)\,dt, \quad S(x)\supset S_0(x), \tag{2.1.1}$$

$$m_0(f)(x)=\inf_{Q(x)}\frac{1}{|Q(x)|}\int_{Q(x)} f(t)\,dt, \quad Q(x)\supset \tfrac{1}{2}S_0(x), \tag{2.1.2}$$

where the infima are taken over all spheres $S(x)$ centered at x such that their radii are greater than r_0 and over all cubes $Q(x)$ centered at x and with edges parallel to the coordinate axes that contain the sphere of radius $\frac{1}{2}r_0$ about x. Under the above assumptions, the following estimates hold:

(i) $\displaystyle\int_{R^n} T(|y|)f(x-y)\,dy \geq \left(\int_{|y|>r_0} T(|y|)\,dy\right)m(f)(x),$

(ii) $\displaystyle\int_{R^n} T(|y|)f(x-y)\,dy \geq \left(\int_{|y|>r_0} T(|y|)\,dy\right)C_n m_0(f)(x).$

Proof. An integration by parts shows

$$\int_{R^n} T(|y|)f(x-y)\,dy \geq -\Gamma_n\int_{r_0}^{\infty}\frac{1}{r^n\Gamma_n}\left(\int_{|x-y|<r} f(x-y)\,dy\right)r^n\,dT(r) \tag{2.1.3}$$

where Γ_n stands for the volume of the n-dimensional unit ball. Using the fact that

$$\frac{1}{r^n \Gamma_n} \int_{|x-y|<r} f(y)\, dy \geq m(f)(x), \quad r > r_0,$$

(2.1.3) gives (i) directly.

The inequality $m(f)(x) \geq C_n m_0(f)(x)$, C_n depending on n only, gives (ii).

2.2. Lemma. *Let f be an L^1 function and $\omega(t)$ its L^1-modulus of continuity. Suppose that there exists a continuous function $\omega_0(t)$, defined for $t \geq 0$, such that*

$$(\beta) \quad \omega_0(0) = 0,$$

$$(2.2.1) \qquad (\beta\beta) \quad \frac{\omega_0(t)}{t} \text{ is nonincreasing,}$$

$$(\beta\beta\beta) \quad \frac{\omega_0(t)}{t} < C \frac{\omega_0(2t)}{2t}.$$

Assume also that $\omega_0(t)$ and $\omega(t)$ satisfy the following integrability conditions:

$$(2.2.2) \qquad (\gamma) \quad \int_1^\infty \omega_0(t) \frac{dt}{t} < \infty; \qquad (\delta) \quad \int_0^1 \omega_0(t)\omega(t) \frac{dt}{t} < \infty.$$

Then f admits the following decomposition. For each positive $\lambda > 0$, there exists a function $\bar{f}$ that satisfies:

(i) $|\bar{f}| < C_1 \lambda$ a.e.,

(ii) *$\bar{f} = f$ on a closed set F; its complement $G(\lambda)$ has measure*

$$|G_\lambda| < \frac{C_2}{\lambda}\left[\|f\|_1 + \iint_{R^n \times R^n} \frac{\omega_0(|x-y|)}{|x-y|^n} |f(x) - f(y)|\, dx\, dy\right].$$

(iii)

$$\iint_{R^n \times R^n} |\bar{f}(x) - \bar{f}(y)| \frac{\omega_0(|x-y|)}{|x-y|^n}\, dx\, dy \leq C_3\left(\|f\|_1 + \iint_{R^n \times R^n} |f(x) - f(y)| \frac{\omega_0(|x-y|)}{|x-y|^n}\, dx\, dy\right).$$

(iv)

$$\iint_{R^n \times R^n} |\bar{f}(x) - \bar{f}(y)|^2 \frac{\omega_0(|x-y|)}{|x-y|^n}\, dx\, dy \leq C_4 \lambda\left[\iint_{R^n \times R^n} |f(x) - f(y)| \frac{\omega_0(|x-y|)}{|x-y|^n}\, dx\, dy + \|f\|_1\right].$$

(v)

$$\int_{R^n} |\bar{f}|^2\, dx \leq C_5 \lambda\left[\|f\|_1 + \iint_{R^n \times R^n} |f(x) - f(y)| \frac{\omega_0(|x-y|)}{x-y|^n}\, dx\, dy\right].$$

Here $C_1, C_2, \ldots, C_5$ do not depend on λ or f.

