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EXACT INTERVALS

BY

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1. Introduction

In a previous paper we characterized those cosimplicial k-spaces $T\colon \Delta \to$ kTop whose left Kan extension $\mathrm{Lan}_R\, T$ along the right Yoneda functor $R\colon \Delta \to$ Simpl (Set) preserves finite products. It was shown that T arises from an "interval" $T\,[1]$. In this paper we extend these results by showing that $\mathrm{Lan}_R\, T$ is an exact functor (preserves finite limits and colimits) if and only if the reflection of $T\,[1]$ into the category of T_0 spaces is a Hausdorff space. In the classical case, where T is the cosimplicial space of affine simplexes, $T\,[1]$ is the standard unit interval I and $\mathrm{Lan}_R\, T$ is the geometric realization functor.

2. Preliminaries

Recall, from [4], that the category Int of Intervals has, as objects, the non-empty, linearly ordered, bounded sets X equipped with a connected compactly generated topology (in the sense of [5]) for which X^n, the n-fold product in kTop, has the weak topology relative to the family $\{gX_n\}$, $g \in S(n)$, the permutation group on n objects, where

$$X_n = \{(x_1, \ldots, x_n) \mid x_1 \le \cdots \le x_n\} \subset X^n$$

and

$$gX_n = \{(x_{g1}, \ldots, x_{gn}) \mid (x_1, \ldots, x_n) \in X_n\} \subset X^n,$$

and has, as morphisms, the continuous, non-decreasing, endpoint preserving maps. Theorem 4.1 of [4] shows that the correspondence $X \to T_X\colon \Delta \to$ kTop, where $T_X[n] = X_n$, defines an equivalence between Int and the full subcategory of cosimplicial k-spaces determined by those $T\colon \Delta \to$ kTop for which $T\,[1]$ is nonempty and connected, and $\mathrm{Lan}_R\, T$ preserves finite products. The aim of this paper is to characterize the category EInt of exact intervals, i.e. to explicitly describe those $X \in$ Int for which $\mathrm{Lan}_R\, T_X$ is exact. Note that $\mathrm{Lan}_R\, T_X$ is exact if and only if it preserves equalizers since, in general, $\mathrm{Lan}_R\, T$ is cocontinuous (it is left adjoint to the singular functor $X \to$ Set (T, X)) and the preservation of finite limits is equivalent to the preservation of finite products and equalizers [2, Section 2, p. 108].

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3. T_0, T_1, T_2 intervals

For each property P of spaces, let PkTop (PInt) be the full subcategory of kTop (Int) determined by those spaces with property P (those intervals with underlying space in PkTop). For P = T_0, the inclusion i: T_0kTop → kTop has a left adjoint Q for which the unit of the adjunction $\eta(X)$: $X \to iQX$ is induced by the quotient map of X onto the quotient space QX of X determined by the equivalence relation that identifies points if and only if they have the same set of neighborhoods. Thus T_0kTop is a reflective subcategory of kTop with reflector Q [3, p. 89]. Further, Q lifts to a functor Int → T_0Int, still denoted by Q, that defines T_0Int as a reflective subcategory of Int (5.1 of [4]). For the separation properties T_1 and T_2 we have:

3.1 THEOREM. $T_1\mathrm{Int} = T_2\mathrm{Int}$.

Proof. The following lemma implies $T_1\mathrm{Int} \subset T_2\mathrm{Int}$.

3.2 LEMMA. *For* $X \in$ Int, *the maps* min *and* max *are continuous.*

Proof. It follows from the definition of interval, for $n = 2$, that

$$X_2 \cap \tau X_2 \overset{u}{\underset{v}{\rightrightarrows}} X_2 \amalg \tau X_2 \to X^2$$

is a coequalizer where the maps are the obvious inclusions and $\tau(x, y) = (y, x)$. The map $X_2 \amalg \tau X_2 \to X$ induced by $P_1(P_2)$ on the first factor and by $P_2(P_1)$ on the second factor (P_i is the i^{th} projection) coequalizes u and v and thus induces a continuous map $X^2 \to X$ that is readily seen to be min (max).

If $a \in X \in T_1\mathrm{Int}$ then, since the set $X - \{a\}$ is open, the sets

$$m_a^{-1}(X - \{a\}) = \{x \mid x < a\} \quad \text{and} \quad M_a^{-1}(X - \{a\}) = \{x \mid a < x\}$$

are also open, where $m_a(x) = \min(a, x)$ and $M_a(x) = \max(a, x)$. This clearly implies $X \in T_2\mathrm{Int}$ and thus 3.1 follows.

