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ON QUASISIMILARITY FOR SUBNORMAL OPERATORS

BY

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In this paper it is shown that if two subnormal operators on separable Hilbert spaces are quasisimilar, then the weak* (or, ultraweakly) closed algebras they generate are isomorphic, and this isomorphism has additional properties. The pure and normal parts of quasisimilar subnormal operators are also investigated, and it is shown that the normal parts must be unitarily equivalent. It is also proved that the pure parts of quasisimilar cyclic subnormal operators must be quasisimilar. These results are then applied to characterize the normal operators all of whose simple parts are quasisimilar. Another application is made to obtain part of a result of W. S. Clary characterizing those subnormal operators that are quasisimilar to the unilateral shift of multiplicity one.

In this paper all Hilbert spaces are separable and all operators are bounded and linear. An operator S on a Hilbert space $\mathscr{H}$ is *subnormal* if there is a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subseteq \mathscr{H}$ and $S = N \mid \mathscr{H}$ (the restriction of N to $\mathscr{H}$). The weak* topology on $\mathscr{B}(\mathscr{H})$ is the topology $\mathscr{B}(\mathscr{H})$ has as the Banach space dual of $\mathscr{B}_1(\mathscr{H})$, the trace class operators [17]. It is customary to call this the ultraweak topology. The term "weak*" not only obviates the misleading term "ultraweak", but also emphasizes that all the results concerning the dual of a separable Banach space are applicable to $\mathscr{B}(\mathscr{H})$ with its weak* topology.

For S in $\mathscr{B}(\mathscr{H})$, $\mathscr{A}(S)$ denotes the weak* closed algebra generated by S and the identity, 1. That is $\mathscr{A}(S)$ is the weak* closure of $\{p(S)\colon p \text{ is a polynomial}\}$. It has recently been shown by Olin and Thomson [13] that, for a subnormal operator S, $\mathscr{A}(S)$ equals the closure of $\{p(S)\colon p \text{ is a polynomial}\}$ in the weak operator topology (WOT).

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, an operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is said to be quasi-invertible if it is injective and has dense range; that is, if $\ker X = (0)$ and $(\operatorname{ran} X)^- = \mathscr{H}_2$. If $S_j \in \mathscr{B}(\mathscr{H}_j)$ $(j = 1, 2)$, then S_1 is *quasisimilar* to S_2 if there are quasi-invertible operators $X_{21}\colon \mathscr{H}_1 \to \mathscr{H}_2$ and $X_{12}\colon \mathscr{H}_2 \to \mathscr{H}_1$ such that $X_{21}S_1 = S_2X_{21}$ and $X_{12}S_2 = S_1X_{12}$. Denote this by $S_1 \sim S_2$. This equivalence relation of quasisimilarity was introduced by Sz.-Nagy and Foias (see

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[12]) and has received considerable attention. Unlike similarity, quasisimilar operators need not have equal spectra [10], though their spectra cannot be disjoint [10]. However, quasisimilar subnormal operators must have equal spectra [4].

In Section 1 of this paper it is shown that if S_1 and S_2 are subnormal operators and X_{21} and X_{12} are quasi-invertible operators such that $X_{21}S_1 = S_2X_{21}$ and $X_{12}S_2 = S_1X_{12}$, then there is an isometric isomorphism $\rho\colon \mathscr{A}(S_1) \to \mathscr{A}(S_2)$ such that $\rho(S_1) = S_2$ and for A_1 in $\mathscr{A}(S_1)$, $X_{21}A_1 = \rho(A_1)X_{21}$, and $X_{12}\rho(A_1) = A_1X_{12}$; moreover, ρ is a weak* homeomorphism. When combined with Theorem 2.1 of [6], this result gives necessary conditions that subnormal operators be quasisimilar in terms of their minimal normal extensions. For $j = 1, 2$, let N_j be the minimal normal extension (mne) of S_j, and let μ_j be a scalar-valued spectral measure for N_j. If $P^\infty(\mu_j)$ denotes the weak* closure of the polynomials in $L^\infty(\mu_j)$, then the above result implies that if S_1 and S_2 are quasisimilar subnormal operators, then the identity map on the polynomials extends to a weak* homeomorphic isomorphism of $P^\infty(\mu_1)$ onto $P^\infty(\mu_2)$; in particular, μ_1 and μ_2 have the same Sarason hulls [16].

It is known that if a subnormal operator S is quasisimilar to a normal operator, then S is normal [14]. From this it is easy to deduce that if S_1 and S_2 are subnormal, $S_1 \sim S_2$, and if S_1 is pure (that is, S_1 has no normal direct summand), then S_2 must be pure. For $j = 1, 2$ let $S_j = N_j \oplus T_j$ on $\mathscr{H}_j = \mathscr{N}_j \oplus \mathscr{T}_j$, where N_j is normal and T_j is pure. In Section 2 it is shown that if $S_1 \sim S_2$, then N_1 and N_2 are unitarily equivalent. An example is produced where $S_1 \sim S_2$, but T_1 and T_2 are not quasisimilar. However, if S_1 and S_2 are cyclic it is shown that $S_1 \sim S_2$ implies that $T_1 \sim T_2$.

In Section 3, the first of two applications of the results of Sections 1 and 2 are given. If N is a normal operator on $\mathscr{K}$ then a *part of* N is the restriction $N \mid \mathscr{H}$ of N to an invariant subspace of N. A part $N \mid \mathscr{H}$ is *simple* if $\mathscr{H}$ is not a reducing subspace; equivalently, if $N \mid \mathscr{H}$ is subnormal but not normal. In Theorem 3.1 it is shown that N has the property that all its simple parts are quasisimilar if and only if N is unitarily equivalent to multiplication by ϕ on L^2 of the circle, where ϕ is a weak* generator of H^∞, the space of bounded analytic functions on the unit disk. This condition is also equivalent to the requirement that all the simple parts of N be unitarily equivalent or similar.

A second application is given in Section 4. Clary [5] has given necessary and sufficient conditions that a subnormal operator be quasisimilar to the unilateral shift of multiplicity one. We will use our Theorem 1.4 to prove that Clary's conditions are necessary.

In this paper, $\square$ denotes the empty set.

