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DIMENSIONS OF THE SETS OF INVARIANT MEANS OF SEMIGROUPS

BY

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1. Introduction

Let S be a semigroup, and $m(S)$ the Banach space of bounded real-valued functions on S with the sup norm. For each $s \in S$, let l_s $[r_s]$ be the left [right] translation linear operator on $m(S)$ defined by $l_s f(t) = f(st)$ $[r_s f(t) = f(ts)]$ for $f \in m(S)$ and $t \in S$. A mean on $m(S)$ is a non-negative element of norm one in the dual space $m(S)^*$. We say that a mean μ is left [right] invariant if $l_s^* \mu = \mu$ $[r_s^* \mu = \mu]$ for every $s \in S$, i.e. for any $f \in m(S)$ we have $\mu(l_s f) = \mu(f)$ $[\mu(r_s f) = \mu(f)]$. Let $Ml(S)$ $[Mr(S)]$ be the set of left [right] invariant means on $m(S)$, and let $M(S) = Ml(S) \cap Mr(S)$. When S has a left [right] invariant mean we say that S is left [right] amenable; moreover, if S has a mean which is both left and right invariant, we say S is amenable. For any $X \subset m(S)^*$ we denote its linear span by $\langle X \rangle$.

The purpose of this paper is to exactly determine the dimension of $\langle Ml(S) \rangle$ for left amenable semigroups, and similarly the dimension of $\langle M(S) \rangle$ for amenable semigroups. The problem has already been settled for left amenable semigroups with a finite dimensional set of left invariant means by the following two theorems: Theorem A. If S is left amenable, then dim $\langle Ml(S) \rangle$ is finite if and only if S has a finite (two-sided) ideal. Theorem B. If S is left amenable then dim $\langle Ml(S) \rangle = n < \infty$ if and only if S contains exactly n disjoint left ideals which are finite groups. Theorem A was proved by Luthar [10] for abelian semigroups. Theorems A and B were proved by Granirer in [5] and [6] for countable semigroups and left cancellative semigroups respectively, and by the author [8] in general.

For left amenable semigroups which have an infinite dimensional set of left invariant means, i.e. which contain no finite ideals, the only exact result known is the following theorem due to Chou [2]: If S is an infinite amenable group then dim $\langle Ml(S) \rangle =$ dim $\langle M(S) \rangle = 2^{2^{|S|}}$. Lower bounds for dim $\langle Ml(S) \rangle$ were also obtained by Chou in [1] for cancellative semigroups, and by the author in [8] for arbitrary semigroups.

Our main result for left amenable semigroups is the following theorem: If S is a left amenable semigroup containing no finite ideals, then

$$\dim \langle Ml(S) \rangle = 2^{2^{\kappa l(S)}}$$

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where $\kappa l(S) = \min \{|B| : \mu(\chi_B) = 1$ for every $\mu \in Ml(S)\}$. As a corollary we show that if S is an infinite left amenable semigroup with either left or right cancellation then $\dim \langle Ml(S) \rangle = 2^{2^{|S|}}$, which extends Chou's result for infinite amenable groups. The proof of this theorem depends heavily on the results on the structure of left thick subsets in left amenable semigroups contained in [8].

For amenable semigroups we prove an analogous theorem: Let S be an amenable semigroup. Then S contains a finite ideal if and only if S has exactly one invariant mean. If S does not contain a finite ideal, then $\dim \langle M(S) \rangle = 2^{2^{\kappa(S)}}$ where $\kappa(S) = \min \{|sSs| : s \in S\}$. The proof closely follows the one for left amenable semigroups, using the structure of the thick subsets of S, which play a role analogous to that of the left thick subsets of a left amenable semigroup. In addition we also give a nicer characterization of $\dim \langle Ml(S) \rangle$ for amenable semigroups. In fact if S is an amenable semigroup with no finite ideals then $\dim \langle Ml(S) \rangle = 2^{2^{\kappa}}$ where κ is the minimum of the cardinalities of the right ideals of S.

Section 2 is devoted to the proof of our main result for left amenable semigroups, and some of its corollaries. In Section 3 we show that the cardinal $\kappa l(S)$ defined above is actually the minimum of the cardinalities of the right ideals of S if S has the finite intersection property of left ideals. This provides the nicer characterization of $\dim \langle Ml(S) \rangle$ when S is amenable; we also show that this characterization holds when S has a finitely supported left invariant mean. We conclude this section by showing that a semigroup has a finitely supported left invariant mean if and only if its right thick subsets form a filter. This answers a question of Rajagopalan and Ramakrishnan [12, Problem 6, p. 19].

