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$\aleph$-PROJECTIVE SPACES

BY

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1. Introduction

By a space we shall always mean a compact Hausdorff space, a map shall always be a continuous map between spaces, and a diagram shall always be a commutative diagram of spaces and maps. A space X is projective if the following lifting property holds. Given spaces Y and Z and maps $\phi: X \to Z$ and $f: Y \to Z$ with f onto, there exists a map $\psi: X \to Y$ satisfying $\phi = f \circ \psi$. In other words, a solution ψ exists in any diagram

(1)
$$\begin{tikzcd} & Y \arrow[d, "f\ \text{(onto)}"] \\ X \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"'] & Z \end{tikzcd}.$$

We call ψ a lifting of ϕ over f. A well known theorem of Gleason characterizes the projective spaces as the extremally disconnected spaces [5][2, p. 51]. A space is extremally disconnected if open sets have open closures.

The weight wt (X) of a space X is the least cardinal of a base of open sets. Let $\aleph$ be an infinite cardinal. We shall say that a space X is $\aleph$-projective if a solution ψ exists in diagram (1) whenever the additional condition wt $(Y) < \aleph$ is satisfied. Since f is onto, wt $(Z) < \aleph$ is also implied; but note that wt (X) is not mentioned. The purpose of this paper is to give the following characterization of $\aleph$-projective spaces.

THEOREM 1. *For $\aleph > \aleph_0$, a compact Hausdorff space X is $\aleph$-projective iff it is a totally disconnected $F_\aleph$-space.*

The following definitions are more or less standard; we follow the conventions of [2]. A cozero set in a space is the complement of the set of zeros of a continuous real valued function, and a set is $\aleph$-open if it is the union of fewer than $\aleph$ cozero sets. A space is an $F_\aleph$-space if any two disjoint $\aleph$-open sets have disjoint closures. An $F_{\aleph_0}$-space is called an F-space. An $\aleph_1$-open set is a cozero set, so an F-space is also an $F_{\aleph_1}$-space. Any space X is $\aleph_0$-projective, and we shall ignore this trivial case from now on.

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A space is totally disconnected if it has an open base of clopen, i.e., closed open, sets. There are F-spaces which are not totally disconnected, which are in fact connected [4, p. 211]! There are also totally disconnected $F_{\aleph}$-spaces which are not $\aleph$-extremally disconnected; examples will appear later. (A space is $\aleph$-extremally disconnected if $\aleph$-open sets have open closures.) Hence Theorem 1 implies that there are $\aleph$-projective spaces X which are not projective. Such a space must have wt $(X) \geq \aleph$. This follows from general facts about spaces [5, Theorem 1.2], but it also follows from Theorem 1 and the following observation: an $F_{\aleph}$-space of wt $< \aleph$ is extremally disconnected, since every open set is $\aleph$-open. The same observation also shows that Gleason's characterization of projective spaces is a consequence of Theorem 1.

In addition, we shall investigate a stronger projectivity associated with the cardinal $\aleph$. We shall say that space X is strongly $\aleph$-projective if ψ exists in diagram (1) whenever the weaker condition wt $(Z) < \aleph$ is satisfied. Our main results in this direction are Theorems 2 and 3.

THEOREM 2. *Assume $GCH_{\aleph}$, the generalized continuum hypothesis at $\aleph$, that $\aleph^+ = 2^{\aleph}$. Then every $\aleph^+$-projective space is strongly $\aleph^+$-projective.*

We regard an infinite cardinal $\aleph$ as an initial ordinal and also as a discrete set of cardinality $\aleph$. We identify the Stone-Čech compactification $\beta\aleph$ of the discrete set $\aleph$ as the space of ultrafilters on $\aleph$. A free ultrafilter $p \in \beta\aleph - \aleph$ is uniform if it contains the generalized Fréchet filter, i.e., if $\{A \subset \aleph : |\aleph - A| < \aleph\} \subset p$. The cofinality cf $(\aleph)$ of $\aleph$ is the least cardinal $\aleph'$ such that $\aleph$ is the sum of $\aleph'$ cardinals smaller than $\aleph$.

THEOREM 3. *The space Δ of uniform ultrafilters in $\beta\aleph - \aleph$ is strongly $\beth^+$-projective for $\beth =$ cf $(\aleph)$.*

Nothing of GCH is involved in Theorem 3. It is known that Δ is not extremally disconnected, so Δ provides an example of a strongly $\beth^+$-projective space which is not projective. In particular, the totally disconnected F-space $\beta\aleph_0 - \aleph_0$ (otherwise $\beta N - N$) is strongly $\aleph_1$-projective, without CH, but is not $\aleph_0$-extremally (=basically) disconnected and so is not projective. Theorem 3, together with Theorem 1, also provides an independent proof that Δ is a totally disconnected $F_{\beth^+}$-space [2, Theorem 14.9].

