1. Introduction

(1.1) Background. Let \( k \) be an algebraically closed field and let \( V \) be a finite-dimensional vector space over \( k \). Let \( G \) be a reductive algebraic subgroup of \( GL(V) \).

Let \( k[V] \) be the algebra of regular functions on \( V \). The group \( G \) acts on \( k[V] \) as follows:

\[
(g \cdot f)(v) = f(g^{-1}v)
\]

for all \( f \in k[V] \), \( g \in G \), and \( v \in V \). The ring of \( G \)-invariant functions on \( V \) is

\[
k[V]^G = \{ f \in k[V] : g \cdot f = f \text{ for all } g \in G \}.
\]

We define an algebraic subvariety \( X \) of \( V \) by

\[
X = \{ v \in V : f(v) = 0 \text{ for each non-constant homogeneous } f \in k[V]^G \}.
\]

A point in \( V \), not in \( X \), is called semi-stable.

In order to describe the points in \( X \), it is useful to introduce the concept of an orbit. Let \( v \) be in \( V \). The orbit of \( v \) with respect to the action of \( G \) is

\[
G \cdot v = \{ g \cdot v : g \in G \}
\]

The Zariski-closure of \( G \cdot v \) will be denoted by \( \text{cl}(G \cdot v) \).

Theorem. Let \( G \) be a connected reductive algebraic subgroup of \( GL(V) \). Let \( v \in V \). The following statements are equivalent:

(a) \( v \) is not semi-stable;
(b) \( 0 \in \text{cl}(G \cdot v) \);
(c) there is a one-parameter subgroup \( \lambda \) of \( G \) so that \( \lambda(\alpha) \cdot v \to 0 \) as \( \alpha \to 0 \).

The notation in (c) will be explained in (2.1). The equivalence of (a), (b), and (c) is proved in [10; Sections 1 and 2] taking into account [5].

(1.2) Summary of results. The purpose of this paper is to prove some results aimed at explicitly describing the set \( X \). The basic theorem is proved in (2.2). As consequences of this theorem, the following corollaries are proved in (2.3) and (3.2).
(1) Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ and let $B = TU$ be a Borel subgroup of $G$. Let $v_0$ be a point in $V$ which is not semi-stable. Then there is a one-parameter subgroup $\lambda_0: \mathbb{G}_m \to B$ such that $\lambda_0(\alpha)v_0 \to 0$ as $\alpha \to 0$.

(2) Let $G$ and $B$ be as in (1). There exist subspaces $W_1, \ldots, W_r$ of $V$ such that the following statements hold:

(a) each $W_i$ is $B$-invariant;
(b) $X = G \cdot W_1 \cup \cdots \cup G \cdot W_r$;
(c) each $G \cdot W_i$ is closed.

(3) Let $G$ be as in (1) and suppose that $G$ acts irreducibly on $V$. Let $v_0 \in V$ be a point which is not semi-stable. Then the highest weight vector of $G$ is in $\text{cl}(G \cdot v_0)$.

(1.3) Existence of semi-stable points. Semi-stable points exist and form an open set if $\dim V > \dim G$ and $G$ is semisimple. This fact follows from several well-known theorems but does not seem to have been stated before. We give the proof now.

**Theorem.** Let $G$ be a connected semisimple algebraic subgroup of $GL(V)$. Let

$$m = \max \{\dim (G \cdot v): v \in V\}.$$ 

If $\dim V > m$, then $V - X$ is not empty.

**Proof.** Let $k(V)$ be the field of rational functions on $V$. The group $G$ acts on $k(V)$ via

$$(g \cdot f)(v) = f(g^{-1}v)$$

for all $f \in k(V)$, $g \in G$, and $v \in V$. Let

$$k(V)^G = \{f \in k(V): g \cdot f = f \text{ for all } g \in G\}.$$ 

We begin by showing that $k(V)^G$ is the quotient field of $k[V]^G$. Let $f = a/b$ be in $k(V)^G$ where $a, b \in k[V]$. Let $a = p_1 \cdots p_r$ and $b = q_1 \cdots q_s$ be the factorizations of $a$ and $b$ into prime elements where we shall assume $a$ and $b$ have no common factors. There is a finite-dimensional subspace $E$ of $k[V]$ which is $G$-invariant and contains each $p_i$ and $q_j$ [6; Proposition, p. 62]. For $h \in k[V]$, let

