ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 4, Winter 1981

BOUNDED SOLUTIONS OF SCALAR, ALMOST PERIODIC LINEAR EQUATIONS

BY

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1. Introduction

Consider a scalar, non-homogeneous differential equation

$$(*) \qquad \dot{x} = a(t)x + b(t),$$

where a and b are Bohr-almost periodic functions. If the mean value of a is not zero, then Cameron [3] showed that (*) admits a unique bounded solution, and that said solution is almost periodic. One can now prove this result by observing that (*) has an exponential dichotomy, and appealing to general theorems (e.g., [5] or [13]). If a has mean value zero, and if $\int_0^t a(s)\,ds$ is bounded (and hence a.p. by Bohr's theorem), then it is easy to prove that one solution of (*) is bounded if and only if all solutions are almost periodic.

Our interest is in the case when a has mean value zero, but $\int_0^t a(s)\,ds$ is unbounded. The example of [12] (which uses that of [4]) shows that (*) may then admit bounded solutions, but *no* almost periodic solutions. Stating our results requires the introduction of the hull Ω of the function $f(t) = (a(t), b(t))$ (see 2.3). The space Ω may be given the structure of a compact, abelian topological group [14]. Let μ_0 be normalized Haar measure on Ω. Let Ω_β be the set of $\omega \in \Omega$ for which the equation $(*)_\omega$ defined by ω (see 3.1) admits a unique bounded solution. Let

$$\Omega_\alpha = \{\omega \in \Omega: \text{the equation } (*)_\omega \text{ admits an } \textit{almost automorphic} \text{ solution}\}$$

(almost automorphy generalizes almost periodicity; see 2.5 and [18]). It turns out that $\Omega_\beta \subset \Omega_\alpha$.

We prove the following: (i) Ω_α and Ω_β are residual subsets of Ω (3.10), and (ii) for "most" functions a, $\mu_0(\Omega_\beta) = 1$ (3.11). An example shows that (iii) $\mu_0(\Omega_\beta)$ may be zero (3.12). We also show that (iv) the example of [12] satisfies $\mu_0(\Omega_\alpha) = 1$ (3.14–3.16). Finally, (v) we indicate how altering the example of [12] might produce an example with $\mu_0(\Omega_\alpha) = 0$ (3.17). It should be noted that (i) is proved in the more general case when Ω is *minimal* (2.1).

We also consider the case when (*) admits *no* bounded solutions. Assuming that Ω is minimal, we show in 4.2 that residually many $\omega \in \Omega$ have the property

Received January 5, 1980.

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that all solutions to $(1)_\omega$ are unbounded above *and* below (i.e., "oscillate"). An example shows that all solutions to $(1)_\omega$ may be bounded above (or below) for all ω in a set $\Omega_0 \subset \Omega$ which is measure-theoretically large (4.3).

The author would like to thank the referee for comments and suggestions leading to improvement of this paper.

2. Preliminaries

2.1 DEFINITIONS. Let X be a compact metric space with metric d. Let $(X, \mathbf{R})$ denote a (real) flow on X. Say that $(X, \mathbf{R})$ is *minimal* if each orbit $\{x \cdot t: t \in \mathbf{R}\}$ is dense in X $(x \in X)$. Say that $(X, \mathbf{R})$ is *almost periodic* if, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(x \cdot t, y \cdot t) < \varepsilon \quad (x, y \in X, t \in \mathbf{R}).$$

If $(X, \mathbf{R})$ is another flow, and if $\pi: X \to Y$ is continuous, then π is a *flow homomorphism* if $\pi(x \cdot t) = \pi(x) \cdot t$ $(x \in X, t \in \mathbf{R})$. If, in addition, π is a homeomorphism, then π is a *flow isomorphism.* See [6].

2.2 DEFINITION. Let $(X, \mathbf{R})$ and $(Y, \mathbf{R})$ be minimal flows, and let $\pi: X \to Y$ be a (necessarily surjective) flow homomorphism. Say that $(X, \mathbf{R})$ is an *almost-automorphic* (a.a.) *extension* of $(Y, \mathbf{R})$ if $\pi^{-1}(y)$ is a singleton for some (hence a residual set of) $y \in Y$. See [18] and [19]. Of course, an a.a. extension of $(Y, \mathbf{R})$ may be isomorphic to $(Y, \mathbf{R})$.