Proof. Let us fix $\lambda > 0$ and consider the sets

(2.2.3) $$G_1(\lambda) = \{x; f^*(x) > \lambda\}, \qquad G_2(\lambda) = \{x; \beta^*(x) > \lambda\},$$

where $f^*(x)$ and $\beta^*(x)$ stand for the maximal functions of $f(x)$ and $\beta(x)$ respectively. The maximal function being used is

(2.2.4) $$A^*(x) = \sup_{Q(x)} \frac{1}{|Q(x)|} \int_{Q(x)} |A(y)|\, dy$$

where the $Q(x)$ are cubes centered at x with edges parallel to the coordinate axes.

The auxiliary function $\beta(x)$ is defined by

(2.2.5) $$\beta(x) = \int_{R^n} \frac{\omega_0(|x-y|)}{|x-y|^n} |f(x) - f(y)|\, dy.$$

The exceptional set $G(\lambda)$ is going to be defined by $G(\lambda) = G_1(\lambda) \cup G_2(\lambda)$. Consider a Whitney covering for G (for details see [9, Chapter VI, Section I]. Thus G is expressed as $\bigcup_1^\infty Q_k$ and the covering possesses the following properties:

(2.2.6) (α) $Q_i^0 \cap Q_j^0 = \phi, \quad i \neq j.$

$(\alpha\alpha)$ diam $(Q_k) \leq$ distance $(Q_k, F) \leq 4$ diam (Q_k).

$(\alpha\alpha\alpha)$ If Q_i and Q_j are adjacent then there exists two universal constants C_1 and C_2 such that C_1 diam $(Q_j) \leq$ diam $(Q_i) \leq C_2$ diam (Q_j).

$(\alpha$v$)$ If Q_i and Q_j do not touch, then distance $(Q_i, Q_j) \geq$ diam (Q_s), $s = i, j$.

Our next step is to define $\bar{f}(x)$:

(2.2.7) $$\begin{aligned} \bar{f}(x) &= f(x) \quad \text{on} \quad F, \\ &= \mu_k \quad \text{on} \quad Q_k, \quad \text{where} \quad \mu_k = 1/|Q_k| \int_{Q_k} f\, dt, \\ &\qquad\qquad\qquad\qquad k = 1, 2, \ldots, m, \ldots \end{aligned}$$

Clearly, from properties (2.2.6), we have $|\mu_k| < C\lambda$, with C depending on n only; hence

(2.2.8) $$|\bar{f}| < C\lambda.$$

Our next steps will be to estimate

$$\int_{R^n} \int_{R^n} \frac{\omega_0(|x-y|)}{|x-y|^n} |\bar{f}(x) - \bar{f}(y)|\, dx\, dy = \int_F \bar{\beta}(x)\, dx + \int_G \bar{\beta}(x)\, dx.$$

Estimate for $\int_F \bar{\beta}(x)\,dx$. From the definition of $\bar{f}(x)$ we get

$$\int_F \bar{\beta}(x)\,dx = \int_F \int_F \frac{\omega_0(|x-y|)}{|x-y|^n}|f(x)-f(y)|\,dy\,dx + \int_F \left(\sum_1^\infty |f(x)-\mu_k| \int_{Q_k} \frac{\omega_0(|x-y|)}{|x-y|^n}\,dy\right)dx. \tag{2.2.9}$$