Finally, we shall show that the α-co-homogeneous α-co-universal spaces of [2, pp. 132–133] provide further examples of strongly $\aleph$-projective spaces which are not projective. The main results of this paper were announced in [6].

2. Partial liftings

Maps ϕ and f being as in diagram (1), we define a partial lifting of ϕ over f to be a triple $(\psi_\alpha, j_\alpha, Y_\alpha)$ satisfying the following diagram:

$$
\begin{tikzcd}
 & Y_\alpha \arrow[dr, "f_\alpha"] & & Y \arrow[ll, "j_\alpha \text{ (onto)}"'] \arrow[dl, "f \text{ (onto)}"] \\
X \arrow[ur, "\psi_\alpha"] \arrow[rr, "\phi"'] & & Z &
\end{tikzcd}
\tag{2}
$$

Note that with j_α onto, Y_α is a quotient space of Y, and that map f_α is determined when f and j_α are given.

We shall say that the partial lifting $(\psi_\alpha, j_\alpha, Y_\alpha)$ is subordinate to the partial lifting $(\psi_\beta, j_\beta, Y_\beta)$ if there exists a connecting map $j = j_{\alpha\beta}$: $Y_\beta \to Y_\alpha$ such that $j \circ j_\beta = j_\alpha$ and $j \circ \psi_\beta = \psi_\alpha$. That is, Y_α is a quotient space of Y_β and ψ_β is a lifting of ψ_α over the quotient map j. Note that the map j is uniquely determined by j_α and j_β, if it exists, and that $f_\alpha \circ j = f_\beta$. We omit the diagram.

By a minimal map f: $Y \to Z$ we mean an onto map with the property that $f(K) \neq Z$ for any closed proper subset K of Y. It will be convenient to assume that map f in diagram (2) is minimal. We shall assume further that space Y is totally disconnected. These assumptions are in force until they are dropped in Section 4. When f is minimal then j_α and f_α in diagram (2) are each necessarily minimal. Moreover, if $(\psi_\alpha, j_\alpha, Y_\alpha)$ is subordinate to $(\psi_\beta, j_\beta, Y_\beta)$ then the connecting map j: $Y_\beta \to Y_\alpha$ is also minimal.

The proof of Theorem 1 depends on the following basic result. The cardinal $\aleph > \aleph_0$ is specified.

LEMMA 1. (Construction Lemma). *In diagram* (2) *let X be a totally disconnected $F_\aleph$-space, let Y be totally disconnected, let f be minimal, and suppose* wt $(Y_\alpha) < \aleph$. *Let E be a clopen subset of Y. Then $(\psi_\alpha, j_\alpha, Y_\alpha)$ is subordinate to a partial lifting $(\psi_\beta, j_\beta, Y_\beta)$ where Y_β is the free union of copies of $j_\alpha(E)$ and $j_\alpha(Y - E)$, with* wt $(Y_\beta) < \aleph$.

The proof of Lemma 1 involves a certain isomorphism property of minimal maps. Recall that a closed set is regular if it is the closure of its interior, and an open set is regular if it is the interior of its closure.

LEMMA 2. *Let g: $V \to W$ be a minimal map of spaces. Suppose $V_1, V_2 \subset V$ are regular closed sets whose interiors are disjoint. Then $g(V_1)$, $g(V_2)$ are regular closed sets whose interiors are disjoint.*

Proof of Lemma 2. This is immediate from [1, Lemma 6]. ▌

Proof of Lemma 1. Let $V_1 = E$, $V_2 = Y - E$, let $W_i = j_\alpha(V_i)$, $i = 1, 2$, and let Y_β be the free union of copies of W_1 and W_2. To construct ψ_β we first produce a clopen partition $X_1 \cup X_2$ of X such that $\psi_\alpha(X_i) \subset W_i$, $i = 1, 2$.

Recall that map j_α is necessarily minimal. The clopen sets V_1 and V_2 are regular, so from Lemma 2, W_1 and W_2 are regular closed subsets of Y_α such that

Int W_1 and Int W_2 in Y_α are disjoint. These interiors are $\aleph$-open, because wt $(Y_\alpha) < \aleph$. Thus $\psi_\alpha^{-1}(\text{Int } W_i)$, $i = 1, 2$, are disjoint $\aleph$-open sets in X. But X is an $F_\aleph$-space, so Cl $\psi_\alpha^{-1}(\text{Int } W_i)$, $i = 1, 2$, are disjoint compact sets in X. Since X is totally disconnected, a clopen covering argument shows that there exists a clopen partition $X_1 \cup X_2$ of X such that Cl $\psi_\alpha^{-1}(\text{Int } W_i) \subset X_i$, $i = 1, 2$.