2.3 DEFINITIONS. Fix n, and let

$$C = \{f: \mathbf{R} \to \mathbf{R}^n : f \text{ is bounded and uniformly continuous}\}.$$

Give C the compact-open topology. If $f \in C$ and $\tau \in \mathbf{R}$, let

$$f_\tau(t) = f(t + \tau) \quad (t \in \mathbf{R}).$$

Define the *hull* Ω of f to be cls $\{f_\tau: \tau \in \mathbf{R}\} \subset C$. Then Ω is compact metric, and the map

$$\Omega \times \mathbf{R} \to \Omega: (\omega, \tau) \to \omega_\tau$$

defines a flow on Ω. The map $F: \Omega \to \mathbf{R}^n$: $F(\omega) = \omega(0)$ "extends f to Ω" in the sense that, if $\omega_0 \equiv f \in \Omega$, then $F(\omega_0 \cdot t) = f(t)$ $t \in \mathbf{R}$). See [15, [17].

2.4 *Remarks.* A function $f \in C$ is Bohr almost periodic if and only if the hull Ω of f is minimal and a.p. [14]. In this case, Ω may be given the structure of a compact abelian topological group [14]. Let μ_0 be normalized Haar measure on Ω. Let F extend f to Ω in the sense of 2.3. Then the *mean value* of F is given by

$$M(f) = \lim_{(t-s) \to \infty} \frac{1}{t-s} \int_0^t f(s)\, ds = \int_\Omega F(\omega)\, d\mu_0(\omega).$$

See [14].

2.5 Definition and Proposition. *Say that $f \in C$ is almost automorphic (in the sense of Bochner; see [2] and [18]) if, whenever (t_n) is a sequence such that $f_{t_n} \to g$ in C, then $g_{-t_n} \to f$ in C. It is shown in [18] that f is almost automorphic if and only if the hull Ω of f has the following property: there is an almost periodic minimal flow $(\Omega_0, \mathbf{R})$ and a flow homomorphism $\pi: \Omega \to \Omega_0$ such that $\pi^{-1}\pi(f) = \{f\}$ (thus $(\Omega, \mathbf{R})$ is an* a.a. *extension of* $(\Omega_0. \mathbf{R})$).

The following result follows easily from Lemma 2.4 of [8].

2.6 Proposition. *Let $(X, \mathbf{R})$ be minimal. Let $g: X \to \mathbf{R}$ be a continuous function such that $\int_0^t g(\bar{x} \cdot s)\, ds$ is bounded for some $\bar{x} \in \bar{X}$. Then there is a continuous $G: X \to \mathbf{R}$ such that $G(x \cdot t) - G(x) = \int_0^t g(x \cdot s)\, ds$ $(x \in X,\ t \in \mathbf{R})$.*

3. Bounded solutions

3.1 *Notation.* Consider a scalar equation

$$\dot{x} + a(t)x = b(t) \quad (x \in \mathbf{R}), \tag{1}$$

where $a, b: \mathbf{R} \to \mathbf{R}$ are bounded and uniformly continuous. Define $f: \mathbf{R} \to \mathbf{R}^2: t \to (a(t), b(t))$. Let Ω be the hull of f, and let $F: \Omega \to \mathbf{R}^2$ "extend f to Ω" in the sense of 2.3. We may write $F(\omega) = (A(\omega), B(\omega))$ $(\omega \in \Omega)$. If $\omega_0 \equiv f \in \Omega$, then $A(\omega_0 \cdot t) = a(t)$, $B(\omega_0 \cdot t) = b(t)$ $(t \in \mathbf{R})$.

Now consider the *collection* of equations

$$\dot{x} + A(\omega \cdot t)x = B(\omega \cdot t) \quad (\omega \in \Omega). \tag{1_ω}$$

Note that equation $(1)_{\omega_0}$ coincides with (1). Equations $(1)_\omega$ generate a flow on $\Sigma = \Omega \times \mathbf{R}$, as follows. Let $(\omega, x_0) \in \Sigma$, and let $x(t)$ be the solution to $(1)_\omega$ satisfying $x(0) = x_0$. Define $(\omega, x_0) \cdot t = (\omega \cdot t, x(t)$ $(t \in \mathbf{R})$. The flow $(\Sigma, \mathbf{R})$ has the following description: let $(\omega, x_0) \cdot t = (\omega \cdot t, x(t))$; then

$$x(t) = x_0 \exp\left(-\int_0^t A(\omega \cdot s)\, ds\right) + \int_0^t B(\omega \cdot s) \exp\left(-\int_s^t A(\omega \cdot r)\, dr\right) ds. \tag{2}$$

Note that the projection $\pi: \Sigma \to \Omega: (\omega, x) \to \omega$ is a flow homomorphism.

3.2 Definition. The *Sacker-Sell spectrum* of equations $(1)_\omega$, denoted by Sp (A), is defined as

$\{\lambda \in \mathbf{R}$: for all $\omega \in \Omega$, the homogeneous equation $\dot{x} + (\lambda + A(\omega \cdot t))x = 0$ does *not* admit an exponential dichotomy$\}$

(see [15], [17]). (This definition also applies to n-dimensional systems.)