1. The main result

A somewhat more general result than the one stated in the introduction will be proved. For a compact subset K of $\mathbf{C}$, let $C(K)$ be the algebra of continuous complex-valued functions on K with the supremum norm, $\|\cdot\|_K$. Let $R(K)$ be the closure in $C(K)$ of the subalgebra of rational functions with poles off K. If S is a subnormal operator and K contains $\sigma(S)$, the spectrum of S, then $f(S)$ is a well defined subnormal operator for each f in $R(K)$. Let $\mathscr{R}(S;K)$ be the weak* closure of $\{f(S): f \in R(K)\}$. It is not difficult to show that $\mathscr{R}(S;K)$ is an algebra. If K is the polynomially convex hull of $\sigma(S)$, then Runge's Theorem implies that $\mathscr{A}(S) = \mathscr{R}(S;K)$.

The next result is due to Clary [4] and will be used repeatedly in this paper, often without further reference.

1.1 THEOREM. *If S_j is a subnormal operator on $\mathscr{H}_j$ $(j = 1, 2)$ and $X: \mathscr{H}_1 \to \mathscr{H}_2$ is a quasi-invertible operator such that $XS_1 = S_2X$, then $\sigma(S_2) \subseteq \sigma(S_1)$.*

1.2 THEOREM. *If S_j is a subnormal operator on $\mathscr{H}_j$ $(j = 1, 2)$, if $X: \mathscr{H}_1 \to \mathscr{H}_2$ is a quasi-invertible operator such that $XS_1 = S_2X$, and if K is a compact set containing $\sigma(S_1)$, then there is a contractive monomorphism $\rho: \mathscr{R}(S_1;K) \to \mathscr{R}(S_2;K)$ such that:*

(a) $\rho(S_1) = S_2$.
(b) *$XA = \rho(A)X$ for every A in $\mathscr{R}(S_1;K)$.*
(c) *ρ is weak* continuous.*

The proof of this theorem is based on the following lemma.

1.3 LEMMA. *If $\mathscr{A}_j$ is a algebra of subnormal operators on $\mathscr{H}_j$ $(j = 1, 2)$, $X: \mathscr{H}_1 \to \mathscr{H}_2$ is a quasi-invertible operator, and $\rho: \mathscr{A}_1 \to \mathscr{A}_2$ is a contractive monomorphism such that $XA = \rho(A)X$ for all A in $\mathscr{A}_1$; then ρ extends to a contractive monomorphism $\tilde{\rho}: \mathscr{B}_1 \to \mathscr{B}_2$ such that $XB = \tilde{\rho}(B)X$ for all B in $\mathscr{B}_1$, where $\mathscr{B}_j$ is the sequential closure of $\mathscr{A}_j$ in the weak operator topology.*

Proof. If $B \in \mathscr{B}_1$, let $\{A_n\}$ be a sequence in $\mathscr{A}_1$ such that $A_n \to B$ (WOT) (WOT = weak operator topology). Thus,

$$M = \sup\{\|A_n\|: n \geq 1\} < \infty.$$

Since ρ is contractive, $\|\rho(A_n)\| \leq M$ for all n. Let $C \in \mathscr{B}_2$ and let $\{A_{n_k}\}$ be a subsequence of $\{A_n\}$ such that $\rho(A_{n_k}) \to C$ (WOT) as $k \to \infty$. Now $XA_{n_k} = \rho(A_{n_k})X$ for all k, so $XB = CX$. Moreover, since B and C are subnormal, $\sigma(C) \subseteq \sigma(B)$ by Theorem 1.1. Hence $\|C\| \leq \|B\|$.

If $\{B_k\}$ is any other sequence in $\mathscr{A}_1$ such that $B_k \to B$ (WOT) and $\rho(B_k) \to C'$ (WOT) as $k \to \infty$, then

$$B_k - A_{n_k} \to 0 \text{ (WOT)} \quad \text{and} \quad \rho(B_k) - \rho(A_{n_k}) \to C' - C \text{ (WOT)} \quad \text{as } k \to \infty.$$

By the above reasoning, $\|C' - C\| \leq 0$; that is, $C = C'$. This means that $\{\rho(A_n)\}$ has a unique WOT cluster point C. Therefore $\rho(A_n) \to C$ (WOT) as $n \to \infty$. Moreover, this argument also shows that C is independent of the choice of a sequence $\{A_n\}$ from $\mathscr{A}_1$ such that $A_n \to B$ (WOT). Therefore, $\tilde{\rho}(B) = C$ gives a well defined map $\tilde{\rho}$: $\mathscr{B}_1 \to \mathscr{B}_2$ and $XB = \tilde{\rho}(B)X$ for all B in $\mathscr{B}_1$.

Clearly $\tilde{\rho}$ is an extension of ρ and $\tilde{\rho}$ is contractive. If B and $B' \in \mathscr{B}_1$ and $\{A_n\}$ and $\{A'_n\}$ are sequences from $\mathscr{A}_1$ such that $A_n \to B$ and $A'_n \to B'$ (WOT) as $n \to \infty$, then, for fixed m, $A_n A'_m \to BA'_m$ (WOT) as $n \to \infty$. Hence, $\rho(A_n)\rho(A'_m) \to \tilde{\rho}(BA'_m)$ as $n \to \infty$. It follows that $\tilde{\rho}(BA'_m) = \tilde{\rho}(B)\rho(A'_m)$. Letting $m \to \infty$, it follows that $\tilde{\rho}$ is multiplicative. Since linearity is an easy exercise, it follows that $\tilde{\rho}$ is a contractive homomorphism such that $XB = \tilde{\rho}(B)X$ for B in $\mathscr{B}_1$. Because $\ker X = (0)$, it follows that $\tilde{\rho}$ is injective. ∎

Proof of Theorem 1.2. For $j = 1, 2$, let

$$\mathscr{R}_0(S_j; K) = \{f(S_j): f \text{ is a rational function with poles off } K\}.$$

Let $\mathscr{R}_1(S_j; K)$ be the WOT sequential closure of $\mathscr{R}_0(S_j; K)$. Define $\mathscr{R}_\alpha(S_j; K)$ inductively for each ordinal number α by letting $\mathscr{R}_\alpha(S_j; K)$ be the WOT sequential closure of $\mathscr{R}_{\alpha-1}(S_j; K)$ when α has an immediate predecessor $\alpha - 1$; and

$$\mathscr{R}_\alpha(S_j; K) = \bigcup \{\mathscr{R}_\beta(S_j; K): \beta < \alpha\}$$

when α is a limit ordinal.