$$\langle h \rangle = \{ch: c \in k\}.$$ 

Now since $g \cdot f = f$, we see that $(gb)a = (ga)b$. Since $a$ and $b$ have no common factors, each $gp_i$ is a multiple of some $p_j$. Thus $G$ permutes $\langle p_1 \rangle \cdots \langle p_r \rangle$. Since $G$ is connected, $G\langle p_i \rangle = \langle p_i \rangle$ for all $i = 1, \ldots, r$. Hence, there are constants $c_g \in k$ satisfying

$$g \cdot p_i = c_gp_i$$

for all $g \in G$. 

The map \( g \to c_g \) is a character of \( G \). But \( G \) is semisimple so each character is trivial. Therefore, \( g \cdot p_i = p_i \) for all \( g \in G \) and \( G \) fixes \( a \). It follows that \( G \) fixes \( b \).

Next, let \( m = \max \{ \dim G \cdot v : v \in V \} \). According to a theorem of M. Rosenlicht [8; pp. 406–407], \( \dim k(V)^G = \dim V - m \). If \( \dim V > m \), then \( \dim k(V)^G > 0 \). By what was proved above, there are non-constant functions in \( k[V]^G \) and, so, \( X \neq V \).

2. The theorem and its corollaries

(2.1) Preliminaries. We begin this section by recalling some notation and definitions along with some of the concepts in [7].

(1) A one-parameter subgroup \( \lambda \) of an algebraic group \( G \) is a homomorphism \( \lambda : G_m \to G \) (where \( G_m \) is the multiplicative group \( k^* = k - \{0\} \)).

Let \( f : G_m \to X \) be a morphism of algebraic varieties. If \( f \) extends to a morphism \( f_0 : G \to X \), then \( y = f_0(0) \) is called the specialization of \( f(\alpha) \) as \( \alpha \) specializes to 0. We shall denote this by \( f(\alpha) \to y \) as \( \alpha \to 0 \).

(2) Let \( \lambda : G_m \to GL(V) \) be a one-parameter subgroup. There is a basis \( v_1, \ldots, v_n \) of \( V \) and integers \( e_1, \ldots, e_n \) so that

\[
\lambda(\alpha)v_i = \alpha^{e_i}v_i
\]

for \( i = 1, \ldots, n \) [6; 16.1]. Let

\[
V(e_i) = \{ v \in V : \lambda(\alpha)v = \alpha^{e_i}v \}.
\]

Next, we define a subspace \( W(\lambda) \) of \( V \) by

\[
W(\lambda) = \{ v \in V : \lambda(\alpha)v \to 0 \text{ as } \alpha \to 0 \}.
\]

Then it is easily verified that \( W(\lambda) \) is the direct sum of those subspaces \( V(e_i) \) where \( e_i > 0 \).

(3) [10; Lemma 3.1]. Let \( G \) be a reductive algebraic group and let \( \lambda : G_m \to G \) be a one-parameter subgroup of \( G \). There is a unique algebraic subgroup \( P(\lambda) \) in \( G \) such that \( p \) is in \( P(\lambda) \) if and only if \( \lambda(\alpha)p\lambda(\alpha^{-1}) \) has a specialization in \( G \) when \( \alpha \) specializes to 0. Moreover, \( P(\lambda) \) is a parabolic subgroup of \( G \).

For \( g \in G \), let \( g\lambda g^{-1} \) denote the one-parameter subgroup of \( G \) defined by

\[
\alpha \to g\lambda(\alpha)g^{-1}.
\]

It is not hard to check that \( P(g\lambda g^{-1}) = gP(\lambda)g^{-1} \).

