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# $L(X)$ AS A SUBALGEBRA OF $K(X)^{**}$

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## 1. Introduction

For $X$ and $Y$ Banach spaces let $L(Y, X)$ and $K(Y, X)$ denote respectively the spaces of bounded and compact operators from $Y$ into $X$. The relationship of $L(Y, X)$ and $K(Y, X)$ as Banach spaces has long been of interest. In some special cases, $L(Y, X)$ is actually equal to $K(Y, X)$ while in others $L(Y, X)$ is equal to $K(Y, X)^{**}$. See [8], [10] and [2], [5], [6], [11]. Recently, Jerry Johnson [9] has extended a weaker result in [6] and shown that if $X$ has the bounded approximation property (metric approximation property), then $L(Y, X)$ can be imbedded isomorphically (isometrically) in $K(Y, X)^{**}$. The purpose of this paper is to study Johnson's imbedding for the case $Y = X$ in which $K(X)$ and $L(X)$ are Banach algebras.

For a Banach of Algebra $\mathcal{A}$, the Arens products (see Section 2) give two ways of regarding $\mathcal{A}^{**}$ as a Banach algebra so that the canonical image of $\mathcal{A}$ in $\mathcal{A}^{**}$ is subalgebra of $\mathcal{A}^{**}$. Specializing Johnson's imbedding to the case $Y = X$, it is natural to consider the operator induced multiplication on the image of $L(X)$ in $K(X)^{**}$. In Section 3, we discuss the imbedding of $L(X)$ into $K(X)^{**}$ under the assumption that $X$ has the bounded approximation property and present an example in which *neither* Arens product coincides with operator induced multiplication. Hence, the imbedding need not be a Banach algebra isomorphism. In Section 4, under the assumption that $K(X)$ has a bounded two-sided weak identity, we show that the Johnson imbedding can be defined as a Banach algebra isomorphism, using the first Arens product on $K(X)^{**}$. We also give a characterization of the image of $L(X)$ which leads to an isomorphic copy of $L(X)$ from $K(X)$, without reference to the underlying Banach space.

## 2. The Arens products

The two Arens products are defined in stages according to the following rules. Let $\mathcal{A}$ be a Banach algebra. Let $A, B \in \mathcal{A}$; $f \in \mathcal{A}^*$; $F, G \in \mathcal{A}^{**}$.

DEFINITION 2.1. $(f *_1 A)B = f(AB)$. This defines $f *_1 A$ as an element of $\mathcal{A}^*$.

$(G *_1 f)A = G(f *_1 A)$. This defines $F *_1 f$ as an element of $\mathcal{A}^*$.

$(F *_1 G)f = F(G *_1 f)$. This defines $F *_1 G$ as an element of $\mathcal{A}^{**}$.

We will call $F *_1 G$ the first or $m_1$ product.

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DEFINITION 2.2. $(f *_2 A)B = f(BA)$, $(F *_2 f)A = F(f *_2 A)$ $(F *_2 G)f = G(F *_2 f)$.

We will call $F *_2 G$ the second or $m_2$ product.

DEFINITION 2.3. A net $A_i$ in $\mathscr{A}$ is called a weak identity if for each $B \in \mathscr{A}$, $BA_i$ and $A_iB$ both approach $B$ in the weak topology on $\mathscr{A}$.

The following proposition summarizes some important properties of the Arens products.

PROPOSITION 2.1. (1) *The first Arens product is left weak star continuous, that is,*

$$F_i \to F\sigma(\mathscr{A}^{**}, \mathscr{A}^*) \quad \textit{implies} \quad F_i *_1 G \to F *_1 G\sigma(\mathscr{A}^{**}, \mathscr{A}^*).$$

(2) *The second Arens product is right weak star continuous, that is,*

$$F_i \to F\sigma(\mathscr{A}^{**}, \mathscr{A}^*) \quad \textit{implies} \quad G *_2 F_i \to G *_2 F\sigma(\mathscr{A}^{**}, \mathscr{A}^*).$$

(3) *The two Arens products agree if one of the factors is in* $\mathscr{A}$.