Using properties (2.2.1) and the fact that $x \in F$ we have

$$\begin{aligned}\sum_1^\infty |f(x)-\mu_k| \int_{Q_k} \frac{\omega_0(|x-y|)}{|x-y|^n}\,dy &\le C' \sum_1^\infty |f(x)-\mu_k|\,|Q_k| \frac{\omega_0(|x-y_k|)}{|x-y_k|^n} \\ &= C' \sum_1^\infty \frac{\omega_0(|x-y_k|)}{|x-y_k|^n} \left| \int_{Q_k} (f(x)-f(y))\,dy \right| \\ &\le C'' \sum_1^\infty \int_{Q_k} \frac{\omega(|x-y|)}{|x-y|^n} |f(x)-f(y)|\,dy\end{aligned} \tag{2.2.10}$$

where y_k stands for the center of Q_k. Taking into account (2.2.9) and (2.2.10) we get

$$\int_F \bar{\beta}(x)\,dx \le (1+C'') \int_F \beta(x)\,dx \tag{2.2.11}$$

where C'' does not depend on λ or f.

Estimates for $\int_G \bar{\beta}(x)\,dx$. Consider

$$\int_G \bar{\beta}(x)\,dx = \int_G \left(\int_F \frac{\omega_0(|x-y|)}{|x-y|^n} |\bar{f}(x)-\bar{f}(y)|\,dy \right) dx + \int_G \left(\int_G \frac{\omega_0(|x-y|)}{|x-y|^n} |\bar{f}(x)-\bar{f}(y)|\,dy \right) dx. \tag{2.2.12}$$

Let us interchange the order of integration in the first term of the right-hand member of (2.2.12). It is readily seen to be dominated by $\int_F \bar{\beta}(y)\,dy$; thus

$$\int_G \left(\int_F |\bar{f}(x)-\bar{f}(y)| \frac{\omega_0(|x-y|)}{|x-y|^n}\,dy \right) dx \le (1+C'') \int_F \beta(y)\,dy. \tag{2.2.13}$$

The second term on the right-hand member of (2.2.12) reduces to

$$\sum_{i,k} |\mu_i - \mu_k| \int_{Q_i} \int_{Q_k} \frac{\omega_0(|x-y|)}{|x-y|^n}\,dy\,dx. \tag{2.2.14}$$

Let us fix i and consider the subindices s such that Q_s touches Q_i and the subindices v such that Q_v does not touch Q_i. First we are going to estimate

(2.2.15)
$$\sum_s |\mu_i - \mu_s| \int_{Q_i} \int_{Q_s} \frac{\omega_0(|x-y|)}{|x-y|^n}\, dy\, dx.$$

From property (2.2.6) it follows that there are at most N different Q_s (here, N depends on the dimension only). Using the fact that $|\mu_i - \mu_s| < 2C\lambda$ we see that (2.2.15) is dominated by

(2.2.16)
$$\sum_s 2C\lambda \int_{Q_s} dy \int \frac{\omega_0(|y-x|)}{|y-x|^n} |\phi_i(y) - \phi_i(x)|\, dx$$

where ϕ_k stands for the characteristic function of Q_k. From properties (2.2.6) it follows that there exists a factor l (depending on the dimension only) such that

(2.2.17)
$$Q_s \subset lQ_i$$

where lQ_i stands for the dialation of Q_i l times about its center. By this last remark, we have (2.2.16) dominated by

(2.2.18)
$$2C\lambda \int_{lQ_i} dy \int \frac{\omega_0(|x-y|)}{|x-y|^n} |\phi_i(x) - \phi_i(y)|\, dx$$

which, in turn, is dominated by

(2.2.19)
$$2C\lambda \int_{|t| \le 4l \operatorname{diam}(Q_i)} \frac{\omega_0(t)}{|t|^n} \left(\int |\phi_i(x) - \phi_i(x-t)|\, dx \right) dt \le 2C\lambda \cdot \text{constant} \cdot |Q_i|.$$