(4) Let \( G \) be a reductive algebraic group and let \( \rho : G \to GL(V) \) be a representation of \( G \). Let \( \lambda : G_m \to G \) be a one-parameter subgroup of \( G \). Let \( W(\lambda) \) and \( P(\lambda) \) be as (2) and (3). Then \( P(\lambda) \cdot W(\lambda) \subset W(\lambda) \).
Proof. Let $p \in P(\lambda)$ and $v \in W(\lambda)$. Then

$$\lambda(\alpha) pv = \lambda(\alpha) p \lambda(\alpha^{-1}) \lambda(\alpha) v \to 0 \quad \text{as} \quad \alpha \to 0.$$ 

(2.2) Theorem. Let $G$ be a connected reductive algebraic subgroup of $GL(V)$. Let $B = TU$ be a Borel subgroup of $G$ and let $W(T) = N(T)/T$ be the Weyl group of $T$. Let $v_0$ be a point in $V$ which is not semi-stable. There is a one-parameter subgroup $\lambda : G_m \to T$ such that the following statements hold:

(a) $B \subset P(\lambda)$, i.e., if $\mu$ is a root of $B$ relative to $T$, then $\langle \mu, \lambda \rangle \geq 0$;

(b) $B \cdot W(\lambda) \subseteq W(\lambda)$;

(c) there are elements $u \in U$, $sT \in W(T)$ such that $v_0 = us \cdot W(\lambda)$.

Proof. According to statement (c) of the theorem in 1.1, there is a one-parameter subgroup $\lambda_0$ of $G$ such that $v_0 \in W(\lambda_0)$. Let $B_0$ be a Borel subgroup in $P(\lambda_0)$ such that each $\lambda_0(\alpha)$ is in $B_0$. There is an element $g \in G$ such that $B = gB_0 g^{-1}$ and an element $b \in B$ so that

$$b(g\lambda_0(\alpha)g^{-1})b^{-1} \in T$$

for all $\alpha \in G_m$. Let $\lambda = (bg)\lambda_0(bg)^{-1}$.

To prove statement (a), we use preliminary (3) above to see that

$$P(\lambda) = (bg) P(\lambda_0)(bg)^{-1} = (bg) B_0 (bg)^{-1} = B.$$ 

Also, we recall that there is an isomorphism $e$ from $G_a$ into $G$ such that for all $t \in T, x \in G_a$, we have

$$te_{\mu}(x)t^{-1} = e_{\mu}(\mu(t)x) \quad \text{[6; Theorem, p. 161]}.$$ 

Hence,

$$\lambda(\alpha)e_{\mu}(x)\lambda(\alpha^{-1}) = e_{\mu}(\alpha^t x)$$ 

where, by definition, $e = \langle \mu, \lambda \rangle$. We now apply (3) again to see that $\langle \mu, \lambda \rangle \geq 0$.

Statement (b) follows from (a) and preliminary (4). To prove (c), we first note that $W(\lambda) = bqW(\lambda_0)$ so that $v_0 \in G \cdot W(\lambda)$. Now, according to the Bruhat decomposition of $G$, we have $G \supseteq \bigcup UsB$ where $sT$ ranges over all the distinct cosets of the Weyl group $W(T) = N(T)/T$. Hence,

$$G \cdot W(\lambda) = \bigcup UsB \cdot W(\lambda) = \bigcup Us \cdot W(\lambda)$$

according to (b). This proves (c).

(2.3) Consequences. Throughout this section, we shall denote by $G$ a connected reductive algebraic subgroup of $GL(V)$ and by $B = TU$ a given Borel subgroup of $G$.

Corollary 1. Let $v_0$ be a point in $V$ which is not semistable. There is a one-parameter subgroup $\lambda_0 : G_m \to B$ such that $\lambda_0(\alpha)v_0 \to 0$ as $\alpha \to 0$. 
Proof. According to (2.2), there is a one-parameter subgroup \( \lambda : G_m \to T \) and elements \( u \in U, sT \in W(T) \) such that \( v_0 = us \cdot W(\lambda) \). Let
\[
\lambda_0(\alpha) = (us)\lambda(\alpha)(us)^{-1}
\]
for all \( \alpha \in G_m \). Then \( \lambda_0 \) is a one-parameter subgroup of \( B \) since
\[
\lambda_0(\alpha) = (us)\lambda(\alpha)(us)^{-1} \subset uTu^{-1} \subset B.
\]
Furthermore, \( \lambda_0(\alpha)v_0 \to 0 \) as \( \alpha \to 0 \). For if \( v_0 = us \cdot w \) with \( w \in W(\lambda) \), then
\[
\lambda_0(\alpha)v_0 = us\lambda(\alpha)s^{-1}u^{-1}usw = us\lambda(\alpha)w.
\]

Lemma. Let \( X \) be a closed subset of \( V \) and let \( P \) be a parabolic subgroup of \( G \). If \( P \cdot X \) is closed, then \( G \cdot X \) is closed.