In Section 4 we introduce thick subsets of semigroups and show that a subset A of an amenable semigroup S is thick if and only if there exists an (two-sided) invariant mean μ on $m(S)$ with $\mu(\chi_A) = 1$. We also include an example of a cancellative right amenable semigroup which is not left amenable (answering a question of Granirer [7, p. 109]), which we use to show that a subset of an amenable group which is both left and right thick, need not be thick.

Section 5 begins with an outline of results on the structure of thick subsets, which correspond to those already established for left thick subsets. We then prove the results determining the dimension of the set of invariant means of an amenable semigroup. We close with some open problems concerning the relationship between $\dim \langle Ml(S) \rangle$, $\dim \langle Mr(S) \rangle$ and $\dim \langle M(S) \rangle$ for an amenable semigroup S.

2. The exact dimension of $\langle Ml(S) \rangle$

Let S be a discrete semigroup. For each $A \subset S$ let χ_A denote the characteristic function of A, that is $\chi_A(s) = 1$ if $s \in A$ and $\chi_A(s) = 0$ if $s \notin A$. We say that a subset $A \subset S$ is left [right] thick if for every finite $F \subset S$ there exists $s \in S$ with $Fs \subset A$ [$sF \subset A$]. A subset $A \subset S$ is strongly left thick if for every $B \subset S$ with $|B| < |A|$, the set $A \setminus B$ is left thick in S.

Remark 2.1. The concept of left thick subsets is due to Mitchell [11], who showed that if S is a left amenable semigroup then $A \subset S$ is left thick if and only if there exists $\mu \in Ml(S)$ with $\mu(\chi_A) = 1$. The importance of strong left thickness lies in the fact that a subset A is strongly left thick if and only if there exists a collection $\{A_\gamma: \gamma \in \Gamma\}$ of pairwise disjoint subsets of A, which are left thick in S, such that $|\Gamma| = |A|$ [8, Theorem 2.2]. Another fact which we will require is that if S is a left amenable semigroup containing no finite ideals, then any left thick subsemigroup of minimal cardinality is strongly left thick (see the proof of Theorem 4.1 in [8]). Finally, notice that if $X \subset m(S)^*$ with $|X| \geq 2^c$ where $c = 2^{\aleph_0}$ then $\dim \langle X \rangle = |X|$ since $|X| \leq 2^{\aleph_0} \dim \langle X \rangle$. Thus to obtain the results in this section it will suffice to determine $|Ml(S)|$.

We begin with a lemma which establishes lower bounds for $\dim \langle Ml(S) \rangle$ using ultrafilters and collections of disjoint subsets of S.

LEMMA 2.2. *Suppose* $\{D_\gamma: \gamma \in \Gamma\}$ *is an infinite collection of disjoint subsets of a left amenable semigroup* S *such that there exists* $\{\mu_\gamma: \gamma \in \Gamma\} \subset Ml(S)$ *with* $\inf\{\mu_\gamma(\chi_{D_\gamma}): \gamma \in \Gamma\} > 0$. *Then* $\dim \langle Ml(S) \rangle \geq 2^{2^{|\Gamma|}}$.

Proof. As noted in the remark above, it suffices to show that $|Ml(S)| \geq 2^{2^{|\Gamma|}}$. Let $\varepsilon = \inf\{\mu_\gamma(\chi_{D_\gamma}): \gamma \in \Gamma\}$. We first show that for any ultrafilter $\mathscr{U}$ on Γ, there exists $\mu_\mathscr{U} \in Ml(S)$ such that

$$\mu_\mathscr{U}(\chi_{\cup\{D_\gamma: \gamma \in \Gamma_1\}}) \geq \varepsilon$$

for every $\Gamma_1 \in \mathscr{U}$. Clearly for any $\Gamma_1 \in \mathscr{U}$, we can choose $\mu_{\Gamma_1} \in Ml(S)$ with $\mu_{\Gamma_1}(\chi_{\cup\{D_\gamma: \gamma \in \Gamma_1\}}) \geq \varepsilon$. Since $Ml(S)$ is weak*-compact, there exists a weak*-limit point, say $\mu_\mathscr{U}$, of the net $\{\mu_{\Gamma_1}: \Gamma_1 \in \mathscr{U}\}$, where as usual the partial order on $\mathscr{U}$ is backwards inclusion. It is easy to check that $\mu_\mathscr{U}$ has the desired property.