3.3 *Remarks.* (a) For the one-dimensional equations $(1)_\omega$, one can show using [15] or [17], that Sp (A) is the interval $[a, b]$, where

$$a = \inf \left\{ c \in \mathbf{R}\text{: given } \varepsilon > 0, \text{ there exists a constant } K_\varepsilon \text{ such that} \right.$$

$$\left. -\int_s^t A(\omega \cdot s)\, ds \geq K_\varepsilon + (c - \varepsilon)(t - s) \text{ for all } t > s \text{ and all } \omega \in \Omega \right\},$$

and

$$b = \sup \left\{ c \in \mathbf{R}\text{: given } \varepsilon > 0, \text{ there exists a constant } K_\varepsilon \text{ such that} \right.$$

$$\left. -\int_s^t A(\omega \cdot s)\, ds \leq K_\varepsilon + (c + \varepsilon)(t - s) \text{ for all } t > s \text{ and all } \omega \in \Omega \right\}.$$

(b) If $a(t)$ is Bohr almost-periodic (i.e., if $(\Omega, \mathbf{R})$ is a.p. minimal), then 3.3(a) and 2.4 imply that Sp $(A) = \{\int_\Omega A(\omega)\, d\mu_0(\omega)\}$ (μ_0 is the normalized Haar measure on Ω).

Using 3.3(a), (b), ([17], Problem 7, p. 184), and ([11], Section 3), we have the following result.

3.4 PROPOSITION. *Suppose* $(\Omega, \mathbf{R})$ *is minimal (i.e., "A is minimal", or "recurrent"), and suppose* $0 \in$ Sp (A). *If* $\int_0^t A(\omega \cdot s)\, ds$ *is unbounded for some (hence all)* $\omega \in \Omega$, *then*

$$\left\{ \omega \in \Omega\text{: } \sup_t \int_0^t A(\omega \cdot s)\, ds = \infty \quad \text{and} \quad \inf_t \int_0^t A(\omega \cdot s)\, ds = -\infty \right\}$$

is a residual subset of Ω. *If* $(\Omega, \mathbf{R})$ *is* a.p. *minimal, we may replace* "$0 \in$ Sp (A)" *by* "$\int_\Omega A(\omega)\, d\mu_0(\omega) = 0$" ($\mu_0$ *is the normalized Haar measure on* Ω).

3.5 *Remark.* If $0 \notin$ Sp (A), then each equation $\dot{x} + A(\omega \cdot t)x = 0$ admits an exponential dichotomy. Hence standard results [5], [13] imply that each equation $(1)_\omega$ admits a unique bounded solution. If $(\Omega, \mathbf{R})$ is minimal, if $0 \in$ Sp (A), and if $\int_0^t A(\omega \cdot s)\, ds$ is bounded for some $\omega \in \Omega$, then 2.6 and formula (2) imply that *every* solution to $(1)_\omega$ is bounded ($\omega \in \Omega$).

In view of Remark 3.5, we will suppose the following.

3.6 *Assumption.* $(\Omega, \mathbf{R})$ is minimal, $0 \in$ Sp (A), and $\int_0^t A(\omega \cdot s)\, ds$ is unbounded for some $\omega \in \Omega$. In particular, if $(\Omega, \mathbf{R})$ is a.p. minimal, we are assuming that the mean value $\int_\Omega A(\omega)\, d\mu_0(\omega) = 0$.

We will study bounded solutions of equations $(1)_\omega$.

3.7 LEMMA. *Let* $\omega \in \Omega$. *Suppose* $\inf_t \int_0^t A(\omega \cdot s)\, ds = -\infty$. *The the equation* $(1)_\omega$ *has at most one bounded solution.*

Proof. If $x_1(t)$ and $x_2(t)$ are two bounded solutions of $(1)_\omega$, then

$$x_2(t) - x_1(t) = [x_2(0) - x_1(0)] \exp\left(-\int_0^t A(\omega \cdot s)\, ds\right).$$

The lemma clearly holds.

3.8 PROPOSITION. *Assume* 3.6, *and suppose that some equation* $(1)_\omega$ *admits a bounded solution. Then there is a unique minimal subflow* $(M, \mathbf{R})$ *of* $(\Sigma, \mathbf{R})$ (*see* 3.1), *and* $(M, \mathbf{R})$ *is an almost automorphic extension* (*see* 2.2) *of* $(\Omega, \mathbf{R})$.

Proof. By 3.4 and 3.7, some equation $(1)_{\bar{\omega}}$ admits a *unique* bounded solution $x(t)$ with, say, $x(0) = x_0$. The orbit closure cls $\{(\bar{\omega}, x_0) \cdot t\colon t \in \mathbf{R}\}$ intersects $\pi^{-1}(\bar{\omega})$ only at $(\bar{\omega}, x_0)$. It also contains a minimal set M, which satisfies $\pi(M) = \Omega$ by minimality of Ω. Hence $M \cap \pi^{-1}(\bar{\omega}) = \{(\bar{\omega}, x_0)\}$, and $(M, \mathbf{R})$ is an a.a. extension of $(\Omega, \mathbf{R})$. Clearly M is the unique minimal subset of Σ.