Since $XS_1 = S_2 X$, it is easy to show that $Xf(S_1) = f(S_2)X$ for f a rational function with poles off K. Define

$$\rho_0: \mathscr{R}_0(S_1; K) \to \mathscr{R}_0(S_2; K)$$

by $\rho_0(f(S_1)) = f(S_2)$. Since $\sigma(S_2) \subseteq \sigma(S_1)$,

$$\|f(S_2)\| = \|f\|_{\sigma(S_2)} \leq \|f\|_{\sigma(S_1)} = \|f(S_1)\|.$$

Hence ρ_0 is a contractive monomorphism and $XA = \rho_0(A)X$ for every A in $\mathscr{R}_0(S_1; K)$.

Using Lemma 1.3 and transfinite induction, for every ordinal α there is a contractive monomorphism ρ_α: $\mathscr{R}_\alpha(S_1; K) \to \mathscr{R}_\alpha(S_2; K)$ such that $XA = \rho_\alpha(A)X$ for every A in $\mathscr{R}_\alpha(S_1; K)$ and ρ_α is an extension of ρ_β if $\beta < \alpha$. But [1, p. 213] $\mathscr{R}(S_j; K) = \mathscr{R}_\alpha(S_j; K)$ for α the first uncountable ordinal; so let $\rho = \rho_\alpha$ for this α.

By the induction process, ρ satisfies (a) and (b). By (b), ρ is WOT sequentially continuous. Since the WOT and the weak* topology agree on bounded subsets of $\mathscr{B}(\mathscr{H})$, ρ is weak* sequentially continuous. An application of the Krein-Smulian Theorem implies that ρ is weak* continuous. ∎

It is easy to obtain the main result as a consequence of Theorem 1.2.

1.4. Theorem. *If S_j is a subnormal operator on $\mathscr{H}_j$ and $X_{ij}: \mathscr{H}_j \to \mathscr{H}_i$ are quasi-invertible operators such that $X_{ij}S_j = S_i X_{ij}$ $(i, j = 1, 2)$, and if K is a compact set containing $\sigma(S_1)$, then there is an isometric isomorphism ρ: $\mathscr{R}(S_1; K) \to \mathscr{R}(S_2; K)$ such that:*

(a) $\rho(S_1) = S_2$.
(b) $X_{21}A = \rho(A)X_{21}$ *and* $X_{12}\rho(A) = AX_{12}$ *for all* A *in* $\mathscr{R}(S_1; K)$.
(c) ρ *is a weak* homeomorphism.*

Proof. By Theorem 1.2, there are contractive monomorphisms

$$\rho\colon \mathscr{R}(S_1; K) \to \mathscr{R}(S_2; K) \quad \text{and} \quad \eta\colon \mathscr{R}(S_2; K) \to \mathscr{R}(S_1; K)$$

such that $\rho(S_1) = S_2$, $\eta(S_2) = S_1$, and ρ and η are weak* continuous. Hence $\rho \circ \eta$ is weak* continuous and $\rho \circ \eta(A) = A$ on the weak* dense subalgebra

$$\{f(S_2)\colon f \text{ is a rational function with poles off } K\}.$$

So $\rho \circ \eta$ is the identity. Similarly, $\eta \circ \rho$ is the identity map, and so $\eta = \rho^{-1}$. ∎

Let S be a subnormal operator and let N be its *mne*. If μ is a scalar-valued spectral measure for N, then $W^*(N)$, the von Neumann algebra generated by N, is isometrically and weak* homeomorphically isomorphic to $L^\infty(\mu)$, by means of the functional calculus $\phi \mapsto \phi(N)$ (see [7, p. 112]). If $\sigma(S) \subseteq K$, then the proof of Theorem 2.1 in [6] can be used to show that

$$\mathscr{R}(S; K) = \{\phi(S)\colon \phi \in R^\infty(\mu; K)\},$$

where $R^\infty(\mu; K)$ is the weak* closure in $L^\infty(\mu)$ of $R(K)$. In particular, $\mathscr{A}(S) = \{\phi(S)\colon \phi \in P^\infty(\mu)\}$, where $P^\infty(\mu)$ is the weak* closure of the polynomials in $L^\infty(\mu)$. Moreover these identifications are isometric isomorphisms that are weak* homeomorphisms.

This allows function-theoretic interpretations of Theorems 1.2 and 1.4. In particular, we will state the interpretation of Theorem 1.4 for the case where K is the polynomially convex hull of $\sigma(S_1)$.

To interpret Theorem 1.4 for $P^\infty(\mu)$, the following characterization of that space by Sarason [16] is needed. Let $\mathscr{P}$ be the collection of all polynomials in one complex variable.

1.5 THEOREM. *If μ is a compactly supported measure in the plane, then there is a pair $(G, \tilde{\mu})$, where G is a bounded open subset of* $\mathbf{C}$, *$\tilde{\mu}$ is a measure such that $\tilde{\mu} \leq \mu$, and the following hold:*

(a) $\tilde{\mu} \perp \mu - \tilde{\mu}$.
(b) $P^\infty(\mu) = L^\infty(\mu - \tilde{\mu}) \oplus P^\infty(\tilde{\mu})$.
(c) *If $\mathscr{P}$ is considered as a subspace of both $H^\infty(G)$ and $P^\infty(\mu)$, then the identity map $p \to p$ on $\mathscr{P}$ extends to an isometric isomorphism $P^\infty(\tilde{\mu}) \approx H^\infty(G)$.*
(d) $\partial G \subseteq \text{support } \tilde{\mu} \subseteq G^-$.
(f) *$\eta\colon P^\infty(\tilde{\mu}) \to \mathbf{C}$ is a weak* continuous multiplicative linear functional if, and only if, there is a z in G such that $\eta(\phi) = \phi(z)$ for all ϕ in $P^\infty(\tilde{\mu})$ $(\approx H^\infty(G))$.*

(g) *If G_1 is a component of G, then G_1 is simply connected. Moreover, if $\tau\colon \mathbf{D} \to G_1$ is a conformal map, then τ is a weak* generator of $H^\infty = H^\infty(\mathbf{D})$.*
(h) *$R(G^-)$ is a Dirichlet algebra.*
(i) *$\tilde{\mu} \mid \partial G$, the restriction of $\tilde{\mu}$ to ∂G, is absolutely continuous with respect to harmonic measure for G^-.*

Although this theorem is not stated in this way in [16], it is easily derivable from the main result there. The reader may also consult [6] and [15]. The pair $(G, \tilde{\mu})$ is called the *Sarason hull* of μ.

The next result is an immediate consequence of Theorem 1.4, Theorem 1.5, and Theorem 2.1 of [6].