We now consider the cardinality of $\{\mu_\mathscr{U}: \mathscr{U}$ an ultrafilter on $\Gamma\}$. For any $\mu \in Ml(S)$ we must have $|\{\mathscr{U}: \mu_\mathscr{U} = \mu\}| \leq 1/\varepsilon$. To see this, suppose that $\mathscr{U}_1, \ldots, \mathscr{U}_n$ are distinct ultrafilters with $\mu_{\mathscr{U}_i} = \mu$ for $i = 1, \ldots, n$. Using induction on n, it is straightforward to prove that since the $\mathscr{U}_i$ are distinct, there exist disjoint subsets $\Gamma_1, \ldots, \Gamma_n$ of Γ such that $\Gamma_i \in \mathscr{U}_i$ for $i = 1, \ldots, n$. It follows that the sets

$$D_i = \bigcup \{D_\gamma: \gamma \in \Gamma_i\}$$

are also disjoint, and so we have

$$1 = \mu(\chi_S) \geq \sum_{i=1}^{n} \mu(\chi_{D_i}) = \sum_{i=1}^{n} \mu_{\mathscr{U}_i}(\chi_{D_i}) \geq n\varepsilon,$$

and hence $n \leq 1/\varepsilon$. Thus

$$|\{\mu_\mathscr{U}: \mathscr{U} \text{ an ultrafilter on } \Gamma\}| \geq \varepsilon |\{\mathscr{U}: \mathscr{U} \text{ an ultrafilter on } \Gamma\}|.$$

Since Γ is infinite there are $2^{2^{|\Gamma|}}$ ultrafilters on Γ so

$$|Ml(S)| \geq \varepsilon(2^{2^{|\Gamma|}}) = 2^{2^{|\Gamma|}}.$$

From this lemma we immediately obtain the following theorem.

THEOREM 2.3. *If S is an infinite strongly left thick left amenable semigroup then* $\dim \langle Ml(S) \rangle = 2^{2^{|S|}}$.

Proof. As $|m(S)^*| = 2^{2^{|S|}}$, clearly $\dim \langle Ml(S) \rangle \leq 2^{2^{|S|}}$. However, there exists a collection $\{D_\gamma : \gamma \in \Gamma\}$ of disjoint left thick subsets of S with $|\Gamma| = |S|$ since S is strongly left thick (see Remark 2.1). Thus by Lemma 2.2, we also have $\dim \langle Ml(S) \rangle \geq 2^{2^{|S|}}$.

COROLLARY 2.4. *If S is an infinite left amenable semigroup with either left or right cancellation, then* $\dim \langle Ml(S) \rangle = 2^{2^{|S|}}$.

Proof. This follows directly from Theorem 2.3 since any infinite left amenable semigroup with either left or right cancellation is strongly left thick [8, 2.6 and 6.3].

After proving one more lemma, we will be ready to prove the main result of this section (Theorem 2.6). First we define

$$\tau l(S) = \min \{ |A| : A \text{ left thick in } S\}.$$

LEMMA 2.5. *If D is a subset of a left amenable semigroup S and $\mu \in Ml(S)$ with $\mu(\chi_D) > 0$, then there exists $D' \subset D$ and $\mu' \in Ml(S)$ such that $|D'| \leq \tau l(S)$ and $\mu'(\chi_{D'}) > 0$.*

Proof. If $\tau l(S)$ is finite then $\mu_0(\chi_{\{t\}}) > 0$ for some $t \in S$ and $\mu_0 \in Ml(S)$. Furthermore $\mu(\chi_D) > 0$ implies that $tS \cap D \neq \emptyset$, and hence we can pick $ts \in D$. Let $D' = \{ts\}$ and let $\mu' = r_s^* \mu_0$. It is easy to check that $\mu' \in Ml(S)$, and we have