Consequently, (2.2.15) is dominated by

(2.2.20)
$$\text{Const}\ \lambda |Q_i|.$$

Our next step will be to estimate

(2.2.21)
$$\int_{Q_i} \left(\sum_v |\mu_i - \mu_v| \int_{Q_v} \frac{\omega_0(|x-y|)}{|x-y|^n}\, dy \right) dx.$$

Now, we shall make use of (2.2.6) (αv) and properties (2.2.1) and get the following estimate for (2.2.21):

(2.2.22)
$$\text{Const} \sum_v \frac{\omega_0(|y_i - y_v|)}{|y_i - y_v|^n} \int_{Q_v} |\mu_i - f(y)|\, dy$$
$$\le \text{Const} \sum_v \int_{Q_i} dx \int_{Q_v} |f(x) - f(y)| \frac{\omega_0(|x-y|)}{|x-y|^n}\, dy$$
$$\le \text{Const} \int_{Q_i} \left(\int_G |f(x) - f(y)| \frac{\omega_0 |x-y|)}{|x-y|^n}\, dy \right) dx.$$

Inequalities (2.2.20)–(2.2.22) give

$$(2.2.23)\qquad \iint_{G\times G} |\bar f(x)-\bar f(y)|\,\frac{\omega_0(|x-y|)}{|x-y|^n}\,dx\,dy \le C\Big(\lambda\,|G_\lambda| + \int_{R^n} \beta(x)\,dx\Big).$$

By the size of G_λ and the estimates (2.2.11) and (2.2.13) we obtain (iii) of the thesis. The other parts are easy consequences of this one and will be left to the reader.

DEFINITION. Let $\phi(t)$ denote the function

$$\left(\int_t^1 \frac{\omega_0(s)}{s}\,ds\right) B(t)$$

where $B(t)$ is the characteristic function of the interval $[0, 1]$. Let $\omega_0(s)$ denote a function coincident with the L^1-modulus of continuity of K if $0<s\le 1$ and extended for values of $s>1$, so that properties (2.2.1) and (2.2.2) (γ) are met.

2.3. LEMMA. *Let $f(x)$, λ and $\bar f(x)$ be the functions and the real parameter of Lemma 2.2. Let $\varphi(x)=f(x)-\bar f(x)$. Then it is possible to find a sequence of cubes $\{A_k\}$ that satisfy:*

(i) $\bigcup_1^\infty A_k \supset G(\lambda)$ *where $G(\lambda)$ is the set introduced in Lemma 2.2.*
(ii) *Each point in R^n belongs to at most N_n different cubes and*

$$\sum_1^\infty |A_k| < C_n^{(1)}\,|G(\lambda)|$$

where the constants $C_n^{(1)}$ and N_n depend on the dimension only.

(iii) $\int_{A_k} |\varphi(y)|\,dy < C_n^{(2)}\lambda\,|A_k|$, $k=1, 2, \ldots$, *where $C_n^{(2)}$ depends on n only.*

(iv) $$\sum_1^\infty \phi(|A_k|^{1/n}) \int_{A_k} |\varphi(y)|\,dy \le C_n^{(3)}\left(\|f\|_1 + \iint_{R^n\times R^n} \frac{\omega_0(|x-y|)}{|x-y|^n}\,|f(x)-f(y)|\,dx\,dy\right)$$

where $C_n^{(3)}$ is independent of λ, $G(\lambda)$ and f.