1.6 THEOREM. *Let S_1, S_2 be subnormal operators with minimal normal extensions N_1, N_2, let μ_1, μ_2 be scalar-valued spectral measures for N_1, N_2, and let $(G_1, \tilde{\mu}_1)$ and $(G_2, \tilde{\mu}_2)$ be the corresponding Sarason hulls. If S_1 and S_2 are quasisimilar, then:*

(a) *$\mu_1 - \tilde{\mu}_1$ and $\mu_2 - \tilde{\mu}_2$ are mutually absolutely continuous.*
(b) *$G_1 = G_2$.*
(c) *The identity map on the polynomials extends to an isometric isomorphism of $P^\infty(\mu_1)$ onto $P^\infty(\mu_2)$ that is a weak* homeomorphism.*

Actually, condition (c) of the preceding theorem is a consequence of (a) and (b); it is only stated for emphasis.

2. The pure and normal parts of quasisimilar subnormal operators

Some of the results of this section will be needed in the following one. In addition, they seem to be sufficiently interesting by themselves to merit separate consideration.

The following result is due to Radjavi and Rosenthal [14, p. 655]. Several generalizations and variations have also appeared in the literature [2], [18].

2.1 LEMMA. *If $S_j \in \mathscr{B}(\mathscr{H}_j)$ $(j = 1, 2)$, $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is quasi-invertible, S_1^* and S_2 are subnormal, and $XS_1 = S_2X$, then S_1 and S_2 are normal and unitarily equivalent.*

An easy consequence of Lemma 2.1 (obtained below) is the fact that if two subnormal operators are quasisimilar and one is pure (that is, it has no normal direct summand), then so is the other. However, it does not follow that the pure parts are quasisimilar, as the following example illustrates.

Let $\Delta = \{z \in \mathbf{C}\colon |z| < \frac{1}{2}\}$ and let $L_a^2(\Delta)$ denote the Bergman space of analytic functions defined on Δ that are square integrable with respect to area measure. Let

$$\mathscr{F} = H^2 \oplus H^2 \oplus \cdots, \qquad \mathscr{G} = L^2(0, \tfrac{1}{2}) \oplus L^2(0, \tfrac{1}{2}) \oplus \cdots,$$

and put

$$\mathscr{H}_1 = \mathscr{F} \oplus L_a^2(\Delta) \oplus \mathscr{G}, \qquad \mathscr{H}_2 = \mathscr{F} \oplus \mathscr{G}.$$

Denote the identity function on $(0, \frac{1}{2})$ by t; that is, $t(a) = a$. If $f = (f_1, f_2, \ldots) \in \mathscr{F}$, $g = (g_1, g_2, \ldots) \in \mathscr{G}$, and $h \in L_a^2(\Delta)$, define $S_j \in \mathscr{B}(\mathscr{H}_j)$ by

$$\begin{aligned} S_1(f \oplus h \oplus g) &= zf \oplus zh \oplus tg \\ &= (zf_1, zf_2, \ldots) \oplus zh \oplus (tg_1, tg_2, \ldots), \\ S_2(f \oplus g) &= zf \oplus tg. \end{aligned}$$

Define $X_{ij}\colon \mathscr{H}_j \to \mathscr{H}_i$ $(i, j = 1, 2)$ by

$$\begin{aligned} X_{21}(f \oplus h \oplus g) &= f \oplus (h|(0, \tfrac{1}{2}), g_1, g_2, \ldots), \\ X_{12}(f \oplus g) &= (f_2, f_3, \ldots) \oplus f_1 \,|\, \Delta \oplus g. \end{aligned}$$

If $X_{21}(f \oplus h \oplus g) = 0$, then $f = 0$, $g = 0$, and $h = 0$ on $(0, \frac{1}{2})$; since h is analytic, $h = 0$. Hence $\ker X_{21} = (0)$. Similarly, $\ker X_{12} = (0)$. Clearly $\operatorname{ran} X_{21}$ contains

$$\{f \oplus (t^n, g_1, g_2, \ldots)\colon f \in \mathscr{F},\ g \in \mathscr{G}, \text{ and } n \geq 0\};$$

so X_{21} is quasi-invertible. Similarly, $\operatorname{ran} X_{12}$ contains

$$\{f \oplus p \oplus g\colon f \in \mathscr{F},\ g \in \mathscr{G}, \text{ and } p \text{ is a polynomial}\};$$

hence, $\operatorname{ran} X_{12}$ is dense and X_{12} is quasi-invertible. It is an easy matter to check that $X_{21}S_1 = S_2X_{21}$ and $X_{12}S_2 = S_1X_{12}$; hence S_1 and S_2 are quasisimilar.

Let T_j be the pure part of S_j $(j = 1, 2)$. So $T_j = S_j \,|\, \mathscr{L}_j$, where $\mathscr{L}_1 = \mathscr{F} \oplus L_a^2(\Delta)$, $\mathscr{L}_2 = \mathscr{F}$. It is claimed that T_1 and T_2 are not quasisimilar. In fact, suppose there is a quasi-invertible operator $Z\colon \mathscr{L}_1 \to \mathscr{L}_2$ such that $ZT_1 = T_2Z$. Let

$$\mathscr{M} = [Z((0) \oplus L_a^2(\Delta))]^- \subseteq \mathscr{L}_2 = \mathscr{F}.$$

It follows that $\mathscr{M} \in \operatorname{Lat} T_2$ and $A_2 = T_2 \,|\, \mathscr{M}$ is an isometry. Let $W\colon L_a^2(\Delta) \to \mathscr{M}$ be defined by $Wh = Z(0 \oplus h)$. Then W is quasi-invertible and $WA_1 = A_2W$, where A_1 is multiplication by z on $L_a^2(\Delta)$. This contradicts Theorem 1.1, and, hence, no such Z can exist. Another example of this phenomenon can be found in Hastings [8].

What is true is that quasisimilar subnormal operators must have unitarily equivalent normal parts. To show this, the following analogue of the Schroeder-Bernstein Theorem is needed. The result is due to Kadison and Singer [11] and the proof will not be presented.

2.2 Lemma. *If A_1, A_2 are operators on $\mathscr{H}_1$, $\mathscr{H}_2$ such that there exists reducing subspaces $\mathscr{L}_1$, $\mathscr{L}_2$ with A_1 unitarily equivalent to $A_2 \,|\, \mathscr{L}_2$ and A_2 unitarily equivalent to $A_1 \,|\, \mathscr{L}_1$, then A_1 and A_2 are unitarily equivalent.*

In the above lemma, if $\mathscr{L}_1$ and $\mathscr{L}_2$ are only assumed to be invariant and not reducing, the above argument is not valid. It might be of interest to study this more restrictive equivalence relation. (Is it more restrictive?) Notice that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are only assumed to be invariant and A_1 and A_2 are normal, then it follows that $\mathscr{L}_1$ and $\mathscr{L}_2$ are reducing subspaces.