$$\mu'(\chi_{D'}) = \mu_0(r_s \chi_{\{ts\}}) \geq \mu_0(\chi_{\{t\}}) > 0.$$

Now suppose that $\tau l(S)$ is infinite. The proof depends on the following result of Mitchell [11, Theorem 4]: For any subset $D \subset S$ there exists $\mu \in Ml(S)$ with $\mu(\chi_D) = \alpha$ if and only if there exists a net $\{T_\delta\}$ of finite averages of right translations such that $T_\delta \chi_D$ converges pointwise to $\alpha \chi_S$. Thus let $\mu(\chi_D) = \alpha > 0$ and $\{T_\delta\}$ such that $T_\delta \chi_D$ converges pointwise to $\alpha \chi_S$. We will construct a subset $D' \subset D$ with $|D'| \leq \tau l(S)$, and a net $\{T'_\gamma\}$ of finite averages of right translations such that $T'_\gamma \chi_{D'}$ also converges pointwise to $\alpha \chi_S$. First we choose a left thick subset A of S with $|A| = \tau l(S)$. For each finite $F \subset S$ choose $t_F \in S$ such that $Ft_F \subset A$, and for each finite $H \subset A$ and $n \in \mathbf{N}$ choose $\delta(H, n)$ so that

$$|T_{\delta(H,n)} \chi_D(s) - \alpha| < 1/n \quad \text{for every } s \in H.$$

Now define $T'_{(F,n)} = T_{\delta(Ft_F,n)} r_{t_F}$ for each finite $F \subset S$ and $n \in \mathbf{N}$. Notice that each $T'_{(F,n)}$ is a finite average of right translations. Next, observe that for any $T_{\delta(H,n)}$ there is a finite subset $D(H, n)$ of D such that $T_{\delta(H,n)} \chi_{D(H,n)}(s) = T_{\delta(H,n)} \chi_D(s)$ for each $s \in H$, because H is finite and $T_{\delta(H,n)}$ is a finite average of right translations.

Thus we define $D' = \bigcup\{D(H, n): H$ a finite subset of A, and $n \in \mathbf{N}\}$. Since A is infinite, $|D'| = |A| = \tau l(S)$; moreover

$$T_{\delta(H,n)}\chi_{D'}(s) = T_{\delta(H,n)}\chi_D(s) \quad \text{for every } s \in H$$

since $T_{\delta(H,n)}\chi_{D(H,n)}(s) \leq T_{\delta(H,n)}\chi_{D'}(s) \leq T_{\delta(H,n)}\chi_D(s)$. Finally consider the net $\{T'_{(F,n)}: F$ a finite subset of S, and $n \in \mathbf{N}\}$ directed by the partial order $(F_1, n_1) \leq (F_2, n_2)$ if $F_1 \subset F_2$ and $n_1 \leq n_2$. For any $s \in S$ and $m \in \mathbf{N}$, if $(F, n) \geq (\{s\}, m)$ we have

$$\begin{aligned} |T'_{(F,n)}\chi_{D'}(s) - \alpha| &= |T_{\delta(Ft_F,n)}r_{t_F}\chi_{D'}(s) - \alpha| \\ &= |T_{\delta(Ft_F,n)}\chi_{D'}(st_F) - \alpha| \\ &= |T_{\delta(Ft_F,n)}\chi_D(st_F) - \alpha| \\ &< 1/n \\ &\leq 1/m. \end{aligned}$$

Thus $T'_{(F,n)D'}$ converges pointwise to $\alpha\chi_S$.

For a left amenable semigroup S we define

$$\kappa l(S) = \min\{|B| : \mu(\chi_B) = 1 \quad \text{for every } \mu \in Ml(S)\}.$$

THEOREM 2.6. *If S is a left amenable semigroup which contains no finite ideals, then* $\dim\langle Ml(S)\rangle = 2^{2^{\kappa l(S)}}$.

Proof. Since S contains no finite ideals, $\dim\langle Ml(S)\rangle$ is infinite, and hence $\kappa l(S)$ must be infinite. Thus choosing $B \subset S$ with $|B| = \kappa l(S)$ and $\mu(\chi_B) = 1$ for every $\mu \in Ml(S)$, it follows that $\dim\langle Ml(S)\rangle \leq |Ml(S)| \leq |m(B)^*| = 2^{2^{\kappa l(S)}}$. Now by Remark 2.1 it will suffice to show that $|Ml(S)| \geq 2^{2^{\kappa l(S)}}$. By Theorem 2.3 this is clear if $\tau l(S) = \kappa l(S)$, since any left thick subsemigroup in S of minimal cardinality must be strongly left thick, infinite, and left amenable (see Remark 2.1). Hence we may assume that $\tau l(S) < \kappa l(S)$. First suppose that $\tau l(S)$ is finite. In this case S contains a left ideal which is a finite group [8, Lemma 4.3]. This implies that S has a right ideal A which is the union of disjoint finite left ideals which are groups (the proof is identical to that of Lemma 3.1 in Granirer [5]). Since $|A| \geq \kappa l(S) \geq \aleph_0$, clearly A is strongly left thick, so again by Theorem 2.3 we have $|Ml(S)| \geq |Ml(A)| = 2^{2^{|A|}} \geq 2^{2^{\kappa l(S)}}$.