Now, Int W_1 and Int W_2 are disjoint regular open sets whose union is dense in Y_α, and from general properties of regular sets we know that $W_i = Y_\alpha -$ Int W_{3-i}, $i = 1, 2$. Since $X_i \cap \psi_\alpha^{-1}(\text{Int } W_{3-i}) = \emptyset$, $i = 1, 2$, we see that $X_i \subset \psi_\alpha^{-1}(W_i)$, $i = 1, 2$. Thus $\psi_\alpha(X_i) \subset W_i$, $i = 1, 2$, as required.

Maps ψ_β, j_β and f_β will be defined by an abuse of language, to avoid proliferation of notation. Recall that Y_β is the free union of copies of W_1 and W_2. Define ψ_β: $X \to Y_\beta$ piecewise to be a copy of ψ_α: $X_1 \to W_1$ on X_1 and a copy of ψ_α: $X_2 \to W_2$ on X_2. The function ψ_β is continuous, since $X_1 \cup X_2$ is a clopen partition. Similarly, define j_β to be j_α on V_1 and to be j_α on V_2. Let j be the inclusion maps on the copies of W_1 and W_2 which comprise Y_β, and let f_β be f_α on W_1 and on W_2. These functions are also continuous. Clearly, $j_\alpha = j \circ j_\beta$, $\psi_\alpha = j \circ \psi_\beta$, and $f = f_\beta \circ j_\beta$. Observe that the construction remains valid, albeit trivial, if $j_\alpha(E)$ is clopen in Y_α. In this event, $W_1 \cup W_2$ is a clopen partition of Y_α, $Y_\beta = Y_\alpha$, j is the identity map, and $j_\beta = j_\alpha, f_\beta = f_\alpha, \psi_\beta = \psi_\alpha$. ∎

3. A lifting lemma

Theorems 1 and 2 will follow rather quickly from the following lifting lemma. Recall that $\aleph$ is a regular cardinal if cf$(\aleph) = \aleph$; that is, if the sum of fewer than $\aleph$ cardinals smaller than $\aleph$ is smaller than $\aleph$. The assumption $\aleph > \aleph_0$ is still in effect.

LEMMA 3. (Lifting Lemma). *In diagram* (1) *let X be a totally disconnected $F_\aleph$-space, let Y be totally disconnected, let f be minimal, and let* wt $(Z) < \aleph$. *Then a solution ψ exists if* wt $(Y) < \aleph$, *or if* wt $(Y) \leq \aleph$ *and $\aleph$ is a regular cardinal.*

Proof. There exists a base of cardinality wt (Y) for the open sets of Y consisting of clopen sets. Let $\{E_\alpha: \alpha < \beth\}$ be a well ordering of such a clopen base, with $\beth = \text{wt}(Y) \leq \aleph$. For each $\beta \leq \beth$ a partial lifting $(\psi_\beta, j_\beta, Y_\beta)$ is determined by the following transfinite recursion. We say that a family of partial liftings $((\psi_\alpha, j_\alpha, Y_\alpha): \alpha < \beta)$ forms a chain if $(\psi_\delta, j_\delta, Y_\delta)$ is subordinate to $(\psi_\alpha, j_\alpha, Y_\alpha)$ whenever $\delta < \alpha < \beta \leq \beth$.

(i) $(\psi_0, j_0, Y_0) = (\phi, f, Z)$. Obviously wt $(Y_0) < \aleph$.

(ii) If $0 < \beta < \beth$ is a successor ordinal, say $\beta = \alpha + 1$, if $((\psi_\delta, j_\delta, Y_\delta): \delta < \beta)$ forms a chain, and if wt $(Y_\alpha) < \aleph$, then $(\psi_\beta, j_\beta, Y_\beta)$ is obtained by applying the construction lemma, Lemma 1, to $(\psi_\alpha, j_\alpha, Y_\alpha)$ and $E = E_\alpha$. Then $((\psi_\delta, j_\delta, Y_\delta): \delta < \beta + 1)$ forms a chain, with wt $(Y_\beta) < \aleph$.