To facilitate the exposition, the following notation is useful. $S_1 \cong S_2$ means S_1 and S_2 are unitarily equivalent; $S_1 \approx S_2$ means S_1 and S_2 are similar; $S_1 \sim S_2$ means S_1 and S_2 are quasisimilar.

2.3 PROPOSITION. *For $j = 1, 2$, let S_j be a subnormal operator on $\mathscr{H}_j$ and suppose $\mathscr{H}_j = \mathscr{N}_j \oplus \mathscr{T}_j$ such that $\mathscr{N}_j$ reduces S_j, $N_j = S_j | \mathscr{N}_j$ is normal, and $S_j | \mathscr{T}_j$ is pure. If $S_1 \sim S_2$, then $N_1 \cong N_2$.*

Proof. Let $X_{ij}\colon \mathscr{H}_j \to \mathscr{H}_i$ $(i, j = 1, 2)$ be quasi-invertible operators such that $X_{ij}S_j = S_i X_{ij}$. It follows that $\mathscr{M}_2 = [X_{21}\mathscr{N}_1]^-$ is an invariant subspace for S_2. Also $X = X_{21} | \mathscr{N}_1\colon \mathscr{N}_1 \to \mathscr{M}_2$ is quasi-invertible and $XN_1 = (S_2 | \mathscr{M}_2)X$. By Lemma 2.1, $S_2 | \mathscr{M}_2$ is normal and $N_1 \cong S_2 | \mathscr{M}_2$. Hence $\mathscr{M}_2 \subseteq \mathscr{N}_2$ and N_1 is unitarily equivalent to a part of N_2. Similarly, N_2 is unitarily equivalent to a part of N_1. By Lemma 2.2, $N_1 \cong N_2$. ∎

This result was obtained independently by Hastings [8] for the more general dominant operators.

If S_1 and S_2 are cyclic subnormal operators and $S_1 \sim S_2$, then not only are the normal parts unitarily equivalent, but the pure parts must be quasisimilar. To prove this the following lemma is needed.

2.4 LEMMA. *If S_1, S_2 are subnormal operators on $\mathscr{H}_1$, $\mathscr{H}_2$, $X_{ij}\colon \mathscr{H}_j \to \mathscr{H}_i$ are operators with dense range such that $X_{ij}S_j = S_i X_{ij}$, and if S_1 is cyclic, then S_2 is cyclic and each X_{ij} is injective. In particular, $S_1 \sim S_2$.*

Proof. If p is a polynomial, then $X_{21}p(S_1) = p(S_2)X_{21}$. So if e_1 is a cyclic vector for S_1, then $X_{21}e_1 = e_2$ is a cyclic vector for S_2. Thus, it may be assumed that S_j is multiplication by z on $H^2(\mu_j)$ for some compactly supported measure on the plane [3].

Now $X_{12}X_{21} \in \{S_1\}'$. By Yoshino's Theorem [19], there is a function ϕ in $H^2(\mu_1) \cap L^\infty(\mu_1)$ such that $X_{12}X_{21} = M_\phi$, multiplication by ϕ. Since X_{12} and X_{21} have dense range, so does M_ϕ. In particular, ϕ cannot vanish on a set with positive μ_1 measure. Therefore, M_ϕ is injective. But $\ker X_{21} \subseteq \ker X_{12}X_{21} = \ker M_\phi = (0)$, so X_{21} is injective. Similarly, X_{12} is injective. ∎

2.5 PROPOSITION. *If S_1, S_2 are cyclic subnormal operators on $\mathscr{H}_1$, $\mathscr{H}_2$, $S_j = N_j \oplus T_j$ on $\mathscr{H}_j = \mathscr{N}_j \oplus \mathscr{T}_j$, where N_j is normal and T_j is pure, then $S_1 \sim S_2$ iff $N_1 \cong N_2$ and $T_1 \sim T_2$.*

Proof. Suppose X_{ij}: $\mathscr{H}_j \to \mathscr{H}_i$ are quasi-invertible operators such that $X_{ij} S_j = S_i X_{ij}$ $(i, j = 1, 2)$. Let Q_j be the projection of $\mathscr{H}_j$ onto $\mathscr{T}_j$, and define Y_{ij}: $\mathscr{T}_j \to \mathscr{T}_i$ by $Y_{ij} = Q_i X_{ij} | \mathscr{T}_j$. If $y \in \mathscr{T}_i$, then there is a sequence $\{x_n\}$ in $\mathscr{H}_j$ such that $X_{ij} x_n \to y$ as $n \to \infty$. Let $x_n = w_n + t_n$, $w_n \in \mathscr{N}_j$ and $t_n \in \mathscr{T}_j$. From the proof of Proposition 2.3, $X_{ij} \mathscr{N}_j \subseteq \mathscr{N}_i$; so $X_{ij} w_n \in \mathscr{N}_i$. Hence

$$Y_{ij} t_n = Q_i X_{ij} t_n = Q_i X_{ij} x_n \to Q_i y = y.$$

So ran Y_{ij} is dense. Also $Q_i \in \{S_i\}'$, so $Y_{ij} T_j = T_i Y_{ij}$. Because S_1 is cyclic, it follows that T_1 is cyclic. By the preceding lemma, Y_{ij} is injective and $T_1 \sim T_2$. Since $N_1 \cong N_2$ by Proposition 2.3, this completes the proof of half this proposition.

The proof of the converse is straightforward. ∎

It should be mentioned that for similarity these difficulties do not arise.

2.6 Proposition. *With the notation the same as in Proposition* 2.5, $S_1 \approx S_2$ *iff* $N_1 \cong N_2$ *and* $T_1 \approx T_2$.

Proof. Suppose $S_1 \approx S_2$, and let R: $\mathscr{H}_1 \to \mathscr{H}_2$ be an invertible operator such that $RS_1 = S_2 R$. As in the proof of Proposition 2.3, $R(\mathscr{N}_1) \subseteq \mathscr{N}_2$. Also $R^{-1}(\mathscr{N}_2) \subseteq \mathscr{N}_1$, so that $R(\mathscr{N}_2) = \mathscr{N}_1$.