Finally we are left with the case $\aleph_0 \leq \tau l(S) < \kappa l(S)$. Choose a maximal collection $\{D_\gamma: \gamma \in \Gamma\}$ of disjoint subsets of S such that for each $\gamma \in \Gamma$ there exists $\mu_\gamma \in Ml(S)$ with $\mu_\gamma(\chi_{D_\gamma}) > 0$ and $|D_\gamma| \leq \tau l(S)$. Let $B = \bigcup\{D_\gamma: \gamma \in \Gamma\}$. By the maximality of the collection $\{D_\gamma: \gamma \in \Gamma\}$ we must have $\mu(\chi_B) = 1$ for every $\mu \in Ml(S)$, since otherwise by Lemma 2.5 there exists $D \subset S\backslash B$ and $\mu \in Ml(S)$ such that $\mu(\chi_D) > 0$ and $|D| \leq \tau l(S)$. Now $|\Gamma| = |B| \geq \kappa l(S)$ because

$|D_\gamma| \le \tau l(S) < \kappa l(S)$ for each $\gamma \in \Gamma$. Furthermore, noting that $\kappa l(S) > \aleph_0$, there must exist $\Gamma' \subset \Gamma$ such that $|\Gamma'| = |\Gamma|$ and

$$\inf \{\mu_\gamma(\chi_{D_\gamma}): \gamma \in \Gamma'\} > 0.$$

Therefore by Lemma 2.2, we have $|Ml(S)| \ge 2^{2^{|\Gamma|}} \ge 2^{2^{\kappa l(S)}}$.

3. Another characterization of $\kappa l(S)$

It is well known that if A is a right ideal of a left amenable semigroup S, then $\mu(\chi_A) = 1$ for every $\mu \in Ml(S)$. Thus it is natural to ask whether $\kappa l(S)$ equals the minimum of the cardinalities of the right ideals of S, i.e. does $\kappa l(S) = \min\{|sS| : s \in S\}$? In general we have been unable to answer this question, though it is obviously true if S is finite or if S is strongly left thick. In this section we show that it also holds if S has the finite intersection property of left ideals (f.i.p.l.i.) or if S has a finitely supported left invariant mean. In these cases (and in particular when S is amenable), this yields a somewhat nicer characterization of dim $\langle Ml(S)\rangle$.

LEMMA 3.1. *If S is an infinite semigroup with f.i.p.l.i. which is not strongly left thick, then there exists $t \in S$ with $|tS| < |S|$.*

Proof. Since S is not strongly left thick there exist subsets F and B of S such that F is finite, $|B| < |S|$, and $Fs \cap B \neq \emptyset$ for each $s \in S$. Thus if $F = \{t_1, \ldots, t_n\}$ we can write $S = \bigcup_{i=1}^n S_i$ where $t_i S_i \subset B$ for $i = 1, \ldots, n$. Now since S has f.i.p.l.i. we can pick $t \in \bigcap_{i=1}^n St_i$, and we have $|tS| = |\bigcup_{i=1}^n tS_i| \le n|B| < |S|$.

PROPOSITION 3.2. *If S is a left amenable semigroup with f.i.p.l.i. then* $\kappa l(S) = \min\{|sS| : s \in S\}$.

Proof. Let tS be a right ideal of minimal cardinality in S. Clearly $\kappa l(S) \le |tS|$, so we need only show $|tS| \le \kappa l(S)$. It is easy to check that tS must also have f.i.p.l.i., thus by Lemma 3.1 tS is either finite or strongly left thick. If tS is finite then trivially $\kappa l(S) \ge |tS|$, since in this case tS is a finite group, and hence for every $s \in tS$ there exists $\mu \in Ml(S)$ with $\mu(\chi_{\{s\}}) > 0$. On the other hand if tS is strongly left thick then for any $B \subset S$ with $|B| < |tS|$ there exists $\mu \in Ml(S)$ with $\mu(\chi_B) = 0$, which shows that $\kappa l(S) \ge |tS|$.