Proof. Let \( P \) act on the right on \( G \times V \) by \( (g, v) \cdot p = (gp, v) \). The quotient variety \( (G \times V)/P \) exists and is \( (G/P) \times V \) [4; 6.6, Corollary]. Let
\[
\pi : G \times V \to (G/P) \times V
\]
be the quotient morphism. Then \( \pi \) is open. Let
\[
A = \{(g, v) \in G \times V : g^{-1}v \in P \cdot X\}
\]
Since \( A \) is the inverse image of \( P \cdot X \) under the morphism \( G \times V \to V \) defined by \( (g, v) \to g^{-1}v \), we see that \( A \) is closed. It is easily verified that \( \pi^{-1}(\pi(A)) = A \) and, so, \( \pi(A) \) is closed in \( (G/P) \times V \) (since \( \pi \) is open). Now \( G/P \) is complete so the image \( G \cdot X \) of \( \pi(A) \) under the projection map \( (G/P) \times V \to V \) is closed in \( V \).

Note. The proof above is a slight extension of one in [3; Lemma 6.3]. A short "transcendental" proof can be given when \( K = C \). For then, \( G = KP \) where \( K \) is compact [11; Theorem 1, p. 102] and, so, \( G \cdot X = K \cdot (P \cdot X) \). But \( K \cdot (P \cdot X) \) is closed since compact transformation groups send closed sets to closed sets.

Corollary 2. Let \( X \) be the set of points in \( V \) which are not semi-stable. There are one-parameter subgroups \( \lambda_1, \ldots, \lambda_r \) of \( T \) such that the following statements hold:
(a) \( B \subset P(\lambda_i) \) and \( B \cdot W(\lambda_i) \subset W(\lambda_i) \) for all \( i = 1, \ldots, r \);
(b) each \( G \cdot W(\lambda_i) \) is closed;
(c) \( X = G \cdot W(\lambda_1) \cup \cdots \cup G \cdot W(\lambda_r) \) and this is the unique decomposition of \( X \) into irreducible components unless there exist \( i, j, i \neq j, sT \in W(T) \) such that \( W(\lambda_i) \subset s \cdot W(\lambda_j) \).

Proof. Let \( T \) have weights \( \chi_1, \ldots, \chi_n \) on \( V \) and let
\[
V(\chi) = \{v \in V : tv = \chi(t)v \quad \text{for all} \quad t \in T\}.
\]
Next let \( \lambda \) be a one-parameter subgroup of \( T \). Let \( \chi \) be one of the weights above and put \( e = \langle \chi, \lambda \rangle \). Then \( \lambda(\alpha)v = \alpha^e v \) for all \( v \in V(\chi) \). Therefore, \( V(\chi) \subset W(\lambda) \) if
and only if \( e > 0 \). It follows that there are (finitely many) one-parameter subgroups \( \lambda_1, \ldots, \lambda_r \) of \( T \) such that (i) \( B \subset P(\lambda_i) \) and (ii) if \( \lambda: G_m \to T \) is any one-parameter subgroup such that \( B \subset P(\lambda) \), then \( W(\lambda) = W(\lambda_i) \) for some \( i = 1, \ldots, r \).

Statements (a), (b), and (c) follow from the theorem and lemma above, except for the decomposition of \( X \).

Let us write \( W_i = W(\lambda_i) \) for \( i = 1, \ldots, r \). Now suppose that

\[
G \cdot W_i \subset G \cdot W_1 \cup \cdots \cup G \cdot W_{i-1} \cup G \cdot W_{i+1} \cup \cdots \cup G \cdot W_r.
\]

Since \( W_i \) is irreducible, there is a \( j \neq i \) so that \( W_i \subset G \cdot W_j \). Applying the Bruhat decomposition of \( G \), we now see that

\[
W_i \subset \cup UsB \cdot W_j = \cup UsW_j.
\]

But \( W_i \) is \( U \)-invariant so \( W_i \subset \cup sW_j \). Since \( W_i \) is irreducible, we obtain the desired result that \( W_i \subset s \cdot W_j \) for some \( sT \in W(T) \).