Proof. Consider f, $\lambda>0$, $G(\lambda)$ and $\bar f$ as introduced in Lemma 2.2 and $\phi(t)$, $\omega_0(t)$ as defined above. We shall define the following covering for $G(\lambda)$: For each $x\in G(\lambda)$ we are going to select a cube centered at x, with edges parallel to the coordinate axes and such that

$$(2.3.1)\qquad \frac{|G\cap Q(x)|}{|Q(x)|} = \left(\frac{1}{10}\right)^n.$$

If $Q'(x)$ is any other cube centered at x such that $Q' \supset Q$ then

$$\frac{|G \cap Q'|}{|Q'|} \le \left(\frac{1}{10}\right)^n \tag{2.3.2}$$

and consequently if Q'' is any cube such that $Q'' \supset Q(x)$ we have

$$\frac{|G \cap Q''|}{|Q''|} \le \left(\frac{2}{5}\right)^n. \tag{2.3.3}$$

From (2.3.1) we have, trivially, $|Q(x)| \le (10)^n\, |G(\lambda)|$. Let us divide R^n into a mesh of cubes that are nonoverlapping and have volume $4^n(10)^n\, |G(\lambda)|$. Call them J_j and consider the sets $G(\lambda) \cap J_j$, $j = 1, 2, \ldots, m, \ldots$. Each set $G(\lambda) \cap J_j$ is bounded and, moreover, is covered by members of the family $\{Q\}$. Apply Lemma 2a in [5, p. 60] to each set $G(\lambda) \cap J_j$ and get

$$\bigcup_{k=1}^{\infty} Q_k^{(j)} \supset G(\lambda) \cap J_j. \tag{2.3.4}$$

Each point of R^n belongs to at most 4^n different cubes $Q_k^{(j)}$. By construction we have

$$G(\lambda) \subset \bigcup_{j,k} Q_k^{(j)}. \tag{2.3.5}$$

Since $|Q_k^{(j)}| \le 4^n(10)^n\, |G(\lambda)| = |J_s|$, each point in J_s could be covered by cubes $\{Q_k^{(s)}\}$ or by cubes associated with the $3^n - 1$ neighboring J_j. Thus, each point in R^n belongs to at most $4^n \cdot 3^n$ different $Q_k^{(j)}$. Let us relabel the cubes $Q_k^{(j)}$ as A_k. By construction, parts (i), (ii) and (iii) are satisfied. It remains to show (iv).

Let us denote by F the complement of G and by $T(|x|)$ the kernel $\omega_0(|x|)/|x|^n$. We have

$$\begin{aligned} \iint_{R^n \times R^n} T(|x-y|)\, |\varphi(x) - \varphi(y)|\, dy &\le \int_G |\varphi(y)| \left\{ \int_F T(|x-y|)\, dx \right) dy \\ &\ge \left(\frac{1}{12}\right)^n \sum_1^\infty \int_{A_k} |\varphi(y)|\, dy \int_F T(|x-y|)\, dx. \end{aligned} \tag{2.3.6}$$

If $y \in A_k$ and $\Psi(x)$ denotes the characteristic function of F, we have

$$\int_{A_k} |\varphi(y)|\, dy \int_F T(|x-y|)\, dx \ge \int_{A_k} |\varphi(y)|\, dy \int T_k(|x-y|)\Psi(x)\, dx \tag{2.3.7}$$

where $T_k(s)=T(s)$ if $|s|>4 \operatorname{diam} A_k$ and $T_k(s)=T(4 \operatorname{diam} A_k)$ if $|s|\leq \operatorname{diam} A_k$. By (2.3.3) and Lemma 2.1 we have

$$\text{(2.3.8)} \quad \int_{A_k} |\varphi(y)|\,dy\left(\int T_k(|x-y|)\Psi(x)\,dx\right.$$
$$\geq C_n\int_{A_k} |\varphi(y)|\left[1-\left(\frac{2}{5}\right)^n\right]\int_{|A_k|^{1/n}}^{1} \frac{\omega_0(t)}{t}\,dt$$
$$= C_n\left(\int_{A_k} |\varphi(y)|\,dy\right)\left(\frac{5^n-2^n}{5^n}\right)\Phi(|A_k|^{1/n}).$$

Combining (2.3.6), (2.3.7) and (2.3.8) we get the thesis.