COROLLARY 3.3. *If S is an amenable semigroup, then either S has a unique left invariant mean, or S contains no finite ideals and* dim $\langle Ml(S)\rangle = 2^{2^\kappa}$ *where* $\kappa = \min\{|sS| : s \in S\}$.

Proof. If S contains a finite ideal then dim $\langle Ml(S)\rangle$ is finite and hence S has a unique invariant mean [8, Cor. 4.6]. The rest of the theorem follows directly from Proposition 3.2 and Theorem 2.6 since every amenable semigroup has f.i.p.l.i.

We say that a mean μ on $m(S)$ is finitely supported if $\mu(\chi_A) = 1$ for some finite subset A of S.

PROPOSITION 3.4. *If S has a finitely supported left invariant mean, then* $\kappa l(S) = \min \{|sS| : s \in S\}$.

Proof. Obviously S has a finite left thick subset, and this implies (as described in the proof of Theorem 2.6) that S has a right ideal A which is the union of left ideals which are finite groups. Now as before, for each $s \in A$ there exists $\mu \in Ml(S)$ with $\mu(\chi_{\{s\}}) > 0$, and hence $\kappa l(S) \geq |A|$.

We now present a theorem giving some equivalent properties satisfied by semigroups which have finitely supported left invariant means. In particular the theorem characterizes the semigroups whose right thick subsets form a filter. This answers a question of Rajagopalan and Ramakrishnan [12, p. 19, Problem 6] who showed that the right thick subsets form an ultrafilter if and only if the semigroup has a unique extremely right invariant mean.

THEOREM 3.5. *For a semigroup S the following are equivalent:*

(a) *S has a finitely supported left invariant mean.*
(b) *S has a finite left ideal which is a group.*
(c) *S has a right ideal which is contained in every right thick subset.*
(d) *The right thick subsets of S form a filter.*
(e) *The intersection of any pair of right thick subsets is non-empty.*

Proof. (b) ⇒ (a) and (c) ⇒ (d) ⇒ (e) are obvious. Also we already noted that (a) ⇒ (b) in the proof of Proposition 3.4. To prove (b) ⇒ (c), let $A = DS$ where D is a finite left ideal of S which is a group, and let B be any right thick subset of S. Clearly A is a right ideal; moreover $A \subset B$ since for every $s \in S$ there exists $t \in S$ with $tDs \subset B$, but $tD = D$ since D is a left ideal and a group.

We complete the proof by showing that (e) ⇒ (b). Let $A = Sa$ be a left ideal of minimal cardinality, and let B and C be right thick subsets of A. Then $B \cap C \neq \emptyset$ since Ba^{-1} and Ca^{-1} are right thick in S and

$$(Ba^{-1} \cap Ca^{-1})a \subset B \cap C$$

(as usual $Xa^{-1} = \{s \in S\colon sa \in X\}$ for any $X \subset S$). Thus A has the finite intersection property of right ideals and except for the trivial case that $|A| = 1$, A cannot be strongly right thick (see Remark 2.1). By a symmetric version of Lemma 3.1 we see that A cannot be infinite since it had minimal cardinality as a left ideal of S. Thus A is a finite right cancellative semigroup with the finite intersection property of right ideals, and hence is a group.

4. Thick sets and amenable semigroups

A subset A of S is said to be thick if for every finite subset F of S there exists $s \in S$ such that $FsF \subset A$. It is easy to see that any thick subset is both left thick and right thick, but we will show that the converse is false (Example 4.4). Thick subsets in an amenable semigroup play the same role as left thick subsets do in a left amenable semigroup, i.e., a subset of an amenable semigroup is thick if and only if there is an (two-sided) invariant mean which takes value one on the characteristic function of the subset (Theorem 4.2). We begin with an easy lemma which gives a useful equivalent definition of thickness. As usual for $A \subset S$ and $t \in S$ we use the notation

$$t^{-1}A = \{s \in S\colon ts \in A\}.$$

LEMMA 4.1. *A subset A of a semigroup S is thick if and only if*

$$\bigcap \{t^{-1}A\colon t \in F\}$$

is right thick for every finite $F \subset S$.