(iii) If $0 < \beta \leq \beth$ is a limit ordinal and $((\psi_\alpha, j_\alpha, Y_\alpha): \alpha < \beta)$ forms a chain, then the inverse limit $Y_\beta = \varprojlim (Y_\alpha, j_{\alpha\delta})$ exists. For each $\alpha < \beta$, Y_β comes equipped with a canonical projection map $j_{\alpha\beta}\colon Y_\beta \to Y_\alpha$ such that $j_{\delta\beta} = j_{\delta\alpha} \circ j_{\alpha\beta}$ whenever $\delta < \alpha$. From the universal mapping property of inverse limits, there exist maps $j_\beta\colon Y \to Y_\beta$ and $\psi_\beta\colon X \to Y_\beta$ such that $j_{\alpha\beta} \circ j_\beta = j_\alpha$ and $j_{\alpha\beta} \circ \psi_\beta = \psi_\alpha$ for all $\alpha < \beta$. Map j_β is onto and hence minimal, because $j_\beta(Y)$ is compact and dense in Y_β [3, p. 430]. It follows that $((\psi_\alpha, j_\alpha, Y_\alpha)\colon \alpha < \beta + 1)$ forms a chain. The weight condition of the induction is clearly satisfied when $\beth < \aleph$. In the case where $\beth = \aleph$ and $\aleph$ is regular, let $\mathscr{B}_\alpha$ for $\alpha < \beta$ be a base of Y_α of cardinality $|\mathscr{B}_\alpha| = \mathrm{wt}\,(Y_\alpha) < \aleph$. Then $\{j_{\alpha\beta}^{-1}(\mathscr{B}_\alpha)\colon \text{all } \alpha < \beta\}$ is a base of Y_β. Evidently $\mathrm{wt}\,(Y_\beta) < \aleph$ when $\beta < \aleph$, because $\aleph$ is regular and $\mathrm{wt}\,(Y_\beta) \le \sum_\alpha \{\mathrm{wt}\,(Y_\alpha)\colon \alpha < \beta\}$ does not exceed the sum of fewer than $\aleph$ cardinals smaller than $\aleph$.

When $\beta = \beth$, at the end of the induction, each clopen set E_α of the base for Y appears in the form $E_\alpha = j_\beta^{-1}(j_\beta(E_\alpha))$, $\alpha < \beta$, from which it is straightforward that $j_\beta\colon Y \to Y_\beta$ is a homeomorphism. The end map $\psi = j_\beta^{-1} \circ \psi_\beta\colon X \to Y$ is the lifting sought. ∎

4. Proof of Theorem 1

($\Leftarrow$) The hard work has already been done in Lemmas 1 and 3, and we need only to reduce the general situation to that covered in Lemma 3. So consider diagram (1) with X a totally disconnected $F_\aleph$-space and $\mathrm{wt}\,(Y) < \aleph$. Now, Y is the continuous image of a totally disconnected space Y_1 such that $\mathrm{wt}\,(Y_1) = \mathrm{wt}\,(Y)$ [2, Corollary 2.38]. A simple Zorn's Lemma argument due to Gleason [5] shows that any onto map has a minimal restriction. Also, a subspace of a totally disconnected space is again totally disconnected. It is straightforward from this combination of remarks that we can reduce to the case where Y is totally disconnected and f is minimal; we omit the diagram. By Lemma 3, the required lifting exists.

($\Rightarrow$) Suppose X is an $\aleph$-projective space. Let U and V be disjoint $\aleph$-open sets in X. We shall prove there exists a clopen partition $X_1 \cup X_2$ of X with $U \subset X_1$, $V \subset X_2$. This will prove that the space X is totally disconnected, and moreover, an $F_\aleph$-space.

Write $U = \bigcup_\alpha (U_\alpha\colon \alpha \in A)$ and $V = \bigcup_\alpha (V_\alpha\colon \alpha \in A)$, where each U_α and each V_α is a cozero set and the index set has cardinality $|A| < \aleph$. For each α there exists a map $\phi_\alpha\colon X \to [-1, 1]$ such that $U_\alpha = \{\phi_\alpha < 0\}$ and $V_\alpha = \{\phi_\alpha > 0\}$. Take the space Z to be $Z = \prod_{\alpha \in A} [-1, 1]_\alpha$, and take the map $\phi\colon X \to Z$ to be $\phi(x) = (\phi_\alpha(x)\colon \alpha \in A)$, $x \in X$. Let

$$Y_1 = \prod_{\alpha \in A} [-1, 0]_\alpha \quad \text{and} \quad Y_2 = \prod_{\alpha \in A} [0, 1]_\alpha,$$

and let $Y = Y_1 \cup Y_2$ as a free union. Define the map: $Y \to Z$ by an abuse of language to be the inclusion map on Y_1 and the inclusion map on Y_2.