(4) *If* $\mathscr{A}$ *has a weak identity* $A_i$ *which converges* $\sigma(\mathscr{A}^{**}, \mathscr{A}^*)$ *to an element* $J \in \mathscr{A}^{**}$, *then* $J$ *is a right identity for the first Arens product and a left identity for the second Arens product.*

*Proof.* See [1], [3].

## 3. Assuming $X$ has the bounded approximation property

The following theorem is a special case of Johnson's imbedding of $L(Y, X)$ into $K(Y, X)^{**}$.

THEOREM 3.1 (J. Johnson). *Let* $X$ *have the* $\lambda$ *bounded approximation property. Then there exists an isomorphism (isometry if* $\lambda = 1$) *of* $L(X)$ *into* $K(X)^{**}$ *whose restriction to* $K(X)$ *is the canonical imbedding.*

The imbedding is constructed by taking a bounded net of finite rank operators $\{A_i\}$ which is $\sigma(K^{**}, K^*)$-convergent and converges to the identity operator in the strong operator topology. For $T \in L(X)$, define $\hat{T} \in K(X)^{**}$ by $\hat{T}(f) = \lim_i f(A_iT)$ where $f \in K^*$.See [9] for the details. It follows easily that $\hat{I}$ is the $\sigma(K^{**}, K^*)$-limit of $\{A_i\}$.

Unfortunately, when $Y = X$, this imbedding is not necessarily a Banach algebra isomorphism.

*Example.* Let $X = l_1$, $\{e_i\}$ be the standard basis, and $A_n$ be the operator whose matrix has ones in the first $n$ places of the diagonal and zeroes everywhere else. $A_n$ converges to $I$ in the strong operator topology and is easily seen to be a Cauchy sequence in the $\sigma(K, K^*)$ topology. Using $A_n$ to define an imbedding, as described above, $A_n$ converges to $\hat{I}\ \sigma(K^{**}, K^*)$.

Let $B$ be the operator whose matrix has all ones in the first row and zeroes everywhere else. Let $D_n$ be the operator which sends $e_n$ to $e_1$ and $e_j$ to 0 for $j \neq n$. Let $f \in K^*$ be 0 on the closed linear span of the $D_n$ and non-zero on $B$. Then,

$$\begin{aligned}(\hat{B} *_2 \hat{I})f &= (\hat{B} *_1 \hat{I})f \quad \text{by Proposition 2.1(3)}\\ &= \hat{B}(\hat{I} *_1 f)\\ &= (\hat{I} *_1 f)B \quad \text{since } K \text{ is imbedded canonically in } K^{**}.\\ &= \hat{I}(f * B)\\ &= \lim_n (f * B)A_n = \lim_n f(BA_n)\\ &= 0 \quad \text{since } BA_n \text{ has only } n \text{ non-zero entries.}\end{aligned}$$

But $(BI)\hat{}(f) = \hat{B}(f) = f(B) \neq 0$
Thus, $(BI)\hat{}$ is not equal to either $\hat{B} *_1 \hat{I}$ or $\hat{B} *_2 \hat{I}$.

## 4. Main results

Throughout this section we assume that $K(X)$ has a bounded weak identity. Under this assumption we show that $L(X)$ can be imbedded as a subalgebra of $K(X)^{**}$ and we characterize the image of $L(X)$.

PROPOSITION 4.1. *Let $A_i$ be a bounded weak identity in $K(X)$. Then $A_i \to I$ in the weak operator topology.*

*Proof.* Let $x \in X$ and $y^* \in X^*$ be arbitrary. Define $f \in K^*$ by $f(C) = y^*(Cx)$ for $C \in K$. Select a finite rank operator $B$: $Bx = x$. Then

$$y^*(x) = f(B) = \lim_i f(A_iB) = \lim_i y^*(A_ix).$$

Hence, $A_ix \to x$ weakly.