3.9 DEFINITION. Assume 3.6. Define

$$\Omega_\beta(A) = \left\{\omega \in \Omega\colon \inf_t \int_0^t A(\omega \cdot s)\, ds = -\infty\right\}.$$

Let

$H = \{B \in C(\Omega)$: for some $\omega \in \Omega$, equation $(1)_\omega$ admits a bounded solution$\}$.

Here we view equation $(1)_\omega$ as depending on B as well as ω. By 3.8(b), if $B \in H$, then equation $(1)_\omega$ admits a bounded solution for *all* $\omega \in \Omega$. For $B \in H$, let M denote the minimal set described in 3.8, and define

$$\Omega_\alpha(A, B) = \{\omega \in \Omega\colon \pi^{-1}(\omega) \cap M \text{ is a singleton}\}.$$

Examining 2.5, 3.4, 3.7, and 3.8, we have:

3.10 PROPOSITION. *Assume* 3.6, *and let* $B \in H$.

(a) $\Omega_\beta(A)$ *is a residual subset of* Ω.
(b) $\Omega_\beta(A) = \{\omega \in \Omega$: *equation* $(1)_\omega$ *admits a unique bounded solution*$\}$.
(c) $\Omega_\beta(A) \subset \Omega_\alpha(A, B)$; *hence* $\Omega_\alpha(A, B)$ *is residual in* Ω.
(d) *If* $(\Omega, \mathbf{R})$ *is* a.p. *minimal, then* $\Omega_\alpha(A, B) = \{\omega \in \Omega$: *equation* $(1)_\omega$ *admits an almost automorphic solution*$\}$ *is residual in* Ω.

Proposition 3.10 states that $\Omega_\beta(A, B)$ and $\Omega_\beta(A)$ are large topologically. We now analyze the measure-theoretic size of these sets when $(\Omega, \mathbf{R})$ is minimal *and a.p.* (i.e., when (1) is a.p.). Let μ_0 be normalized Haar measure on Ω. Observe that $\Omega_\alpha(A, B)$ and $\Omega_\beta(A)$ are invariant, hence (by the Birkhoff ergodic theorem [14]) both sets either have measure zero or measure 1.

Our results are as follows.

(I) For "most" functions A, one has $\mu_0(\Omega_\beta(A)) = 1$, and hence $\mu_0(\Omega_\alpha(A, B)) = 1$ for all $B \in H$.

(II) There is an irrational twist flow on the 2-torus K^2 and a continuous $A: K^2 \to \mathbf{R}$ such that $\mu_0(\Omega_\beta(A)) = 0$. We do not know whether there is a.p. minimal flow $(\Omega, \mathbf{R})$ and $A \in C(\Omega)$, $B \in H$ such that $\mu_0(\Omega_\alpha(A, B)) = 0$ (but see 3.17).

(III) If A and B are chosen as in [12], then one has $\mu_0(\Omega_\alpha(A, B)) = 1$. We prove this by showing that $(M, \mathbf{R})$ is uniquely ergodic [14]. Compare this fact with [10]: the a.a. minimal sets in the Millionščikov and Vinograd examples each have 2 ergodic measures.

3.11 THEOREM. *Assume* $(\Omega, \mathbf{R})$ *is* a.p. *minimal. Let*

$$C_0(\Omega) = \left\{ f \in C(\Omega) : \int_\Omega f(\omega)\, d\mu_0(\omega) = 0 \right\}.$$

There is a residual subset $C_1 \subset C_0(\Omega)$ *such that, if* $A \in C_1$, *then* 3.6 *holds and* $\mu_0(\Omega_\beta(A)) = 1$.

Proof. Suppose A has the property that $\mu_0(\Omega_\beta(A)) = 0$. Let

$$Q = \left\{ \omega \in \Omega : \int_0^t A(\omega \cdot s)\, ds \text{ is bounded below} \right\}.$$

Then $\mu_0(Q) = 1$ (3.7). Let $L(\omega) = -\inf_t \int_0^t A(\omega \cdot s)\, ds$ $(\omega \in Q)$. Then L is μ_0-measurable, is finite on Q, and

$$L(\omega \cdot t) - L(\omega) = \int_0^t A(\omega \cdot s)\, ds \quad (\omega \in Q,\ t \in \mathbf{R}).$$

Now, by [9, Section 4], there is a residual subset C_1' of $C_0(\Omega)$ such that, if $A \in C_1'$, then A admits no such L. It is easily seen that

$$C_1 = \{A \in C_1' : 3.6 \text{ holds}\}$$

is also residual.