We wish to use the $\aleph$-projectivity of X to obtain a lifting $\psi\colon X \to Y$. Clearly $\mathrm{wt}\,(Y) < \aleph$, but the map f is not onto. However, we shall show that f is onto the range of ϕ, which is all that is essential. Now,

$$U = \{x \in X : \phi_\alpha(x) < 0 \text{ for some } \alpha \in A\}$$

and

$$V = \{x \in X : \phi_\alpha(x) > 0 \text{ for some } \alpha \in A\}.$$

These sets are disjoint, so

$$\begin{aligned} U \subset X - V &= \{x \in X : \phi_\alpha(x) \leq 0 \text{ for all } \alpha \in A\}, \\ V \subset X - U &= \{x \in X : \phi_\alpha(x) \geq 0 \text{ for all } \alpha \in A\}, \\ X - (U \cup V) &= \{x \in X : \phi_\alpha(x) = 0 \text{ for all } \alpha \in A\}. \end{aligned}$$

It follows that $f(Y) \supset \phi(X)$, so a lifting $\psi : X \to Y$ of ϕ over f does exist.

It is apparent that $\phi(U) \subset Z - f(Y_2)$, and since $\phi(U) = f(\psi(U))$, it must be the case that $\psi(U) \subset Y_1$; in the same way, $\psi(V) \subset Y_2$. Since $Y_1 \cup Y_2$ is a clopen partition and ψ is continuous, $X_i = \psi^{-1}(Y_i)$, $i = 1, 2$, is a clopen partition with the property $U \subset X_1$, $V \subset X_2$. ∎

5. Proof of Theorem 2

We drop the restriction $\aleph > \aleph_0$, so from now on $\aleph$ denotes any infinite cardinal. The case $\aleph = \aleph_0$ will often be trivial, however, since any space X is strongly $\aleph_0$-projective. The following simple lemma will facilitate the proof of Theorem 2. Denote the Gleason minimal projective cover of Z by gZ, and denote the associated minimal map by $\pi : gZ \to Z$ [5].

LEMMA 4. *The space X is strongly $\aleph$-projective iff a diagram*

(3)
$$\begin{tikzcd} & gZ \arrow[d, "\pi"] \\ X \arrow[ur, dashed, "\psi_1"] \arrow[r, "\phi"'] & Z \end{tikzcd}$$

has a solution ψ_1 whenever wt $(Z) < \aleph$.

Proof. ($\Leftarrow$) In the augmentation of diagram (1)

$$\begin{tikzcd} & & Y \arrow[d, "f \text{ (onto)}"] \\ X \arrow[urr, "\psi"] \arrow[r, "\psi_1"] \arrow[drr, "\phi"'] & gZ \arrow[ur, "\eta"] \arrow[dr, "\pi"] & Z \end{tikzcd},$$

the map ψ_1 exists by hypothesis, and η exists by projectivity of gZ. The map $\psi = \eta \circ \psi_1$ is the required lifting of ϕ over f.

($\Rightarrow$) This part is vacuous. ∎

The next lemma is an easy consequence of Lemma 4. However, we shall not need it, so we offer it without proof.

LEMMA 5. *The space X is strongly $\aleph$-projective iff it has the following property. For each onto map $\phi\colon X \to Z$ with* wt $(Z) < \aleph$, *and each minimal restriction $\phi_1\colon X_1 \to Z$ of ϕ, the domain X_1 is homeomorphic to gZ and is a retract of X under a lifting of ϕ over ϕ_1.* ∎

Proof of Theorem 2. Let X be $\aleph^+$-projective. By Theorem 1, X is a totally disconnected $F_{\aleph^+}$-space. Let Z be a space with wt $(Z) < \aleph^+$, i.e., wt $(Z) \leq \aleph$. The space gZ can be realized as the Stone space of the complete Boolean algebra $\mathcal{G}^{\mathrm{reg}}(Z)$ of regular open subsets of Z [5]. Thus wt $(gZ) = |\mathcal{G}^{\mathrm{reg}}(Z)| \leq 2^{\aleph}$. By $GCH_{\aleph}$, $2^{\aleph} = \aleph^+$ so wt $(gZ) \leq \aleph^+$. As a successor cardinal, $\aleph^+$ is regular. In diagram (3) we apply Lemma 3, the lifting lemma, to find that the solution ψ_1 of diagram (3) exists. By Lemma 4, X is strongly $\aleph^+$-projective. ∎

$GCH_{\aleph}$ is essential to this method of proof because it can happen that wt $(gZ) = 2^{\aleph}$. For example, if Z is the one point compactification of the discrete set $\aleph$ then wt $(Z) = \aleph$ and wt $(gZ) = 2^{\aleph}$, because $gZ = \beta\aleph$ has weight wt $(\beta\aleph) = |\mathcal{P}(\aleph)| = 2^{\aleph}$. It is an open question as to whether $GCH_{\aleph}$ is an essential hypothesis in Theorem 2.