Proof. Simply notice that for any $F \subset S$ and $s \in S$ we have $FsF \subset A$ if and only if $sF \subset \bigcap \{t^{-1}A\colon t \in F\}$.

THEOREM 4.2. *If S is an amenable semigroup then $A \subset S$ is thick if and only if there exists $\mu \in M(S)$ such that $\mu(\chi_A) = 1$.*

Proof. Suppose $A \subset S$ and $\mu \in M(S)$ with $\mu(\chi_A) = 1$. For any finite $F \subset S$ we have $\mu(\chi_{t^{-1}A}) = 1$ for each $t \in F$, and hence

$$\mu(\chi_{\cap \{t^{-1}A: t \cdot F\}}) = 1$$

also. Thus $\bigcap \{t^{-1}A\colon t \in F\}$ is right thick which shows that A is thick by Lemma 4.1. Now let A be a thick subset of S. For every finite $F \subset S$ choose $\upsilon_F \in Mr(S)$ such that $\upsilon_F(\chi_{\cap \{t^{-1}A: t \in F\}}) = 1$. Since $Mr(S)$ is weak*-compact we can find $\upsilon_2 \in Mr(S)$ such that υ_2 is a weak*-limit point of the net $\{\upsilon_F\colon F$ finite in $S\}$ directed by inclusion. Note that for every $t \in S$ we must have $\upsilon_2(\chi_{t^{-1}A}) = 1$. Choose any $\upsilon_1 \in Ml(S)$ and let $\mu = \upsilon_1 \odot \upsilon_2$, where $\odot$ denotes the Arens product (i.e. for any $f \in m(S)$ the function $\upsilon_2 \odot f \in m(S)$ is defined by $\upsilon_2 \odot f(t) = \upsilon_2(l_t f)$ for each $t \in S$, and $\upsilon_1 \odot \upsilon_2 \in m(S)^*$ is defined by $\upsilon_1 \odot \upsilon_2(f) = \upsilon_1(\upsilon_2 \odot f)$). It is easy to check that $\mu \in M(S)$ because $\upsilon_1 \in Ml(S)$ and $\upsilon_2 \in Mr(S)$. Moreover

$$\mu(\chi_A) = \upsilon_1(\upsilon_2 \odot \chi_A) = \upsilon_1(\chi_S) = 1$$

since $\upsilon_2 \odot \chi_A(t) = \upsilon_2(l_t \chi_A) = \upsilon_2(\chi_{t^{-1}A}) = 1$ for each $t \in S$.

In order to give an example of a set which is both left thick and right thick but not thick, we first give an example of a cancellative semigroup which is right amenable but not amenable. It is well known that any right amenable cancellative semigroup can be embedded as a right thick subset of an amenable group (see [13, Corollaries 3.2 and 3.6]). Thus this provides an example of a right amenable subsemigroup of an amenable group which is not left amenable, hence negatively answering the question of Granirer [7, p. 109].

Example 4.3. Let $B = \{(m, n) \in Z \times Z: m \geq 0, \ n \geq 1\}$ with the multiplication

$$(m_1, n_1)(m_2, n_2) = (m_1 + 2^{n_1}m_2, n_1 + n_2).$$

It can easily be verified that B is a cancellative semigroup, but B is not left amenable since $(0, 1)B \cap (1, 1)B = \emptyset$. To see that S is right amenable, notice that B can be represented as the semidirect product $U \times_\rho T$, where $U[T]$ is the semigroup of non-negative [positive] integers under addition, and $\rho: T \to \text{End}(U)$ is defined by $(\rho(n))(m) = 2^n m$. Now, because both U and T are right amenable, B must be also [9, Prop. 3.10]. For more details about this semigroup, see Remark 3.6.iii in [9].

Example 4.4. From the above example and the comments which precede it, it follows that there exists an amenable group S with a subsemigroup B which is right thick but not left amenable. Thus $\mu(\chi_B) = 0$ for every $\mu \in M(S)$. Moreover, the set $B^{-1} = \{s \in S: s^{-1} \in B\}$ is left thick, but not right amenable, and hence $\mu(\chi_{B^{-1}}) = 0$ for every $\mu \in M(S)$. Now we see that the set $B \cup B^{-1}$ is both left and right thick; however $B \cup B^{-1}$ is not thick since $\mu(\chi_{B \cup B^{-1}}) \leq \mu(\chi_B) + \mu(\chi_{B^{-1}}) = 0$ for every $\mu \in M(S)$.