3.3 *Remarks.* (1) $X \in T_2\mathrm{Int}$ if and only if X is a non-empty, linearly ordered, bounded set equipped with a connected k-topology that contains the order topology (5.3 of [4]).

(2) Each interval X has the structure of a topological monoid under max. Thus X can be used as the monoid in Boardman and Vogt's bar construction for theories [1, p. 72]. As they pointed out in [1, Remark 3.2, p. 74], most of their results do not hold for a general monoid without further restrictions. However, the restrictions imposed on the monoid X by the requirement that X be an interval do allow for certain extensions of their results.

4. Exact intervals

This section deals with the main result:

4.1 THEOREM. $X \in$ EInt if and only if $X \in$ Int and $QX \in T_1$Int.

Proof. We begin with a number of preliminary results. A space $A \in$ kTop is said to have the f-induced k-topology for $(f\colon A \to B) \in$ kTop if the continuity of fg implies that of g for any function $g\colon C \to A$, with $C \in$ kTop. Let

$$(*) \qquad F \xrightarrow{j} G \underset{g}{\overset{f}{\rightrightarrows}} H$$

be an equalizer in Simpl (Set).

4.2 LEMMA. *For* $X \in$ Int, *the functor* $|?|_X = \mathrm{Lan}_R\, T_X$ *preserves the equalizer* (*) *if and only if* $|F|_X$ *has the* $|j|_X$*-induced k-topology.*

Proof. The way in which equalizers are computed in kTop clearly implies that $|F|_X$ has the $|j|_X$-induced k-topology if (*) is preserved. To show the converse it thus suffices to show that, on the underlying set level,

$$|F|_X \to |G|_X \rightrightarrows |H|_X$$

is an equalizer. Since the underlying set functor U: kTop $\to$ Set is both continuous and cocontinuous, it readily follows from the coend formula

$$|?|_X = \int^n R[n] \otimes T_X[n]$$

that $U|?|_X = |?|_{UX}$ and that $|?|_{UX}$ preserves finite products. Thus it is sufficient to show that

$$(**) \qquad |F|_{UX} \to |G|_{UX} \rightrightarrows |H|_{UX}$$

is an equalizer in Set. That (**) is indeed an equalizer can be proved by appropriately modifying a proof (in particular the one given in [2, p. 5.1, Section 3.3] of the corresponding classical ($X = I$) result. We begin by observing that for any point

$$y = (y_1, \ldots, y_n) \in X_n^0 = \{(x_1, \ldots, x_n) \,|\, 0 < x_1 < \cdots < x_n < 1\} \subset X_n$$

(0, 1 are the endpoints of X) and any point $z = (z_1, \ldots, z_n) \in X_n$ there is an endpoint preserving, nondecreasing function (not necessarily mono, epi or continuous) $S\colon UX \to UX$ for which $S(y_i) = z_i$, $i = 1, \ldots, n$. Further, S extends to a natural transformation $T_S\colon T_{UX} \to T_{UX}$ that in turn extends to a natural transformation $|?|_S\colon |?|_{UX} \to |?|_{UX}$ for which the map

$$S_n = |R[n]|_S\colon |R[n]|_{UX} = UX_n \to UX_n$$

satisfies $S_n(y) = (Sy_1, \ldots, Sy_n) = z$ (S_n acts this way since $|\,?\,|_{UX}$ preserves finite products). One now proceeds as in the above mentioned proof in [2] noting that the replacement of I by UX results in the replacement of the affine n-simplex and its interior by UX_n and $U\mathring{X}_n$ respectively. An essential fact needed in that proof is that if $|f|_{UX}(x) = |g|_{UX}(x)$ for $x \in |G|_{UX}$ then $|f|_{UX}$ and $|g|_{UX}$ agree on the "cell" determined by x. That this fact indeed obtains follows as in [2] if one notes that the above observation about S is sufficient to give the necessary results of Section 1.6 [2]. It should be noted that while the group of continuous endpoint preserving homeomorphisms of I is sufficient to obtain the pertinent results of Section 1.6 [2] when $X = I$, a larger class of endomorphisms of UX is needed to obtain the analogous results involving UX. The rest of the proof of 4.3 now follows as in [2].