**Corollary 3.** Suppose that there is an element \( sT \) in the Weyl group of \( G \) so that \( s\chi = -\chi \) for all weights \( \chi \) of \( T \). Let \( X \) be the set of points in \( V \) which are not semi-stable. Then \( \dim X < \frac{1}{2} \dim V + \dim U \).

**Proof.** Let us use the notation for \( V(\chi) \) introduced in the proof of Corollary 2. Let \( \lambda_i: G_m \to T \) be as in Corollary 2. If \( \chi \) is a weight of \( T \) on \( V \) and if \( V(\chi) \subset W(\lambda_i) \), then \( V(\chi) \cap W(\lambda_i) = \{0\} \). Since \( s \cdot V(\chi) = V(-\chi) \), we have \( \dim V(\chi) = \dim V(-\chi) \). Thus \( \dim W(\lambda_i) \leq \frac{1}{2} \dim V \). The statement about \( \dim X \) now follows from the Bruhat decomposition of \( G \) and the fact that \( B \cdot W(\lambda_i) \) is contained in \( W(\lambda_i) \). For \( G \cdot W(\lambda_i) = \cup UsB \cdot W(\lambda_i) = \cup UsW(\lambda_i) \).

**Notes.** Let \( G \) be a simple algebraic group, not of type \( A_n, D_n \) \( (n \text{ odd}) \), or \( E_6 \). Then there is an element \( sT \) in the Weyl group of \( G \) satisfying \( s\chi = -\chi \) for all weights \( \chi \) of \( T \) [11; p. 226].

(2.4) **Properly stable points.** Let \( G \) be a reductive algebraic subgroup of \( GL(V) \). A point \( v \) in \( V \) is called properly stable if the orbit \( G \cdot v \) is closed and has dimension equal to that of \( G \). A point \( v \) in \( V \) is not properly stable if and only if there is a one-parameter subgroup \( \lambda: G_m \to G \) such that \( \lambda(\alpha) \cdot v \) has a specialization in \( V \) as \( \alpha \) specializes to 0 [10; Section 2].

In case \( \text{char } k = 0 \), one may prove the following result analogous to the Theorem of (1.3): Let \( G \) be a connected semisimple algebraic group and let \( \rho: G \to GL(V) \) be a finite-dimensional representation of \( G \). There is an integer \( M \) so that if \( \dim V > M \), then the set of properly stable points in \( V \) contains a non-empty open set [12] and [1]—the first paper holds in any characteristic.

Let us assume \( \text{char } k \geq 0 \) and let \( \lambda: G_m \to G \) be a one-parameter subgroup of \( G \). Let

\[
W'(\lambda) = \{v \in V: \lambda(\alpha) \cdot v \text{ has a specialization in } V \text{ as } \alpha \to 0\}.
\]
Then we may show that $P(\lambda) \cdot W'(\lambda) \subseteq W'(\lambda)$ as in (2.1) and prove the following theorem and corollaries just as in (2.2) and (2.3).

**Theorem.** Let $G$ be a connected reductive algebraic subgroup of $GL(V)$. Let $B = TU$ be a Borel subgroup of $G$ and let $W(T) = N(T)/T$ be the Weyl group of $T$. Let $v_0$ be a point in $V$ which is not properly stable. There is a one-parameter subgroup $\lambda : G_m \to T$ such that the following statements hold:

(a) $B \subseteq P(\lambda)$, i.e., if $\mu$ is a root of $B$ relative to $T$, then $\langle \mu, \lambda \rangle \geq 0$;

(b) $B \cdot W'(\lambda) \subseteq W'(\lambda)$;

(c) there are elements $u \in U$, $sT \in W(T)$ such that $v_0 = us \cdot W'(\lambda)$.

**Corollary 1.** There is a one-parameter subgroup $\lambda_0 : G_m \to B$ such that $v_0 \in W'(\lambda_0)$.

**Corollary 2.** Let $X'$ be the set of points in $V$ which are not properly stable. There are one-parameter subgroups $\lambda_1, \ldots, \lambda_r$ of $T$ such that the following statements hold:

(a) $B \subseteq P(\lambda_i)$ and $B \cdot W'(\lambda_i) \subseteq W'(\lambda_i)$ for all $i = 1, \ldots, r$;

(b) each $G \cdot W'(\lambda_i)$ is closed;

(c) $X' = G \cdot W'(\lambda_1) \cup \cdots \cup G \cdot W'(\lambda_r)$ and this is the unique decomposition of $X'$ into irreducible components unless there exist $i, j, s \in T$ $W(T)$ such that $W'(\lambda_i) \supset s \cdot W'(\lambda_j)$.