3. Proof of Theorem A

Let $\lambda>0$ be a fixed real number and construct $G(\lambda)$ and $\bar{f}$ as in Lemma 2.2. Define φ by

$$\text{(3.1.1)} \qquad f=\bar{f}+\varphi.$$

Let $\{A_k\}$ be the family of cubes constructed in Lemma 2.3. Let $K_0(x)$ be the kernel that equals K if $|x|\leq 1$ and is zero otherwise and consider the truncated integral

$$\text{(3.1.2)} \qquad \int_{|x-y|>\epsilon} K_0(x-y)f(y)\,dy \quad \text{where} \quad x\in R^n-\bigcup_1^\infty 20A_k.$$

Clearly, we have

$$\text{(3.1.3)} \qquad \left|\bigcup_1^\infty 20A_k\right|<\frac{C_n}{\lambda}\left(\|f\|_1+\iint_{R^n\times R^n} \frac{\omega_0(|x-y|)}{|x-y|^n}|f(x)-f(y)|\,dx\,dy\right)$$

where C_n depends on n only. Let $\theta_k(y)$ be the characteristic functions of the A_k's and let

$$n_k(y)=\frac{\theta_k(y)}{\sum_{j=1}^{\infty}\theta_j(y)}.$$

Let k' be the indices of the cubes that do not touch the ball of radius ϵ about x and let k'' be the indices corresponding to the cubes that intersect the sphere of radius ϵ about x. Let

$$\mu_k=\frac{1}{|Q_k|}\int_{A_k} \varphi(y)\eta_k(y)\,dy.$$

Since $(12)^{-n}<\eta_k(y)\leq 1$ over A_k we have

$$\text{(3.1.4)} \qquad |\mu_k|<C_n\lambda$$

where C_n depends on n only. Let $\bar{\varphi}(y)=\sum_1^\infty \mu_k\theta_k(y)$. Then $|\bar{\varphi}(y)|<C_n 12^n\lambda$.

Let us write the truncated integral (3.1.2) as

$$\int_{|x-y|>\epsilon} K_0(x-y)f(y)\,dy = \sum_{k'} \int_{A_k} K_0(x-y)(\varphi(y)\eta_k(y)-\mu_k)\,dy$$
$$(3.1.5) \qquad +\sum_{k''} \int_{|x-y|>\epsilon} K_0(x-y)[\varphi(y)\eta_k(y)-\mu_k\theta_k(y)]\,dy$$
$$+\int_{|x-y|>\epsilon} K_0(x-y)\bar{\varphi}(y)\,dy.$$

Majorization for

$$\sum_{k'} \int_{A_k} K_0(x-y)[\varphi(y)\eta_k(y)-\mu_k\theta_k(y)]\,dy.$$

We are going to use the fact that $\varphi(y)\eta_k(y)-\mu_k\theta_k(y)$ has mean value zero over A_k. Let y_k be the center of A_k. We have

$$(3.1.6) \quad \sum_{k'} \int_{A_k} K_0(x-y)[\varphi(y)\eta_k(y)-\mu_k\theta_k(y)]\,dy$$
$$= \sum_{k'} \int_{A_k} [K_0(x-y)-K_0(x-y_k)][\varphi(y)\eta_k(y)-\mu_k\theta_k(y)]\,dy.$$

Now consider the expression

$$(3.1.7) \quad M_1(x) = \sum_1^\infty \int_{A_k} |K_0(x-y)-K_0(x-y_k)|\,(|\varphi(y)|\,\eta_k(y)+|\mu_k|\,\theta_k(y))\,dy.$$

Clearly $M_1(x)$ dominates (3.1.6).