We return to statement (II). The construction in 3.12 was suggested by those of Anosov in [1].

3.12 *Example.* Let $\xi \in \mathbf{R}$ be an irrational number such that, for some constant $\sigma_1 > 0$, one has $|n\xi + m| \geq \sigma_1/n$ for all $n, m \in \mathbf{Z}$ with $n \neq 0$. The set of all such irrationals has Hausdorff dimension 1 and Lebesgue measure zero [16]. View the 2-torus K^2 as the square $[0, 1] \times [-\frac{1}{2}, \frac{1}{2}]$ with opposite sides identified; give it the Euclidean metric d. Define a flow on K^2 by

$$(x, y) \to (x + t, y + \xi \cdot t) \quad ((x, y) \in K^2;\ t \in \mathbf{R})$$

(obvious identifications are made in defining this flow). Let $\omega_0 = (0, 0) \in K^2$; then $\omega_0 \cdot t = (t, \xi t)(t \in \mathbf{R})$. By assumption on ξ, there exists $\sigma_2 > 0$ such that

$$(*) \qquad |t| \geq \tfrac{1}{2} \Rightarrow d(\omega_0 \cdot t, \omega_0) \geq \sigma_2/|t|.$$

Now let $(\gamma_n)_{n=1}^{\infty}$ be a sequence such that $0 < \gamma_n < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Let $g_n\colon \mathbf{R} \to [0, 1]$ be a continuously differentiable function, supported on

$$I_n = [-2\gamma_n^{-2}, 2\gamma_n^{-2}],$$

satisfying (i) $g_n(0) = 1$; (ii) $|\dot{g}_n(t)| \le \gamma_n^2$ for all $t \in \mathbf{R}$ $(n \ge 1)$. By choice of ξ, there exists σ_3 such that (iii) $0 < \sigma_3 < 1$; (iv) σ_3 depends on ξ but not on n; (v) if

$$Q_n = \{0\} \times [-\sigma_3 \gamma_n^3/4, \sigma_3 \gamma_n^3/4] \subset K^2,$$

then the map $i_n\colon Q_n \times I_n \to K^2\colon (\omega, t) \to \omega \cdot t$ is *injective* $(n \ge 1)$. Finally, for $c > 0$ and $x \in \mathbf{R}$, define

$$r(x, c) = \begin{cases} 0, & |x| > c \\ 1 - \dfrac{|x|}{c}, & |x| \le c, \end{cases}$$

and let $r_n(x) = r(x, \sigma_3 \gamma_n^3/4)$ $(x \in \mathbf{R}, n \ge 1)$.

Define functions $L_n\colon K^2 \to \mathbf{R}$ as follows $(n \ge 1)$. If $\omega \in K^2$, and there exists $p = (0, x) \in Q_n$ and $t \in I_n$ such that $i_n(p, t) = \omega$, then

$$L_n(\omega) = \gamma_n^{-1} r_n(x) g_n(t);$$

otherwise, $L_n(\omega) = 0$. Then L_n is continuous, $L_n(\omega) \ge 0$ for all $\omega \in K^2$, and

$$\int_{K^2} L_n(\omega)\, d\mu_0(\omega) \ge \gamma_n^{-1} \cdot \sigma_3 \gamma_n^3 \cdot \gamma_n^{-2}/2 = \sigma_3/2.$$

Also, the function

$$A_n(\omega) = \frac{d}{dt} L_n(\omega \cdot t)\Big|_{t=0}$$

is continuous, and

$$|A_n(\omega)| < \gamma_n^{-1} \cdot \gamma_n^2 = \gamma_n.$$

In addition, if $J_n = i_n\ (Q_n \times I_n)$, then $\mu_0(J_n) \le \sigma_3 \gamma_n^3 \cdot 4\gamma_n^{-2} = 4\sigma_3 \gamma_n$. Hence $\sum_{n=1}^{\infty} \mu_0(J_n) < \infty$.

This last statement implies that, if $V = \{\omega \in K^2\colon \omega$ belongs to at most finitely many of the $J_n\}$, then $\mu_0(V) = 1$ (the Borel-Cantelli lemma). The function $L(\omega) = \sum_{n=1}^{\infty} L_n(\omega) \ge 0$ is defined (i.e., finite) for all $\omega \in V$. Since

$$\int_{K^2} L(\omega)\, d\mu_0(\omega) = \sum_{n=1}^{\infty} \int_{K^2} L_n(\omega)\, d\mu_0(\omega) = \infty,$$

L is certainly not equal μ_0-a.e. to a continuous function on K^2.