3. Borel subgroups and semi-stable points

(3.1) **Theorem.** Let $B$ be a connected solvable algebraic group acting on an affine variety $X$. Let $x \in X$ and $Z = cl(B \cdot x)$. Then either $B \cdot x$ is closed or there is an $f \in k[Z]$ such that

$$Z - B \cdot x = \{z \in Z : f(z) = 0\}.$$  

In the latter case, there is an element $c$ in $k^*$ so that the mapping $\chi : B \to k$ given by $\chi(b) = cf(b \cdot x)$ is a character of $B$.

**Proof.** The group $B$ operates on $k[Z]$ via $(b \cdot f)(z) = f(b^{-1} \cdot z)$ for all $f \in k[Z]$, $z \in Z$, and $b \in B$. Let $I$ be the ideal in $k[Z]$ vanishing on $Z - B \cdot x$. Then $I$ is $B$-invariant, i.e., $b \cdot I \subseteq I$ for all $b \in B$. Suppose now that $I \neq \{0\}$ and let $f$ be any non-zero element in $I$. There is a finite-dimensional $B$-invariant subspace $E \subseteq I$ such that $f \in E$ [6; Proposition, p. 62]. By the Lie–Kolchin theorem, there is a non-zero common eigenvector $h$ in $E$ for $B$ [6; 17.6, p. 113]. Let $b \cdot h = c_b h$. Then

$$h(b^{-1} \cdot x) = (b \cdot h)(x) = c_b h(x).$$

If $h(x) = 0$, then $h = 0$ on $B \cdot x$ and $h = 0$. Hence, $h(x) \neq 0$ and $h$ is non-zero on $B \cdot x$. Since $h$ is in $I$, $h$ is $0$ on $Z - B \cdot x$.

The mapping $b \mapsto h(b \cdot x)$ is non-zero on $B$ and, so is a character of $B$ if $h(e \cdot x) = 1$ [9; Proposition 3, p. 29]. Modifying $h$ by a constant, we obtain the theorem.
Corollary (Kostant, Rosenlicht). Let $U$ be a unipotent group acting on an affine variety $X$. For every $x \in X$, the orbit $U \cdot x$ is closed.

Proof. The corollary follows at once from the theorem since the only character of $U$ is trivial.

Notes. The corollary above was first proved by B. Kostant. A shorter proof was found by M. Rosenlicht. Another proof was found by A. Borel [3; Theorem 12.1]. A modification of Borel's proof gives the theorem above.

(3.2) Theorem. Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ and let $B = TU$ be a Borel subgroup of $G$. Suppose that $0$ is the only point in $V$ fixed by $G$. Let $v_0$ be a non-zero vector in $V$ which is not semi-stable. There is a non-zero vector $v \in cl(B \cdot v_0)$ such that $U \cdot v = v$.

Proof. According to Corollary 1 in Section 2.3, the point $0$ is in $cl(B \cdot v_0)$. Let $w \in cl(B \cdot v_0)$ be chosen so that $B \cdot w$ has the smallest possible positive dimension. Then $cl(B \cdot w) - B \cdot w$ consists of points fixed by $B$. Since $G/B$ is complete, each of these points is fixed by $G$. However, by our assumption, then, $cl(B \cdot w) - B \cdot w = \{0\}$. The theorem in (3.1) now implies that $dim(B \cdot w) = 1$.

Now $U$ must fix $w$. For otherwise, $U \cdot w$ is a closed subset (by the corollary above) of $B \cdot w$ having dimension 1. This would imply that $U \cdot w = B \cdot w$ and $B \cdot w$ is closed.

Corollary. Let $G$ be a connected reductive algebraic subgroup of $GL(V)$ which acts irreducibly on $V$. Let $B$ be a Borel subgroup of $G$. Let $v_0$ be a non-zero vector in $V$ which is not semi-stable. Then the highest weight vector of $G$ on $V$ (relative to $B$) is $cl(G \cdot v_0)$.