The problem of constructing $\aleph$-projective covers for spaces is in general open. However, if Z is a space with wt $(X) < \aleph$, it is easy to see that gZ is the strongly-$\aleph$-projective cover of Z. For, any onto map $\phi\colon X \to Z$ from a strongly $\aleph$-projective space X factors through gZ, by Lemma 4. Similarly, if $GCH_{\aleph}$ is assumed, then the proof of Theorem 2 shows that gZ is the $\aleph^+$-projective cover of any space Z for which wt $(Z) < \aleph^+$.

6. Proof of Theorem 3

We divide the proof into several lemmas. Let $\aleph$ be fixed, and let Δ and $\beth = \mathrm{cf}(\aleph)$ be as in the statement of Theorem 3.

LEMMA 6. *Suppose Δ has the property that each map $\phi\colon \Delta \to Z$ with* wt $(Z) < \beth^+$ *can be extended to a map $\check{\phi}\colon \beta\aleph \to Z$. Then Δ is strongly $\beth^+$-projective.*

Proof. In diagram (1) assume that $X = \Delta$ and that wt $(Z) < \beth^+$, and consider the augmented diagram

$$\begin{tikzcd}
 & & Y \arrow[dd, "f\ (\text{onto})"] \\
\Delta \arrow[r, hook, "\iota"] \arrow[urr, "\psi"] \arrow[drr, "\phi"'] & \beta\aleph \arrow[ur, "\eta"] \arrow[dr, "\check{\phi}"'] & \\
 & & Z
\end{tikzcd}$$

where ι is the inclusion map. The map $\check{\phi}$ exists by hypothesis. With $\check{\phi}$ given, the map η exists because $\beta\aleph$ is projective. The solution ψ is then $\psi = \eta \circ \iota$. ∎

We shall construct the extension $\check{\phi}$ by a transfinite recursion similar to the one employed in the proof of Lemma 3. With $\phi\colon \Delta \to Z$ given, a partial extension $(\check{\phi}_\alpha, j_\alpha, Z_\alpha)$ is defined to be a triple satisfying the diagram

(4)
$$\begin{tikzcd} & & Z \arrow[d, "j_\alpha \text{ (onto)}"] \\ \Delta \arrow[urr, "\phi"] \arrow[r, hook, "\iota"] \arrow[drr, "\phi_\alpha"'] & \beta\aleph \arrow[dr, "\check{\phi}_\alpha"] & \\ & & Z_\alpha \end{tikzcd}.$$

We remark that Z_α is a quotient space of Z, that $\phi_\alpha = j_\alpha \circ \phi$ is determined by j_α, and that $\check{\phi}_\alpha$ is an actual extension of ϕ_α. We shall assume for the time being that Z and Z_α are totally disconnected.

We shall say that $(\check{\phi}_\alpha, j_\alpha, Z_\alpha)$ is subordinate to a partial extension $(\check{\phi}_\beta, j_\beta, Z_\beta)$ if there exists a (necessarily unique) connecting map $j_{\alpha\beta}\colon Z_\beta \to Z_\alpha$ such that $j_{\alpha\beta} \circ j_\beta = j_\alpha$ and $j_{\alpha\beta} \circ \check{\phi}_\beta = \check{\phi}_\alpha$. We omit the diagram.

LEMMA 7. *In diagram* (4) *let Z and Z_α be totally disconnected, with* wt $(Z_\alpha) < \beth$. *Let E be a clopen subset of Z. Then $(\check{\phi}_\alpha, j_\alpha, Z_\alpha)$ is subordinate to a partial extension $(\check{\phi}_\beta, j_\beta, Z_\beta)$ where Z_β is the free union of copies of $j_\alpha(E)$ and $j_\alpha(Z - E)$, with Z_β totally disconnected and* wt $(Z_\beta) < \beth$.