If $\mathscr{L}_2 = R(\mathscr{T}_1)$, then $\mathscr{L}_2$ is an invariant subspace for S_2 and $T_1 \approx S_2 | \mathscr{L}_2$. Because R is invertible and $\mathscr{N}_2 = R\mathscr{N}_1$, $\mathscr{L}_2 \cap \mathscr{N}_2 = (0)$ and $\mathscr{H}_2 = \mathscr{N}_2 + \mathscr{L}_2$ (not an orthogonal sum). Let Q_2 be the orthogonal projection of $\mathscr{H}_2$ onto $\mathscr{T}_2$, and define A: $\mathscr{T}_1 \to \mathscr{T}_2$ by $A = Q_2 R | \mathscr{T}_1$. It is easy to check that A is injective. If $t_2 \in \mathscr{T}_2$, then $t_2 = y_2 + n_2$ for unique vectors y_2 in $\mathscr{L}_2$ and n_2 in $\mathscr{N}_2$. Let $x_1 = R^{-1} t_2 = t_1 + n_1$, where $t_1 \in \mathscr{T}_1$ and $n_1 \in \mathscr{N}_1$, $y_2 + n_2 = t_2 = Rx_1 = Rt_1 + Rn_1$, and $Rt_1 \in \mathscr{L}$ and $Rn_1 \in \mathscr{N}_2$. Hence $t_2 = Q_2 t_2 = Q_2 R x_1 = Q_2 R t_1 = A t_1$, and so A is surjective. Therefore A is invertible. Because $Q_2 \in \{S_2\}'$, $A T_1 A^{-1} = T_2$. So $T_1 \approx T_2$.

The converse is straightforward. ∎

This result also follows from Lemma 1 of [14].

3. Parts of a normal operator

If N is a normal operator on a Hilbert space $\mathscr{K}$, a *part* of N is the restriction of N to one of its invariant subspaces $\mathscr{H}$. So if $N | \mathscr{H}$ is a part of N, $N | \mathscr{H}$ is subnormal. It is a standard fact that $N | \mathscr{H}$ is normal iff $\mathscr{H}$ reduces N. If $\mathscr{H}$ does not reduce N, then $N | \mathscr{H}$ is called a *simple* part of N.

We make the following definitions. Say that N satisfies: (a) Condition (U) if any two simple parts are unitarily equivalent; (b) Condition (S) if any two simple parts are similar; (c) Condition (QS) if any two simple parts are quasisimilar.

Notice that it is pointless to make a restriction on the nonsimple parts of N. Indeed, if it were required that *all* parts of N be unitarily equivalent, then, by considering the restriction of N to its spectral subspaces, it would follow that N is a multiple of the identity.

There are normal operators that satisfy Condition (U) in a vacuous way; that is, they have no simple parts. Such operators are called *reductive* since each invariant subspace reduces N. A necessary and sufficient condition for N to be reductive is that $P^\infty(\mu) = L^\infty(\mu)$, where μ is a scalar-valued spectral measure for N. In other words, N is reductive iff the Sarason hull of its scalar-valued spectral measure is $(\square, 0)$.

A nontrivial example of a normal operator that satisfies Condition (U) is the bilateral shift of multiplicity one. If ω is normalized arc length measure on the unit circle $\partial\mathbf{D}$ and N is multiplication by z on $L^2(\omega)$, then a simple part $N \mid \mathscr{H}$ of N must be an isometry that has N as a unitary extension. Since $N \mid \mathscr{H}$ is not unitary, it must be a unilateral shift of multiplicity one. Hence N satisfies Condition (U).

If $\phi \in H^\infty$, then ϕ is a weak* generator of H if the weak* closed algebra generated by ϕ is H^∞. Thus, ϕ is a weak* generator of H^∞ iff there is a net of polynomials $\{p_\alpha\}$ such that $p_\alpha(\phi) \to z$ weak* in $H^\infty \subseteq L^\infty(\omega)$. The weak* generators of H^∞ where characterized by Sarason [15]. If N is the bilateral shift of multiplicity one and ϕ is a weak* generator of H^∞, then it is easy to see that a subspace $\mathscr{H}$ of $L^2(\omega)$ is invariant for N iff it is invariant for $\phi(N)$. Also, $\mathscr{H}$ is reducing for N if $\mathscr{H}$ is reducing for $\phi(N)$. It follows that $\phi(N)$ also satisfies Condition (U), since all its simple parts are unitarily equivalent to the analytic Toeplitz operator T_ϕ on H^2. It will now be shown that, up to unitary equivalence, these are the only normal operators satisfying Condition (U). In fact, these are the only normal operators satisfying Condition (S) or Condition (QS).

3.1 THEOREM. *If N is a nonreductive normal operator, then the following are logically equivalent statements:*

(a) *N satisfies Condition* (U)
(b) *N satisfies Condition* (S)
(c) *N satisfies Condition* (QS)
(d) *There is a weak* generator ϕ of H^∞ such that N is unitarily equivalent to multiplication by ϕ on $L^2(\omega)$.*

Proof. Using the remarks preceding the statement of the theorem and disposing of certain trivial implications, it is easy to see that to complete the proof it is only necessary to prove that (c) implies (d). So suppose N satisfies Condition (QS). Let μ be a scalar-valued spectral measure for N. So $\mathscr{A}(N) = \{\phi(N): \phi \in P^\infty(\mu)\}$.

3.2 *Claim.* If $(G, \tilde{\mu})$ is the Sarason hull of μ, then G is connected and $\tilde{\mu} = \mu$.

If $\tilde{\mu} \neq \mu$, then there is a nontrivial reducing subspace $\mathscr{R}$ for N such that $\mu - \tilde{\mu}$ is a scalar-valued spectral measure for $N \mid \mathscr{R}$ and $\tilde{\mu}$ is a scalar-valued spectral measure for $N \mid \mathscr{R}^{\perp}$ (see Theorem 7.1 of [6], for example). Since $P^{\infty}(\mu - \tilde{\mu}) = L^{\infty}(\mu - \tilde{\mu})$, $N \mid \mathscr{R}$ is reductive. Also $N \mid \mathscr{R}^{\perp}$ has no reductive direct summand. By Theorem 9.3 of [6], there is a nonreducing invariant subspace $\mathscr{M}$ for $N \mid \mathscr{R}^{\perp}$. Moreover, $\mathscr{M}$ can be chosen such that $N \mid \mathscr{R}^{\perp}$ is the minimal normal extension of $N \mid \mathscr{M}$. Hence [6, Theorem 2.1] $\mathscr{A}(N \mid \mathscr{M})$ and $\mathscr{A}(N \int \mathscr{R}^{\perp})$ are isomorphic algebras, and they are isomorphic to $P^{\infty}(\tilde{\mu})$. But $N \mid \mathscr{M}$ and $N \mid (\mathscr{R} \oplus \mathscr{M})$ are both simple parts of N, and $\mathscr{A}(N \mid \mathscr{M})$ and $\mathscr{A}(N \mid \mathscr{R} \oplus \mathscr{M}) = \mathscr{A}(N \mid \mathscr{R}) \oplus \mathscr{A}(N \mid \mathscr{M})$ are not isomorphic. By Theorem 1.4, $N \mid \mathscr{M}$ and $N \mid \mathscr{R} \oplus \mathscr{M}$ cannot be quasisimilar, contradicting Condition (QS). Therefore $\tilde{\mu} = \mu$.