THEOREM 4.1. *Let $A_i$ be a bounded weak identity in $K(X)$ which converges $\sigma(K^{**}, K^*)$. Then $L$ can be imbedded isomorphically (isometrically, if $\|A_{i_j}\| \to 1$ for some subnet of $A_i$) in $K^{**}$.*

*Proof.* For $T \in L$, define $\hat{T}$, as before, by $\hat{T}(f) = \lim f(A_iT)$. This limit exists by essentially the same technique as in [9], since $\lim_i (A_iT) = \lim_i g(A_i)$ where $g$ is the functional in $K^*$ defined by $g(C) = f(CT)$ for $C \in K$. It follows easily from Proposition 4.1 that $\|\hat{T}\| \geq \|T\|$. Also, the image of $B$ is the canonical image since $\hat{B}(f) = \lim_i f(A_iB) = f(B)$.

PROPOSITION 4.2. *Let $A_i$ be as in Theorem 4.1. Then, using the imbedding determined by $A_i$:*

(1) *$A_i \to \hat{I}\sigma(K^{**}, K^*)$.*

(2) *If $T_i \to T$ in the weak operator topology and $T_i \to \hat{S}$ $(K^{**}, K^*)$ where $S \in L$, then $S = T$.*

*Proof.* Similar to Proposition 4.1 and Theorem 4.1.

*Remark.* Unfortunately, if a net $T_i \in K$ converges to $T$ in $L(X)$ even in the strong operator topology and to $F \in K^{**}$ in the $\sigma(K^{**}, K^*)$ topology, it is not necessarily true that $\hat{T} = F$. For example, let $A_n$ be the operator in $K(c_0)$ with ones in the first $n$ entries down the diagonal and zeros elsewhere, and let $T_i$ have entries $1/i$ in the first $i$ entries of the first row and zeros elsewhere. Then under the imbedding defined by the weak identity $A_n$, $T_i$ converges to 0 in the strong operator topology but to a non-zero element $\sigma(K^{**}, K^*)$..

PROPOSITION 4.3. *Let $A_i$ be as in Theorem 4.1. Then, for $B \in K$, $T \in L$, $f \in K^*$, $\lim_j f(BA_jT) = f(BI)$.*

*Proof.* Define $g \in K^*$ by $g(C) = f(CT)$ for $C \in K$. Then

$$\lim_j f(BA_jT) = \lim_j g(BA_j) = g(B) = f(BT).$$

THEOREM 4.2. *Let $A_i$ be as in Theorem 4.1. Then, for $S$, $T \in L$, $\widehat{ST} = \hat{S} *_1 \hat{T}$.*

*Proof.* Let $f \in K^*$.

$$\widehat{ST}(f) = \lim_i f(A_iST),$$

$$\begin{aligned}(\hat{S} *_1 \hat{T})(f) = \hat{S}(\hat{T} *_1 f) &= \lim_i [(\hat{T} * f)(A_iS)] \\ &= \lim_i \hat{T}(f * A_iS) = \lim_i \lim_j [(f * A_iS)A_jT] \\ &= \lim_i \lim_j f(A_iSA_jT) \\ &= \lim_i f(A_iST) \quad \text{by Proposition 4.3}\end{aligned}$$

COROLLARY 4.1. *$\hat{I}$ is a two-sided identity with respect to the first Arens product, on the image of L.*

We now give a characterization of the image of $L(X)$ under the imbedding.