Majorization for

$$\sum_{k''} \int_{|x-y|>\epsilon} K_0(x-y)\{\varphi(y)\eta_k(y)-\mu_k\theta_k(y)\}\,dy.$$

It can be readily seen that for the cubes whose subindices have been labeled $\{k''\}$ we have

$$(3.1.8) \qquad A_k \subset \{y;\, \epsilon/2 < |x-y| < 2\epsilon\}.$$

Let $\gamma_k(x) = |\varphi(y)|\,\eta_k(y)+|\mu_k|\,\theta_k(y)$ and let ν_k be the mean value of $\gamma_k(x)$ over A_k. Then, we have

$$(3.1.9) \quad \left|\sum_{k''} \int_{|x-y|>\epsilon} K_0(x-y)\{\varphi(y)\eta_k(y)-\mu_k\theta_k(y)\}\,dy\right|$$
$$\leq \int_{\epsilon/2<|x-y|<2\epsilon} |K_0(x-y)|\left(\sum_1^\infty \nu_k\theta_k(y)\right)dy$$
$$+\sum_{k''} \int_{A_k} |K_0(x-y)|\,(\gamma_k(y)-\nu_k\theta_k(y))\,dy$$
$$\leq \text{Const}\,\lambda + \sum_1^\infty \int_{A_k} |K_0(x-y)-K_0(x-y_k)|\,(\gamma_k(y)+\nu_k\theta_k(y))\,dy.$$

Let

$$M_2(x)=\sum_1^\infty \int_{A_k} |K_0(x-y)-K_0(x-y_k)|\,(\gamma_k(y)+\nu_k\theta_k(y))\,dy.$$

Collecting estimates we get

$$(3.1.10)\qquad \left|\sum_{k''} \int_{|x-y|>\epsilon} |K_0(x-y)\{\varphi(y)\eta_k(y)-\mu_k\theta_k(y)\}\,dy\right| \le \text{Const}\,\lambda + M_2(x).$$

Estimates for the functions $M_1(x)$ and $M_2(x)$. A calculation using the homogeneity of $K_0(x)$ shows

$$(3.1.11)\qquad \int_{|x|>2|h|} |K_0(x+h)-K_0(x)|\,dx < C\int_{|h|}^1 \omega_0(t)\frac{dt}{t} \quad \text{if} \quad |h|<1/2$$

where $\omega_0(t)$ is the modulus of continuity of the kernel K as defined in (1.2) and C is independent of h. By the definition of $\eta_k(y)$, μ_k, ν_k and $\gamma_k(g)$ we have

$$(3.1.12)\qquad \begin{aligned} &\int_{A_k} \gamma_k(y)\,dy < C_n \int_{A_k} |\varphi(y)|\,dy,\\ &(\nu_k+|\mu_k|)\,|A_k| \le C_n \int_{A_k} |\varphi(y)|\,dy. \end{aligned}$$

Consequently

$$(3.1.13)\qquad \int_{R^n-\bigcup_1^\infty 20A_k} (M_1(x)+M_2(x))\,dx \le C_n \sum_{k=1}^\infty C \int_{|A_k|^{1/n}}^1 \frac{\omega_0(t)}{t}\,dt \int_{A_k} |\varphi(y)|\,dy.$$

Notice that if $|A_k|^{1/n}>1/2$ then $(K_0 * \phi_k)(x)=0$ because $x\in C(20A_k)$. From Lemma 2.3 and 3.1.13) we get

$$(3.1.14)\qquad |E(M_1(x)+M_2(x)>\lambda)| < \frac{C_n}{\lambda}\left(\|f\|_1 + \int_0^1 \omega_0(t)\omega(t)\frac{dt}{t}\right).$$

Let

$$K_0^*(f)=\sup_{\epsilon>0}\left|\int_{|x-y|>\epsilon} K_0(x-y)f(y)\,dy\right|.$$

So far we have

$$(3.1.15)\qquad K_0^*(f) \le K_0^*(\bar f) + K_0^*(\bar\varphi) + C_n\lambda + M_1(x) + M_2(x).$$