5. The dimension of $\langle M(S) \rangle$

In this section we will present results which completely determine the dimension of the set of invariant means of an amenable semigroup. Most of the details of the proofs are omitted because of their similarity with those for the analogous results on left amenable semigroups.

A set $A \subset S$ is said to be strongly thick if for every $B \subset S$ with $|B| < |A|$, the set $A \setminus B$ is thick.

THEOREM 5.1. *A subset $A \subset S$ is strongly thick if and only if there exists a collection $\{D_\gamma: \gamma \in \Gamma\}$ of disjoint subsets of A which are thick in S, such that $|\Gamma| = |A|$.*

A proof using transfinite induction can easily be obtained by mimicking the proof of Theorem 2.2 in [8].

LEMMA 5.2. *If $\{D_\gamma: \gamma \in \Gamma\}$ is a collection of pairwise disjoint thick subsets of an amenable semigroup S, then* $\dim \langle M(S) \rangle \geq 2^{2^{|\Gamma|}}$.

The proof is identical to that of Lemma 2.2.

For any semigroup S we define $\kappa(S) = \min \{|sSs| : s \in S\}$.

THEOREM 5.3. *If S is an amenable semigroup such that $\kappa(S)$ is infinite, then* $\dim \langle M(S) \rangle = 2^{2^{\kappa(S)}}$.

Proof. Choose $a \in S$ so that $|aSa| = \kappa(S)$. We will show that aSa is strongly thick. If not, then there must exist subsets F and B of S such that F is finite, $|B| < |aSa|$, and $FsF \not\subset aSa \setminus B$ for each $s \in S$. Let $F = \{s_1, \ldots, s_n\}$. Then we can write

$$S = \bigcup \{A_{ij}: 1 \leq i, j \leq n\} \quad \text{where } s_i A_{ij} s_j \subset B \text{ for each } i, j.$$

Since S is amenable there is some $d \in \bigcap \{s_i S s_i: 1 \leq i \leq n\}$. Now

$$|dSd| = |\bigcup \{dA_{ij} d: 1 \leq i, j \leq n\}| \leq n^2 |B| < |aSa|$$

because $|aSa|$ is infinite, which contradicts $|aSa| = \kappa(S)$. Thus combining the fact that aSa is strongly thick with Theorem 5.1 and Lemma 2.2, we obtain $\dim \langle M(S)\rangle \geq 2^{2^{\kappa(S)}}$. Furthermore, since $\mu(\chi_{aSa}) = 1$ for every $\mu \in M(S)$, we have $\dim \langle M(S)\rangle \leq |m(aSa)^*| = 2^{2^{\kappa(S)}}$.

THEOREM 5.4. *For an amenable semigroup S, the following are equivalent:*

(a) $\dim \langle M(S)\rangle$ *is finite.*
(b) *S has a finite thick subset.*
(c) *S has a finite two-sided ideal.*
(d) $|Ml(S)| = |Mr(S)| = |M(S)| = 1$.

Proof. (a)$\Rightarrow$(b). Choose $s \in S$ so that sSs is finite, which is possible since $\kappa(S)$ must be finite by Theorem 5.3. Now sSs is also thick since $\mu(\chi_{sSs}) = 1$ for any $\mu \in M(S)$.

(b)$\Rightarrow$(c). For some $a \in S$ and $\mu \in M(S)$ we must have $\mu(\chi_{\{a\}}) > 0$, but then SaS is finite.

(c)$\Rightarrow$(d). Since (c) implies that $\dim \langle Ml(S)\rangle$ is finite, this follows from Corollary 4.6 in [8].

(d)$\Rightarrow$(a). Obvious.

We close with two problems concerning the relationship between $\dim \langle Ml(S)\rangle$ and $\dim \langle Mr(S)\rangle$ for an amenable semigroup S.

Problem 1. Characterize the amenable semigroups S such that $\dim \langle Ml(S)\rangle = \dim \langle Mr(S)\rangle$.

We have been unable to find an example where the dimensions were different.

Problem 2. If S is amenable, is it always true that

$$\dim \langle M(S)\rangle = \min \{\dim \langle Ml(S)\rangle, \dim \langle Mr(S)\rangle\}?$$

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