Let $A(\omega) = \sum_{n=1}^{\infty} A_n(\omega)$ $(\omega \le K^2)$. Then A is continuous. Let E be a countable dense subset of $\mathbf{R}$ containing 0, and let $V_0 = \cap \{V \cdot t\colon t \in E\}$. Then $V_0 \subset V$, and $\mu_0(V_0) = 1$. For each $t \in \mathbf{R}$ and $\omega \in K^2$,

$$\int_0^t A(\omega \cdot s)\, ds = \sum_{n=1}^{\infty} \int_0^t A_n(\omega \cdot s)\, ds = \sum_{n=1}^{\infty} [L_n(\omega \cdot t) - L_n(\omega)].$$

If $\omega \in V_0$ and $t \in E$, then $\int_0^t A(\omega \cdot s)\, ds = L(\omega \cdot t) - L(\omega)$. Hence if $\omega \in V_0$, then $\int_0^t A(\omega \cdot s)\, ds$ is bounded below by $-L(\omega)$. Using 2.6 and the fact that L is not equal a.e. to a continuous function, it is easily seen that $\int_0^t A(\omega \cdot s)\, ds$ is unbounded for all $\omega \in \Omega$. Since $\int_{K^2} A_n(\omega)\, d\mu_0(\omega) = 0$, $\int_{K^2} A(\omega)\, d\mu_0(\omega) = 0$ (see also [1]). Using formula (2), one shows that, if $\omega \in V_0$ and equation $(1)_\omega$ admits one bounded solution, then *all* solutions of $(1)_\omega$ are bounded.

In conclusion: $(K^2, \mathbf{R})$ is a.p. minimal, A satisfies 3.6, and $\mu_0(\Omega_\beta(A)) = 0$. These are the conditions we wanted.

3.13 *Remark.* It is now easy to construct A and B so that $\mu_0(\Omega_\beta(A)) = 0$ and $\mu_0(\Omega_\alpha(A, B)) = 1$. Let A be as in 3.12, let $x: K^2 \to \mathbf{R}$ be a C^1 function, and let

$$B(\omega) = \frac{d}{dt} x(\omega \cdot t)\Big|_{t=0} + A(\omega)x(\omega).$$

Then $\Omega_\alpha(A, B) = K^2$.

Finally, we consider (III). Suppose that equation (1) is the a.p. equation constructed in [12] (actually, a *class* of equations is constructed in [12], so pick one). Construct Ω, A, B, and Σ as in 3.1. By 2.6 and [12], $\int_0^t A(\omega \cdot s)\, ds$ is unbounded for all $\omega \in \Omega$, and $\int_\Omega A(\omega)\, d\mu_0(\omega) = 0$. Let $(M, \mathbf{R})$ be the unique minimal subflow of $(\Sigma, \mathbf{R})$; by 3.8, it is almost automorphic. By [12], it is *not* almost periodic.

3.14 LEMMA. *$\mu_0(\Omega_\alpha(A, B)) = 0$ if and only if $(M, \mathbf{R})$ is uniquely ergodic (has a unique invariant measure; see [14]).*

Proof ($\Rightarrow$) Suppose $\mu_0(\Omega_\alpha(A, B)) = 0$. Let $M_\omega = \{x \in \mathbf{R}: (\omega, x) \in M\}$ $(\omega \in \Omega)$. Then for μ_0-almost all ω, card $M_\omega > 1$. Let

$$g_1(\omega) = \max\{x: x \in M_\omega\}, \qquad g_2(\omega) = \min\{x: x \in M_\omega\}.$$

The maps $\mu_i: C(M) \to \mathbf{R}: f \to \int_\Omega f(\omega, g_i(\omega))\, d\mu_0(\omega)$ $(i = 1, 2)$ define distinct invariant measures μ_1, μ_2, on M. Hence M is not uniquely ergodic.

($\Leftarrow$) We prove the contrapositive. Suppose $\mu_0(\Omega_\alpha(A, B)) = 1$. Let

$$M_0 = \{m \in M: \{m\} = \pi^{-1}\pi(m) \cap M\}.$$

Let μ be an invariant measure on M. Since $\pi(\mu) = \mu_0$, we must have $\mu(D) = \mu_0(\pi(D))$ for all Borel sets D in M such that $D \subset M_0$. It follows that μ is unique.

3.15. Next, we observe that an arbitrary minimal flow $(\tilde{M}, \mathbf{R})$ is uniquely ergodic if and only if there is an $m \in \tilde{M}$ with the following property. If $f \in C(\tilde{M})$ and $\varepsilon > 0$ are given, then there exists $T > 0$ such that $|t - s| > T$, $|t' - s'| > T$ imply

$$\left| \frac{1}{t - s} \int_s^t f(m \cdot r)\, dr - \frac{1}{t' - s'} \int_{s'}^{t'} f(m \cdot r)\, dr \right| < \varepsilon.$$

This is easily proved using techniques from [14, pp. 498–511].