Since $\bar{f}$ and $\bar{\varphi}$ belong to L^2 we have

$$(3.1.16)\quad |E(K_0^*(\bar{f})+K_0^*(\bar{\varphi})>\lambda)|\le\frac{C}{\lambda^2}(\|\bar{f}\|_2^2+\|\bar{\varphi}\|_2^2)$$
$$\le C\frac{1}{\lambda}\left(\|f\|_1+\int_0^1\omega_0(t)\omega(t)\frac{dt}{t}\right)$$

where C does not depend on λ or f. Select a constant $L>C_n$ and evaluate $|E(K_0^*(f)>L\lambda)|$. From (3.1.15) we have

$$(3.1.17)\quad |E(K_0^*(f)>L\lambda|\le|E(K_0^*(\bar{f})+K_0^*(\bar{\varphi})+M_1(x)+M_2(x)>(L-C_n)\lambda)|$$
$$\le\frac{C}{(L-C_n)}\frac{1}{\lambda}\left(\|f\|_1+\int_0^1\omega_0(t)\omega(t)\frac{dt}{t}\right).$$

In order to finish the proof consider

$$(3.1.18)\quad \int_{|x-y|>1}K(x-y)f(y)\,dy=\int_{|x-y|>1}K(x-y)\bar{f}(y)\,dy$$
$$+\int_{|x-y|>1}K(x-y)\varphi(y)\,dy.$$

Since $\bar{f}(y)$ belongs to $L^2(R_n)$ the first term of the right-hand member of (3.1.18) does not represent any difficulty. Now let $K_1(x)$ be the function that equals $K(x)$ if $|x|>1$ and zero otherwise. Let us integrate the absolute value of $K_1*\varphi$ over a sphere S centered at the origin and such that diam $(S)\ge$ diam (A_k) for all k. Let A_k' be the cubes A_k such that distance $(A_k', S)<$ 10 diam S. For those cubes we have

$$(3.1.19)\quad \int_S dx\int_{A_k}|K_1(x-y)|\,\eta_k(y)\,|\varphi(y)|\,dy$$
$$\le C_n|\log(20\operatorname{diam}S)|\int_E|K(\alpha)|\,d\alpha\int_{A_k'}|\varphi(y)|\,dy.$$

For the cubes A_k'' such that distance $(A_k'', S)\ge 10$ diam S we have

$$(3.1.20)\quad \int_S\left(\int_{A_k''}|K_1(x-y)|\,\eta_k(y)\,|\varphi(y)|\,dy\right)$$
$$\le C_n\int_{A_k}|\varphi(y)|\,dy\int_{|x|<\operatorname{diam}S}|K_1(x-y)|\,dx$$
$$\le B_0C_n\left(\int_{A_k}|\varphi(y)|\,dy\right)$$

where

$$(3.1.21)\quad B_0=\sup_{\substack{r>8d(S),\\ r-d(S)<|y|<r+d(S)}}\int|K(y)|\,dy$$

with $d(S)=\operatorname{diam}(S)$. This finishes the proof of Theorem A.

References

1. A. P. Calderon, M. Weiss and A. Zygmund, *On the existence of singular integrals*, Proc. Symp. Pure Math., vol. 10 (1967), pp. 56–73
2. A. P. Calderon and A. Zygmund, *Addendum to the paper "On singular integrals"*, Studia Math., vol. 46 (1973), pp. 297–299
3. L. Carleson, *Selected problems in exceptional sets*, van Nostrand, Princeton, New Jersey, 1967
4. Williams Connett, *The Dini condition and the existence of singular integrals*, to appear
5. Mischa Cotlar, *A general interpolation theorem for linear operators*, Revista Matematica Cuyana, vol. 1 (1955), pp. 57–84
6. R. Fefferman, *A theory of entropy in Fourier analysis*, Doctoral dissertation, Princeton University 1975
7. ———, *A theory of entropy in Fourier analysis* Advances in Mathematics
8. N. M. Rivière, *Singular integrals and multiplier operators*, Arch. Math., vol. 9 (1971), pp. 243–274
9. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, New Jersey, 1970

University of Illinois at Chicago Circle
Chicago, Illinois