4.3 LEMMA. *$|F|_X$ has $|j|_X$-induced k-topology if $X \in \mathrm{T_1Int}$.*

Proof. If $X \in \mathrm{T_1Int}$ then, by 3.1, $X \in \mathrm{T_2Int}$ and consequently $\dot{X}_n = X_n - \mathring{X}_n$ is a closed subset of X_n. Hence, by an obvious modification of the argument of 2, p. 50, Section 3.2, $|j|_X$ is a closed injection.

4.4 LEMMA. *The image of $|\,?\,|_X$ is in $\mathrm{T_0kTop}$ if $X \in \mathrm{T_1Int}$.*

Proof. Since $\dot{X}_n$ is a closed subset of the T_2 space X_n, it follows from Fig. 14, p. 44 of [2] that $i_n\colon |Sk^{n-1}F|_X \to |Sk^nF|_X$ is a closed injection and, inductively, that $|Sk^nF|_X$ is T_0, for any simplicial set F. Since $|F|_X = \text{colimit } i_n$, 4.4 readily follows.

4.5 LEMMA. *The functors $iQ|\,?\,|_X$ and $|\,?\,|_{iQX}$ are naturally equivalent if $QX \in \mathrm{T_1Int}$.*

Proof. Since $Q\colon \mathrm{ktop} \to \mathrm{T_0kTop}$ is cocontinuous, product preserving and $Qi = \mathrm{id}$ one has

$$\begin{aligned} iQ|\,?\,|_X &= iQ \int^n R[n] \otimes T_X[n] \\ &\approx i \int^n R[n] \otimes QT_X[n] \\ &= i \int^n R[n] \otimes QiQT_X[n] \\ &\approx iQ \int^n R[n] \otimes iQT_X[n] \\ &\approx iQ \int^n R[n] \otimes T_{iQX}[n] \\ &= iQ|\,?\,|_{iQX} = |\,?\,|_{iQX}, \end{aligned}$$

where the last equality follows from 4.4.

4.6 PROPOSITION. $X \in \text{EInt}$ *if* $QX \in \text{T}_1\text{Int}$.

Proof. Since the horizontal arrows (from the unit of the (i, Q) adjunction) in the commutative square

$$\begin{array}{ccc} |F|_X & \to & iQ|F|_X \\ {\scriptstyle |j|_X}\downarrow & & \downarrow{\scriptstyle iQ|j|_X} \\ |G|_X & \to & iQ|G|_X \end{array}$$

induce the k-topology on their respective domains and, by 4.5, $iQ|j|_X$ is equivalent to $|j|_{iQX}$ and, by 4.3, $|F|_{iQX}$ has the $|j|_{iQX}$-induced k-topology it readily follows that $|F|_X$ has the $|j|_X$-induced k-topology. Thus 4.2 gives 4.6.

4.7 PROPOSITION. $QX \in \text{T}_1\text{Int}$ *if* $X \in \text{EInt}$.

Proof. In Set, and consequently in Simpl (Set), each mono i imbeds in a cocartesian square

$$A \underset{i}{\overset{i}{\rightrightarrows}} B \rightrightarrows C$$

that is also cartesian; i.e. each mono is an equalizer. Clearly there is a mono

$$i\colon (R[1])^2 \to (R[1])^3$$

for which $\alpha = |i|_X\colon |R[1]^2|_X = X^2 \to X^3 = |R[1]^3|_X$ satisfies $\alpha(x, y) = (x, x, y)$(resp. (y, x, x) if $x \leq y$ (resp. $y \leq x$). Thus X^2 has the α-induced k-topology if $X \in \text{EInt}$. If $QX \notin \text{T}_1\text{Int}$ then there is a subspace $S = \{a, b\} \subset X$ that is neither discrete nor indiscrete. If $a < b$ then the map $x \mapsto (a, x, b)\colon X \to X^3$ induces a continuous map $\beta\colon S \to X^3$ that factors, by $\gamma\colon a, b \mapsto (a, b), (b, a)\colon S \to X^2$, through α. Since S is clearly a k-space, γ is continuous and consequently every neighborhood of (a, b) in X^2 contains (b, a) or vice versa. This implies, in either case, that S is indiscrete, a contradiction. Hence $QX \in \text{T}_1\text{Int}$ and 4.7 is proved.

Theorem 4.1 now follows from 4.6 and 4.7.

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