3.16. Finally, we sketch the proof that $(M, \mathbf{R})$ is uniquely ergodic. We assume the reader has [12] before him, and use facts discussed there. Let $\omega_0 \in \Omega$ be that point such that equation $(1)_{\omega_0}$ is the same as equation (1). Using formula (2) and the fact that $\int_0^t A(\omega_0 \cdot s)\, ds \to \infty$, one can show that the element $(\omega_0, 0)$ of Σ is actually in M.

Define the *density* $D(F)$ of a set $F \subset \mathbf{R}$ to be

$$\overline{\lim_{0 \le t-s \to \infty}} \frac{1}{t-s} l([s, t] \cap F),$$

where l is Lebesgue measure on $\mathbf{R}$. Let $\varepsilon > 0$ be given. Let $x(t)$ be the solution of (1) such that $x(0) = 0$, and let $x_n(t)$ $(n \ge 3)$ be the approximating solutions defined in [12]. Using details of the construction of $x_n(t)$ and $x(t)$, one can show that, given $\delta > 0$, there is an N such that

$$n \ge N \Rightarrow D\{t \in \mathbf{R}: |x_n(t) - x(t)| > \delta\} < \delta.$$

Let $f \in C(M)$. Given $\varepsilon > 0$, choose $\delta < \varepsilon/6\|f\|$ such that

$$\max\{|f(\omega, z) - f(\omega, z')| : (\omega, z), (\omega, z') \in M\} < \varepsilon/6 \quad \text{if } |z - z'| < \delta.$$

Note that $(\omega_0, 0) \cdot t = (\omega_0 \cdot t, x(t)) \in M$ $(t \in \mathbf{R})$. Now

$$\left| \frac{1}{t-s} \int_s f(\omega_0 \cdot r, x(r))\, dr - \frac{1}{t'-s'} \int_{s'}^{t'} f(\omega_0 \cdot r, x(r)\, dr \right|$$

$$\le \frac{1}{|t-s|} \int_s^t |f(\omega_0 \cdot r, x(r)) - f(\omega_0 \cdot r, x_n(r))|\, dr$$

$$+ \left| \frac{1}{t-s} \int_s^t f(\omega_0 \cdot r, x_n(r))\, dr - \frac{1}{t'-s'} \int_{s'}^{t'} f(\omega_0 \cdot r, x_n(r))\, dr \right|$$

$$+ \frac{1}{|t'-s'|} \int_{s'}^{t'} |f(\omega_0 \cdot r, x_n(r)) - f(\omega_0 \cdot r, x(r))|\, dr.$$

Choose N so that

$$n \ge N \Rightarrow D\{t \in \mathbf{R}: |x_n(t) - x(t)| > \delta\} < \delta.$$

We can then find T_1 such that $|t - s| > T_1$, $|t' - s'| > T_1$ imply that the first and third terms are less than $\varepsilon/3$. Since $x_n(t)$ is *periodic*, the map

$$r \to f(\omega_0 \cdot r, x_n(r))$$

is almost periodic, hence there is a $T \ge T_1$ such that, if $|t - s| > T$, $|t' - s'| > T$, then the middle term is less than $\varepsilon/3$. It follows that $(M, \mathbf{R})$ is uniquely ergodic.

3.17 *Remark.* One might be able to find an a.p. equation such that $\mu_0(\Omega_x(A, B)) = 0$ by altering the functions A and B of [12] so as to make $(M, \mathbf{R})$

non-uniquely ergodic (and, of course, using 3.15 and 3.16 to verify non-unique ergodicity).

4. Unbounded solutions

In Section 4, we consider equations $(1)_\omega$ for some fixed $A, B \in C(\Omega)$. We are interested in the behavior of solutions to equations $(1)_\omega$ when $(\Omega, \mathbf{R})$ is minimal and some equation $(1)_\omega$ admits *no* bounded solutions. Note this implies that $0 \in \text{Sp}(A)$ (3.5).

4.1 DEFINITION. Define a flow on $\Sigma_c = \Omega \times [0, \pi]$ as follows. If $\theta_0 = 0$ or $\theta_0 = \pi$, then

$$(\omega, \theta_0) \cdot t = (\omega \cdot t, \theta_0) \quad (\omega \in \Omega, t \in \mathbf{R});$$

if $0 < \theta_0 < \pi$, then

$$(\omega, \theta_0) \cdot t = (\omega \cdot t, \theta(t)),$$

where

$$\cot \theta(t)$$
$$= (\cot \theta_0) \exp\left(-\int_0^t A(\omega \cdot s)\, ds\right) + \int_0^t B(\omega \cdot s) \exp\left(-\int_s^t A(\omega \cdot r)\, dr\right) ds.$$

Note that, if $x(t)$ is the solution to $(1)_\omega$ with $x(0) = \cot \theta_0$, then $x(t) = \cot \theta(t)$. Also note that $\inf_t x(t) = -\infty$ iff $\sup_t \theta(t) = \pi$, and $\sup_t x(t) = \infty$ iff $\inf_t \theta(t) = 0$.