A set is $\beth$-clopen if it can be expressed as the union of fewer than $\beth$ clopen sets. The proof of Lemma 7 will involve the following:

LEMMA 8. *Suppose U is a $\beth$-clopen set in $\beta\aleph$. Then*
$$\Delta \cap \mathrm{Cl}_{\beta\aleph} U = \mathrm{Cl}_\Delta(\Delta \cap U).$$

Proof. This is implicit in the proof of [2, Lemma 14.7]. ∎

Proof of Lemma 7. Let $E_1 = E$, $E_2 = Z - E$, and let Z_β be the free union of copies of $j_\alpha(E_1)$ and $j_\alpha(E_2)$. The space Z_β is totally disconnected and wt $(Z_\beta) < \beth$, clearly. By an abuse of language, define $j_\beta\colon Z \to Z_\beta$ to be j_α from E_i to the piece $j_\alpha(E_i)$ of Z_β and define $j_{\alpha\beta}\colon Z_\beta \to Z_\alpha$ to be the inclusion map of the piece $j_\alpha(E_i)$, $i = 1, 2$. Evidently j_β and $j_{\alpha\beta}$ are continuous, and $j_{\alpha\beta} \circ j_\beta = j_\alpha$. We note that $\phi_\alpha = j_{\alpha\beta} \circ \phi_\beta$ for $\phi_\alpha = j_\alpha \circ \phi$, $\phi_\beta = j_\beta \circ \phi$.

Now we define a partition $H_1 \cup H_2 = \Delta$ by $H_i = \phi^{-1}(E_i)$, $i = 1, 2$. Also $H_i = \phi_\beta^{-1}(j_\beta(E_i))$ because $\phi_\beta^{-1} = \phi^{-1} \circ j_\beta^{-1}$ and $j_\beta^{-1}(j_\beta(E_i)) = E_i$, $i = 1, 2$. To determine the map $\check{\phi}_\beta$ it will suffice to find a clopen partition $W_1 \cup W_2$ of $\beta\aleph$ such that $W_i \supset H_i$ and $\check{\phi}_\alpha(W_i) \subset j_\alpha(E_i)$, $i = 1, 2$. For then we may define $\check{\phi}_\beta$ by abuse of language to be $\check{\phi}_\alpha$ from W_i to the piece $j_\alpha(E_i)$ of Z_β, $i = 1, 2$. This function will be continuous, and $j_{\alpha\beta} \circ \phi_\beta = \phi_\alpha$. It will also restrict to ϕ_β on Δ, since $W_i \cap \Delta = H_i$, $\phi_\alpha(H_i) = j_\alpha(E_i)$, $i = 1, 2$, and ϕ_β is piecewise a copy of ϕ_α.

Let us now construct W_1 and W_2. The sets $Z_\alpha - j_\alpha(E_{3-i})$, $i = 1, 2$, are disjoint open sets in Z_α, so $U_i = \check{\phi}_\alpha^{-1}(Z_\alpha - j_\alpha(E_{3-i}))$, $i = 1, 2$, are disjoint open sets in $\beta\aleph$. Since $\beta\aleph$ is extremally disconnected, $V_i = \mathrm{Cl}_{\beta\aleph}\, U_i$, $i = 1, 2$, are disjoint clopen sets. If $\check{\phi}_\beta$ exists it necessarily maps V_i into $j_\beta E_i$, so we must have $W_i \supset V_i$, $i = 1, 2$.

On the other hand, $W_i \cap \Delta = H_i$ is also required. Let us show next that $V_i \cap \Delta \subset H_i$, $i = 1, 2$. The set U_i is $\beth$-clopen, since $\mathrm{wt}\,(Z_\alpha) < \beth$. By Lemma 8, $V_i \cap \Delta = \mathrm{Cl}_\Delta\,(U_i \cap \Delta)$, $i = 1, 2$. We use the fact that $\check{\phi}_\alpha$ restricts to ϕ_α, to find

$$\begin{aligned} U_i \cap \Delta = \Delta \cap \check{\phi}_\alpha^{-1}(Z_\alpha - j_\alpha(E_{3-i})) &= \phi_\alpha^{-1}(Z_\alpha - j_\alpha(E_{3-i})) \\ &= \Delta - \phi^{-1}(j_\alpha^{-1}(j_\alpha(E_{3-i}))) \subset H_i, \quad i = 1, 2; \end{aligned}$$

we have also used $\phi_\alpha^{-1} = \phi^{-1} \circ j_\alpha^{-1}$. Since H_i is closed,

$$V_i \cap \Delta = \mathrm{Cl}_\Delta\,(U_i \cap \Delta) \subset H_i, \quad i = 1, 2.$$