If G is not connected and G_1, G_2 are distinct components of G, then their characteristic functions belong to $P^{\infty}(\mu)$ by Theorem 1.5. Therefore, there are reducing subspaces $\mathscr{R}_1$ and $\mathscr{R}_2$ of N such that $\mathscr{A}(N \mid \mathscr{R}_1)$ and $\mathscr{A}(N \mid \mathscr{R}_2)$ are isomorphic to $H^{\infty}(G_1)$ and $H^{\infty}(G_2)$. But [6, Theorem 9.3] there are simple invariant subspaces $\mathscr{M}_1$, $\mathscr{M}_2$ of $\mathscr{R}_1$, $\mathscr{R}_2$ such that $N \mid \mathscr{R}_1$ and $N \mid \mathscr{R}_2$ are the minimal normal extensions of $N \mid \mathscr{M}_1$ and $N \mid \mathscr{M}_2$. Hence $\mathscr{A}(N \mid \mathscr{M}_1)$ is isomorphic to $\mathscr{A}(N \mid \mathscr{R}_1)$, and $\mathscr{A}(N / \mathscr{M}_2)$ is isomorphic to $\mathscr{A}(N \mid \mathscr{R}_2)$. Since $G_1 \cap G_2 = \square$, $\mathscr{A}(N \mid \mathscr{M}_1)$ and $\mathscr{A}(N \mid \mathscr{M}_2)$ cannot be isomorphic. By Theorem 1.4, $N \mid \mathscr{M}_1$ and $N \mid \mathscr{M}_2$ cannot be quasisimilar, contradicting (c).

Therefore, Claim 3.2 is established.

3.3 *Claim.* N is cyclic.

If $N = \int z \, dE(z)$ is the spectral decomposition of N, there is a vector e_0 in $\mathscr{K}$ such that $\mu(\Delta) = \|E(\Delta)e_0\|^2$ for every Borel subset Δ of $\mathbf{C}$ [7, p. 110]. Let

$$\mathscr{K}_0 = [N^{*k}N^n e_0 \colon k, n \geq 0].$$

Thus, $\mathscr{K}_0$ reduces N, $N \mid \mathscr{K}_0$ is a cyclic normal operator, and μ is a scalar-valued spectral measure for $N \mid \mathscr{K}_0$. In particular, both $\mathscr{A}(N)$ and $\mathscr{A}(N \mid \mathscr{K}_0)$ are isomorphic to $H^{\infty}(G)$, or $P^{\infty}(\mu)$. By Proposition 9.6 of [6], there is an invariant subspace $\mathscr{M}$ of $N \mid \mathscr{K}_0$ such that $N \mid \mathscr{M}$ is a pure subnormal operator and $N \mid \mathscr{K}_0$ is the minimal normal extension of $N \mid \mathscr{M}$. Now $N \mid \mathscr{M}$ and $N \mid (\mathscr{M} \oplus (\mathscr{K} \ominus \mathscr{K}_0))$ are both simple parts of N, the first is pure, and the second is not unless $\mathscr{K} \ominus \mathscr{K}_0 = (0)$. Hence they cannot be quasisimilar unless $\mathscr{K} \ominus \mathscr{K}_0 = (0)$ (Proposition 2.3). By (c), $\mathscr{K}_0 = \mathscr{K}$ and N is cyclic.

3.4 *Claim.* $\mu(G) = 0$.

If $\mu(G) > 0$, then there is a compact subset K of G with $\mu(K) > 0$. Let U be an open set with $K \subseteq U \subseteq U^{-} \subseteq G$. If $\nu = \mu \mid (G^{-} \backslash U)$, then the Sarason hull of ν is (G, ν) [14]. Let $\mathscr{M}$ be an invariant subspace for N such that $\mathscr{M} \subseteq L^2(\nu)$ and $N \mid \mathscr{M}$ is pure [6, Proposition 9.6]. If $\mathscr{L} = \mathscr{M} \oplus L^2(\mu \mid K)$, then $N \mid \mathscr{M}$ and $N \mid \mathscr{L}$ are two parts that cannot be quasisimilar by Proposition 2.3. This establishes the claim.

Let $\alpha \in G$ and let η be harmonic measure for G^- defined at the point α. Let ϕ be the conformal map of $\mathbf{D}$ onto G such that $\phi(0) = \alpha$ and $\phi'(0) > 0$. Denoting normalized arc length measure on $\partial\mathbf{D}$ by ω, ϕ induces an isomorphism of the measure space $(\partial\mathbf{D}, \omega)$ with $(\partial G, \eta)$ [16, p. 6]. Let M_ϕ denote multiplication by ϕ on $L^2(\omega)$. Since ϕ is a weak* generator of H^∞ (Theorem 1.5), M_ϕ is cyclic. By the preceding comments, η is a scalar-valued spectral measure for M_ϕ. Since M_ϕ is cyclic, M_ϕ is unitarily equivalent to the position operator on $L^2(\eta)$. But since (G, μ) is the Sarason hull of μ and $\mu(G) = 0$, $\mu = \mu | \partial G$ and η must be mutually absolutely continuous. So N is unitarily equivalent to the position operator on $L^2(\eta)$, and hence $N \cong M_\phi$. ∎

4. Quasisimilar and similar cyclic subnormal operators

It is known [3] that a cyclic subnormal operator is unitarily equivalent to S_μ, the operator given by multiplication by z on $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, for some compactly supported measure on the plane. The next result says that if two cyclic subnormal operators are quasisimilar, then the corresponding measures can be chosen in such a way that the quasi-invertible intertwining operators have simple representations.