4.2 THEOREM. *Suppose $(\Omega, \mathbf{R})$ is minimal, and suppose some equation $(1)_\omega$ admits no bounded solutions. Then there is a residual subset $Q \subset \Omega$ such that, if $\omega \in Q$, then all solutions to equation $(1)_\omega$ satisfy* $\inf_t x(t) = -\infty$, $\sup_t x(t) = \infty$.

Proof. The first part of the proof is similar to that of [11, Theorem 3.8]. Let

$$\Sigma_n = \{(\omega, \theta) \in \Sigma_c\colon 0 \le \theta \le \pi - 1/n\}.$$

Let

$$K_n = \{\omega, \theta) \in \Sigma_c\colon (\omega, \theta) \cdot t \in \Sigma_n \quad \text{for all } t \in \mathbf{R}\}.$$

Then K_n is closed and invariant. Let $f_n(\omega) = \max\{\theta\colon (\omega, \theta) \in K_n\}$; thus $f_n(\omega)$ is the "upper endpoint" of $K_n \cap (\{\omega\} \times [0, \pi])$. Then f_n is upper semi-continuous, hence has a residual set Ω_n of continuity points.

Suppose ω_0 is a continuity point of f_n such that $f_n(\omega_0) > 0$. There is a neighborhood V of ω_0 such that $f_n(\omega) > 0$ if $\omega \in V$. Since Ω is minimal, $\Omega = \bigcup_{i=1}^m V \cdot t_i$ for some integer m. Invariance of K_n now implies that there is a $\delta > 0$ such that $f_n(\omega) \ge \delta > 0$ for all $\omega \in \Omega$.

Now, let $M = \{(\omega, f_n(\omega))\colon \omega \in \Omega\}$. It is easily seen that M is invariant. Also,

$M \subset \Omega \times [\delta, \pi - 1/n]$. This means that, if $(\omega, \theta_0) \in M$ and $x_0 = \cot^{-1} \theta_0$, then $\cot^{-1} (\pi - 1/n) \leq x(t) \leq \cot^{-1} \delta$, where $x(t)$ is the solution to $(1)_\omega$ such that $x(0) = x_0$. This contradicts our assumption; hence $f_n(\omega) = 0$ for all $\omega \in \Omega_n$. That is, if $\omega \in \Omega_n$, then all solutions to $(1)_\omega$ are at some time less than or equal to $\cot^{-1} (\pi - 1/n)$.

Let $Q_1 = \bigcap_{n \geq 1} \Omega_n$. Then if $\omega \in Q_1$, every solution to $(1)_\omega$ is unbounded below. Similarly, one can find a residual subset Q_2 of Ω such that, if $\omega \in Q_2$, then all solutions to $(1)_\omega$ are unbounded above. Let $Q = Q_1 \cap Q_2$.

4.3 *Remark.* Suppose $(\Omega, \mathbf{R})$ is a.p. minimal, and let μ_0 be normalized Haar measure on Ω. Suppose the assumptions of 4.2 are satisfied. Then $\mu_0(Q)$ may be zero. For example, let $A: K^2 \to \mathbf{R}$ be the function constructed in 3.12. Consider the equations $(1)_\omega$ $\dot{x} = A(\omega)$ $(\omega \in K^2)$. Then solutions to $(1)_\omega$ are of the form $x(t) = x_0 + \int_0^t A(\omega \cdot s)\, ds$, hence are bounded below if ω is in a subset V_0 of K^2 satisfying $\mu_0(V_0) = 1$.

4.4 *Remark.* Suppose $(\Omega, \mathbf{R})$ is minimal, and suppose $0 \in \operatorname{Sp}(A)$. Let

$$G = \{B \in C(\Omega): \text{some equation } (1)_\omega \text{ admits no bounded solution}\}.$$

Then G is a residual subset of $C(\Omega)$. To see this, first assume that $\int_0^t A(\omega \cdot s)\, ds$ is unbounded for some $\omega \in \Omega$. By [17, Exercise 7, p. 204], there is an $\bar{\omega} \in \Omega$ such that $\int_0^t A(\bar{\omega} \cdot s)\, ds$ is bounded below. It follows that, if $B(\omega) = 1$ for all $\omega \in \Omega$, then $B \in G$. From this, one can show that G is residual in $C(\Omega)$. If $\int_0^t A(\omega \cdot s)\, ds$ is bounded for some $\omega \in \Omega$, the proof is even easier.

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