THEOREM 4.3. *Let $A_i$ be as in Theorem 4.1. The image of L under the associated imbedding is equal to*

$$\{F \in K^{**}: F * A_i,\ A_i * F \text{ are in } K \text{ for all } i, \text{ and } \hat{I} *_1 F = F\}.$$

*Proof.* Suppose $F$ satisfies the stated conditions. We must associate an operator $T$ with $F$ such that $\hat{T}=F$. First, define $T$ on $\{A_i x\colon \text{all } i, x\}$ by $T(A_i x)=\text{weak}_j \lim (A_j * F)(A_i x)$. This limit exists since $A_j(F * A_i)x$ converges weakly by Proposition 4.1. Note that $T$ is well defined on this set since if $A_i x = A_k Z$, then $(A_j * F)(A_i x)=(A_j * F)A_k Z$ for all $j$. Extend $T$ to finite linear combinations by

$$T(A_{i_1}x_1+A_{i_2}x_2+\cdots+A_{i_n}x_n)=\text{weak}\lim_j A_j * F(A_{i_1}x_1+\cdots+A_{i_n}x_n).$$

$T$ is bounded in norm by $\sup \|A_i\|\,\|F\|$ on this weakly dense, convex set. Thus $T$ can be extended to $X$.

Now we must show that $F=\hat{T}$. For $f\in K^*$,

$$\begin{aligned} F(f)&=(\hat{I} *_1 F)f \quad \text{by the condition on } F.\\ &=\lim_j (A_j * F)f \quad \text{by the left weak star continuity of } m_1.\\ &=\lim_j f(A_j * F),\\ \hat{T}(f)&=\lim_j f(A_jT) \quad \text{by the definition of } T.\end{aligned}$$

Thus, it is sufficient to show $A_j * F=A_jT$, for each $j$. For each $i$,

$$\sigma(K^{**},K^*)\text{–}\lim_j (A_j * F * A_i)=\hat{I} *_1 (F * A_i)=F * A_i \quad \text{by Corollary 4.1}$$

Also, $A_j * F * A_i \to T * A_i$ in the weak operator topology. Hence, by Proposition 4.2(2), $F * A_i = T * A_i$, for each $i$. This implies that $A_j * F$ and $A_j * T$ agree on finite linear combinations of elements from the set $\{A_i x\colon \text{all } i, x\}$. hence, $A_j * F = A_j * T$, and this concludes the proof.

PROPOSITION 4.4. *Let $K$ be the space of compact operators in some Banach space $X$. Let $J$ be an element of $K^{**}$ such that $J * C = C * J = C$ for all $C\in K$, and let $B_i \to J$ $\sigma(K^{**},K^*)$ with $B_i\in K$ and $\|B_j\|\le\|J\|$. Then $B_i$ is a weak identity.*

*Proof.* $C=J * C$ implies $f(C)=f(J *_1 C)=\lim_i f(B_iC)$; and $C=C *_1 J$ implies $f(C)=f(C *_1 J)=f(C *_2 J)=\lim_i f(CB_i)$ by Proposition 2.1(2).

THEOREM 4.4. *Let $K$, the space of all compact operators on some Banach space $X$, have a weak identity. Then an isomorphic copy of $L$ can be constructed from $K$ without reference to the underlying Banach space $X$.*

*Proof.* Let $J$ be any element of $K^{**}$ such that $J * C = C * J = C$, for all $C\in K$. There is at least one such element $J$, namely, the image of $I$ under the imbedding determined by any weak identity. Let $B_i \to J\,\sigma(K^{**},K^*)$.

Then

$L \approx \{F \in K^{**}: F * B_i, B_i * F$ are in $K$ for all $i$, and $B_i * F \to F \sigma(K^{**}, K^*)\}$.

Note that for this construction it is not necessary for the weak identity to be explicitly given. Also, the isomorphism is an isometry if $\|J\| = 1$.

DEFINITION 4.1. A basis $\{x_i\}$ for a Banach space $X$ is called shrinking if the coordinate functionals $\{x_i^*\}$ form a basis for $X^*$.

We then have the following straightforward theorem whose proof we omit.

THEOREM 4.5. *Let $X$ have a shrinking basis, and $E_n$ be the operator with ones down the first $n$ entries of the diagonal and zeros elsewhere. Then $E_n$ is an approximate identity (and a fortiori a weak identity) for $K(X)$.*

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