4.1 Proposition. *If S_1 and S_2 are cyclic subnormal operators, then $S_1 \sim S_2$ iff there are compactly supported measures μ_1 and μ_2 such that $S_1 \cong S_{\mu_1}$, $S_2 \cong S_{\mu_2}$, and there are constants c_1 and c_2 and a function ϕ in $H^2(\mu_1) \cap L^\infty(\mu_1)$, such that:*

(a) *$\{\phi p$: p is a polynomial$\}$ is dense in $H^2(\mu_1)$;*
(b) *for every polynomial p,*

$$c_2 \int |\phi|^2 |p|^2 \, d\mu_1 \le \int |p|^2 \, d\mu_2 \le c_1 \int |p|^2 \, d\mu_1.$$

Proof. Suppose $S_1 \sim S_2$ and let $Y_{ij}\colon \mathscr{H}_j \to \mathscr{H}_i$ be quasi-invertible operators such that $Y_{ij} S_j = S_i Y_{ij}$. If e_1 is a cyclic vector for S_1, then it follows that $e_2 = Y_{21} e_1$ is a cyclic vector for S_2. Choose compactly supported measures μ_1 and μ_2 on $\mathbf{C}$ such that there are isomorphisms $U_j\colon \mathscr{H}_j \to H^2(\mu_j)$ with $U_j e_j = 1$ and $U_j S_j U_j^{-1} = S_{\mu_j}$. Let $X_{ij} = U_i Y_{ij} U_j^{-1}$. So $X_{ij}\colon H^2(\mu_j) \to H^2(\mu_i)$ is quasi-invertible. Moreover, it is straightforward to verify that $X_{ij} S_{\mu_j} = S_{\mu_i} X_{ij}$.

If p is a polynomial, then

$$X_{21} p = X_{21} p(S_{\mu_1}) 1 = p(S_{\mu_2}) X_{21} 1 = p(S_{\mu_2}) U_2 Y_2 e_1 = p(S_{\mu_2}) 1 = p.$$

If $c_1 = \|X_{21}\|^2$, this shows that $\int |p|^2 \, d\mu_2 \le c_1 \int |p|^2 \, d\mu_1$.

To find the constant c_2, notice that $X_{12} X_{21}$ commutes with S_{μ_1}. By Yoshino's Theorem [19], there is a function ϕ in $H^2(\mu_1) \cap L^\infty(\mu_1)$ such that $X_{12} X_{21} f = \phi f$ for every f in $H^2(\mu_1)$. Hence, for a polynomial p, $\phi p = X_{12} X_{21} p = X_{12} p$. Let $c_2 = \|X_{12}\|^{-2}$.

For the converse, suppose μ_1, μ_2, c_1, c_2, and ϕ are given. If $X_{21}p$ and $X_{12}p$ are defined for polynomials p by $X_{21}p = p$ and $X_{12}p = \phi p$, then it is easy to see that X_{21} and X_{12} extend to bounded linear maps X_{ij}: $H^2(\mu_j) \to H^2(\mu_i)$ such that $X_{ij}S_{\mu_j} = S_{\mu_i}X_{ij}$. Clearly ran X_{ij} is dense. By Lemma 2.4, the operators X_{ij} are injective. Hence $S_1 \sim S_2$. ∎

In his thesis [5], Clary proved the following theorem. We give a different proof that the conditions are necessary.

4.2 THEOREM. *A subnormal operator S is quasisimilar to the unilateral shift iff $S \cong S_\mu$ where μ is a measure such that:*

(a) support $\mu \subseteq \mathbf{D}^-$,
(b) $\nu = \mu|\partial\mathbf{D} \ll \omega$,
(c) $\log (d\nu/d\omega) \in L^1(\omega)$.

Suppose $S \sim S_\omega$. By Proposition 4.1 there are compactly supported μ_1 and μ_2 such that $S_\omega \cong S_{\mu_1}$, $S \cong S_{\mu_2}$, and μ_1 and μ_2 satisfy the restrictions enunciated there. Let $\mu = \mu_2$, $\nu = \mu|\partial\mathbf{D}$. By Theorem 1.6, the Sarason hull of μ is $(\mathbf{D}, \mu)$ and $\nu \ll \omega$ [16]. So (a) and (b) hold.

Because $S_{\mu_1} \cong S_\omega$, μ_1 and ω are mutually absolutely continuous, and Szego's Theorem [9] implies that $\log (d\mu_1/d\omega) \in L^1(\omega)$. If $n \geq 1$ and p is a polynomial without constant term, then

$$\int_{\partial\mathbf{D}} |\phi|^2 |1-p|^2 \, d\mu_1 = \int_{\partial\mathbf{D}} |\phi|^2 |1-p|^2 |z|^{2n} \, d\mu_1$$
$$\leq \frac{1}{c_2} \int |1-p|^2 |z|^{2n} \, d\mu.$$

Letting $n \to \infty$, we get

$$\int_{\partial\mathbf{D}} |\phi|^2 |1-p|^2 \, d\mu_1 \leq \frac{1}{c_2} \int_{\partial\mathbf{D}} |1-p|^2 \, d\nu.$$

By Szego's Theorem,

$$\exp \int \log \left(|\phi|^2 \frac{d\mu_1}{d\omega} \right) d\omega = \inf \left\{ \int |\phi|^2 |1-p|^2 \, d\mu_1 \colon p(0) = 0 \right\}$$
$$\leq \frac{1}{c_2} \inf \left\{ \int |1-p|^2 \, d\nu \colon p(0) = 0 \right\}$$
$$= \frac{1}{c_2} \exp \int \log \left(\frac{d\nu}{d\omega} \right) d\omega.$$

But

$$\log \left(|\phi|^2 \frac{d\mu_1}{d\omega} \right) = 2 \log |\phi| + \log \left(\frac{d\mu_1}{d\omega} \right) \in L^1(\omega),$$

since $S_{\mu_1} = S_\omega$ implies that $H^2(\mu_1) \cap L^\infty(\mu_1) = P^\infty(\mu_1) = H^\infty$, and, hence, that $\log |\phi| \in L^1(\omega)$. Thus, (c) holds.

The converse is proved as in [5]. ∎

For similarity, a statement analogous to, but stronger than, Proposition 4.1 is possible.

4.3 PROPOSITION. *If S_1 and S_2 are cyclic subnormal operators, then $S_1 \approx S_2$ iff there are compactly supported measures μ_1 and μ_2 and constants c_1 and c_2, such that $S_1 \cong S_{\mu_1}$, $S_2 \cong S_{\mu_2}$, and for every polynomial p*

$$c_2 \int |p|^2 \, d\mu_1 \leq \int |p|^2 \, d\mu_2 \leq c_1 \int |p|^2 \, d\mu_1.$$

The proof of this proposition follows the lines of the proof of Proposition 4.1 and will be left to the reader.

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