4. Examples

(4.1) The adjoint representation. Let $G$ be a connected reductive algebraic group and let $L(G)$ denote the Lie algebra of $G$. Then $G$ acts on $L(G)$ via the adjoint representation.

Let $B = T \cdot U$ be a Borel subgroup of $G$. Let $L(T)$, $L(U)$, and $L(B)$ be the Lie algebras of $T$, $U$, and $B$, respectively. We shall denote the roots of $T$ acting on $L(U)$ by $\alpha$, $\beta$, $\gamma$, . . . . Then there is a basis $\{e_\alpha\}$ of $L(U)$ so that $t \cdot e_\alpha = \alpha(t)e_\alpha$ for all $t \in T$.

Next, let $W$ be a subspace of $L(G)$ which is $B$-invariant. If $W$ contains $e_\beta$ (where $e_\beta \in L(U)$), then $w = [e_\beta, e_\beta]$ is a non-zero element in $W$ which is fixed by $T$.

Let $\lambda: G_m \rightarrow T$ be a one-parameter subgroup of $T$ such that $W(\lambda)$ is $B$-invariant. Then $W(\lambda) \subset L(U)$ by the argument just given. Also, there is a one-parameter subgroup $\lambda$ of $T$ so that $\langle \lambda, \alpha \rangle > 0$ if $\alpha > 0$ [4; Theorem, p. 317]. For this one-parameter subgroup, we have $W(\lambda) = L(U)$ and $P(\lambda) = B$. 
Finally, let $X$ be the set of points in $L(G)$ which are not semi-stable. According to the remarks above and Corollary 2 in (2.3), we have

$$X = G \cdot L(U).$$

It is known that $G \cdot L(U)$ is precisely the set of nilpotent elements in $L(G)$. Hence, we obtain a result of B. Kostant: a point $v$ in $L(G)$ is not semi-stable if and only if $v$ is nilpotent.

(4.2) Certain actions of $SL_n$ Let $SL_n$ be the group of all $n \times n$ matrices with entries in $k$ and having determinant 1. Let

$$T = \{ t = (t_{ij}) \in SL_n : t_{ij} = 0 \text{ for } i \neq j \}.$$

We shall denote a typical matrix $t = (t_{ij})$ in $T$ by $t = [t_{11}, \ldots, t_{mn}]$. Let us define characters $\chi_1, \ldots, \chi_n$ of $T$ by

$$\chi_i(t_{11}, \ldots, t_{mn}) = t_{ii} \quad \text{for each } i = 1, \ldots, n$$

(so $\chi_1 + \cdots + \chi_n = 0$). Let

$$B = \{ (b_{ij}) \in SL_n : b_{ij} = 0 \text{ for } i > j \}.$$

Then $B$ is a Borel subgroup with maximal torus $T$. A simple system of roots for $T$ on $B$ is $\{ \mu_1, \ldots, \mu_{n-1} \}$ where $\mu_i = \chi_i - \chi_{i+1}$. If $\lambda$ is a one-parameter subgroup of $T$, then there are integers $u_1, \ldots, u_n$ so that

$$\lambda(x) = [x^{u_1}, \ldots, x^{u_n}]$$

and $u_1 + \cdots + u_n = 0$. The subgroup $B$ is contained in $P(\lambda)$ if and only if each $\langle \mu, \lambda \rangle \geq 0$, that is, if and only if

$$u_i \geq u_{i+1} \text{ for } i = 1, \ldots, n - 2 \quad \text{and} \quad 2u_{n-1} + u_1 + \cdots + u_{n-2} \geq 0.$$

The group $SL_n$ acts on the vector space $k^n$ of all $n \times 1$ column matrices in the natural way, namely, $g \cdot v = gv$ for all $g \in SL_n, v \in k^n$. This action gives rise to an action on $k[x_1, \ldots, x_n]$, the algebra of regular functions on $k^n$, via

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \quad \text{for all } g \in SL_n, v \in k^n, f \in k[x_1, \ldots, x_n].$$

Let $S_m$ be the vector space consisting of all those polynomials in $k[x_1, \ldots, x_n]$ which are homogeneous of degree $m$. Then $S_m$ is a finite-dimensional subspace of $k[x_1, \ldots, x_n]$ which is stable under the action of $SL_n$. We shall study the variety $X$ in $S_m$.