The rest of $\beta\aleph$ is easily disposed of. Let $M_1 \cup M_2$ be any clopen partition of $\beta\aleph$ such that $H_i = M_i \cap \Delta$, $i = 1, 2$. Such a partition exists because each clopen subset of Δ is of the form $M \cap \Delta$ for some clopen $M \subset \beta\aleph$ [2, Lemma 7.12]. Let

$$W_i = V_i \cup \{[\beta\aleph - (V_1 \cup V_2)] \cap M_i\}, \quad i = 1, 2.$$

Clearly, $W_1 \cup W_2$ is a clopen partition of $\beta\aleph$ satisfying $W_i \cap \Delta \subset H_i$, $i = 1, 2$. But then $W_i \cap \Delta = H_i$, $i = 1, 2$, because $H \cup H_2$ is a partition. From

$$\check{\phi}_\alpha[\beta\aleph - (V_1 \cup V_2)] \subset j_\alpha(E_1) \cap j_\alpha(E_2)$$

it follows that $\check{\phi}_\alpha(W_i) \subset j_\alpha(E_i)$, $i = 1, 2$, as required. ∎

Proof of Theorem 3. Let $\phi\colon \Delta \to Z$ be a map such that $\mathrm{wt}\,(Z) < \beth^+$. We wish to extend ϕ to a map $\check{\phi}\colon \beta\aleph \to Z$. By [2, Lemma 2.37] there exist a totally disconnected space T with $\mathrm{wt}\,(T) \leq \mathrm{wt}\,(Z)$, and maps $\sigma\colon \Delta \to T$ and $\theta\colon T \to Z$, such that $\phi = \theta \circ \sigma$; we omit the diagram. Thus we may assume without loss of generality that Z is totally disconnected. Now construct $\check{\phi}$ by the same transfinite recursion used to prove Lemma 3, except for the following details: partial extensions replace partial liftings, Lemma 7 replaces Lemma 1, Z_0 is a singleton, an inverse limit of totally disconnected spaces is totally disconnected, and the cardinal $\beth = cf(\aleph)$ is necessarily regular. By Lemma 6, Δ is strongly $\beth^+$-projective. ∎

According to Lemma 5, any minimal restriction of an onto such $\phi\colon \Delta \to Z$ necessarily has a copy of $gZ \subset \Delta$ as domain, and this domain is a retract in Δ which respects ϕ. This is a refinement of [2, Lemma 7.14], at least when $\aleph$ is a regular cardinal.

7. Another example

The α-co-universal α-co-homogeneous spaces of [2, pp. 132–133] provide further examples of strongly $\aleph$-projective spaces which are not projective. If α is an infinite cardinal, $\alpha^{\underset{\smile}{\alpha}}$ denotes the cardinal sum $\alpha^{\underset{\smile}{\alpha}} = \sum_\lambda \{\alpha^\lambda : \lambda < \alpha\}$.

THEOREM 4. *Let $\alpha \geq \aleph_0$ be such that $\alpha = \alpha^{\underset{\smile}{\alpha}}$, so that the α-co-universal α-co-homogeneous space $\mathscr{S}_\alpha$ exists (for continuous maps of compact Hausdorff spaces). Space $\mathscr{S}_\alpha$ is strongly α-projective, but not projective.*

Proof. Let $\phi : \mathscr{S}_\alpha \to Z$ be an onto map with wt $(Z) < \alpha$. Let $\pi : gZ \to Z$ be the minimal projective covering of Z. The remaining maps in the diagram

$$\begin{tikzcd}
\mathscr{S}_\alpha \arrow[r, two heads, "\theta"] \arrow[dr, two heads, "\phi'" near start] & gZ \arrow[d, two heads, "\pi"] \\
\mathscr{S}_\alpha \arrow[u, "h"] \arrow[ur, "\psi" near start] \arrow[r, two heads, "\phi"'] & Z
\end{tikzcd}$$

are determined as follows. The map θ exists because $\mathscr{S}_\alpha$ is α-co-universal and wt $(gZ) \leq 2^{\underset{\smile}{\alpha}} \leq \alpha^{\underset{\smile}{\alpha}} = \alpha$. Next, ϕ' is defined by $\phi' = \pi \circ \theta$. Now the map h exists satisfying $\phi = \phi' \circ h$, because $\mathscr{S}_\alpha$ is α-co-homogeneous and wt $(Z) < \alpha$. Finally, the lifting ψ of ϕ over π is determined as $\psi = \theta \circ h$. By Lemma 4, $\mathscr{S}_\alpha$ is strongly α-projective. It follows from [2, Lemma 6.3, 6.5] that $\mathscr{S}_\alpha$ is not $\aleph_0$-extremally disconnected. ▮

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