Let $v = x_1^{e_1} \cdots x_n^{e_n}, e_1 + \cdots + e_n = m$, be in $S_m$ and let

$$\lambda(x) = [x^{u_1}, \ldots, x^{u_n}]$$

be a one-parameter subgroup of $T$. Then

$$\lambda(x) \cdot v = x^{e}v \quad \text{where } e = u_1(e_n - e_1) + \cdots + u_{n-1}(e_n - e_{n-1}).$$

To summarize, we have seen that:

1. a one-parameter subgroup $\lambda$ of $T$ may be identified with a point $(u_1, \ldots, u_{n-1})$ where each $u_i$ is an integer;
(2) \( B \subset P(\lambda) \) if and only if
\[ u_1 - u_2 \geq 0, \ldots, u_{n-2} - u_{n-1} \geq 0, \quad \text{and} \quad 2u_{n-1} + u_1 + \cdots + u_{n-2} \geq 0; \]
(3) \( x_1^{u_1} \cdots x_n^{u_n} \in W(\lambda) \) if and only if
\[ u_1(e_n - e_1) + \cdots + u_{n-1}(e_n - e_{n-1}) > 0. \]

We turn now to the cases \( n = 2 \) and \( n = 3. \)

**SL2.** Let us put \( \lambda(x) = [x^n, x^{-n}] \) where we may assume that \( u > 0 \) (by (2)). Then, by (3), \( x_1^u x_2^{m-n} \) is in \( W(\lambda) \) if and only if \( u(m - 2e) > 0 \), i.e., \( e < m/2. \)

If \( m = 2s \), then \( W(\lambda) \) is spanned by \( x_2^m, x_1 x_2^{m-1}, \ldots, x_1^{s-1} x_2^{m-s+1} \). In each of these monomials, the multiplicity of \( x_2 \) is \( \geq s + 1 \). Hence, \( G \cdot W(\lambda) \) consists of all those polynomials in \( S_m \) having a linear factor whose multiplicity is \( \geq s + 1 \).

If \( m = 2s + 1 \), we arrive at a conclusion just like the one just given: \( G \cdot W(\lambda) \) consists of all those polynomials in \( S_m \) having a linear factor whose multiplicity is \( \geq s + 1 \).

In both cases above, \( X \) has only one component and \( P(\lambda) = B \).

**SL3.** Let us change notation here and write \( u, t \) instead of \( u_1, u_2 \) and \( a, b, c \) instead of \( e_1, e_2, e_3 \). According to (2) and (3) above, we should study pairs \( u, t \) so that \( u \geq t \) and \( u + 2t \geq 0 \). (If \( \lambda \) is to be non-trivial, we should take \( u > 0 \).) Then \( x_1^u x_2^b x_3^c \) is in \( W(\lambda) \) if and only if \( u(c - a) + t(c - b) > 0 \). Let us distinguish two types of one-parameter subgroups of \( T \), namely:

(I) \( u > 0, u \geq t \geq 0 \);
(II) \( u > 0, t \leq 0, u + 2t \geq 0 \).

The chart below summarizes the conditions \( u, t \) must satisfy for \( x_1^u x_2^b x_3^c \) to be in \( W(\lambda) \).

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The chart summarizes the conditions \( u, t \) must satisfy for \( x_1^u x_2^b x_3^c \) to be in \( W(\lambda) \):
To illustrate how this chart may be used, let us look at the case $m = 8$. Using Corollary 2c, (2.3), one may show that

$$X = G \cdot W(\lambda_1) \cup G \cdot W(\lambda_2) \cup G \cdot W(\lambda_3) \cup G \cdot W(\lambda_4)$$

is the unique decomposition of $X$ into irreducible components where

- $\lambda_1$ is of type I with $0 < t/u < 1/6$;
- $\lambda_2$ is of type I with $2/3 < t/u < 1$;
- $\lambda_3$ is of type II with $1/4 < -t/u < 1/3$;
- $\lambda_4$ is of type II with $1/3 < -t/u < 1/2$.

In each case, $P(\lambda_i) = B$.

REFERENCES

11. R. Steinberg, Lectures on Chevalley groups, Yale University, Department of Mathematics, New Haven, 1967.