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HYPERCOMPLEX FOURIER AND LAPLACE TRANSFORMS I

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Introduction

In [7], a hypercomplex function theory has been introduced, which generalizes the theory of holomorphic functions of one complex variable to the $(m + 1)$-dimensional space. The functions in this theory, which are called monogenic functions, are $\mathscr{A}$-valued, $\mathscr{A}$ being the Clifford algebra constructed over an n-dimensional real quadratic vector space $(n \geq m)$. Hence if one wants to apply this function theory to analysis in a natural way, the role of the complex field as range of the functions and distributions under consideration is taken over by the Clifford algebra.

This means that our theory deals with the left and right modules $\mathscr{A}$-valued testfunctions, C_∞-functions, rapidly decreasing C_∞-functions, etc., and their corresponding dual modules, the elements of which are called $\mathscr{A}$-distributions, $\mathscr{A}$-distributions with compact support, tempered $\mathscr{A}$-distributions, etc.

The aim of this paper is to study the Fourier and Laplace transforms in the context of monogenic functions and $\mathscr{A}$-distributions, by making use of the exponential function $E(\mathbf{t}, x)$, $(\mathbf{t}, x) \in \mathscr{R}^m \times \mathscr{R}^{m+1}$ introduced in [12], which itself is a natural generalization of e^{-itz}, $(t, z) \in \mathscr{R} \times \mathscr{C}$ and which for fixed $\mathbf{t}$ is monogenic in $\mathscr{R}^{m+1}$. The restriction $E(\mathbf{t}, \mathbf{x})$, $(\mathbf{t}, \mathbf{x}) \in \mathscr{R}^m \times \mathscr{R}^m$ of $E(\mathbf{t}, x)$ to the hyperplane $x_0 = 0$ replaces in our theory the Fourier kernel functions e^{-itx}, $(t, x) \in \mathscr{R} \times \mathscr{R}$.

We first introduce the Fourier transform $\mathscr{F}\phi(\mathbf{x}) = \int_{\mathscr{R}^m} E(\mathbf{t}, \mathbf{x})\phi(\mathbf{t})\, d\mathbf{t}$ of rapidly decreasing $\mathscr{A}$-valued C_∞-functions, which leads to the definition of the Fourier transforms of tempered $\mathscr{A}$-distributions.

Next we investigate the generalized Fourier transform $\mathscr{F}\phi(x) = \int_{\mathscr{R}^m} E(\mathbf{t}, x)\phi(\mathbf{t})\, d\mathbf{t}$ of $\mathscr{A}$-valued testfunctions. In this way we generalize the Gelfand-Schilow $\mathscr{Z}$-space of one complex variable (see e.g. [1], [2], [8] and [10]), which consists of all entire functions of the form $f(z) = \int_{\mathscr{R}} e^{-itz}\phi(t)\, dt$, for some $\phi \in \mathscr{D}(\mathscr{R})$. Moreover the classical result is extended, stating that $\mathscr{Z}$ coincides with the space of entire functions such that for some $R > 0$ and for every $k \in \mathscr{N}$ there exists $C_k > 0$ for which $|z^k f(z)| \leq C_k e^{R|y|}$.

In the third section we introduce the generalized Fourier transform

$$\langle T_{\mathbf{t}}, E(\mathbf{t}, x)\rangle$$

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of $\mathscr{A}$-distributions with compact support and we prove the analogue of the classical Paley-Wiener-Schwartz theorem (see [1], [2], [9] and [10]).

Finally we define the Laplace transform $\mathscr{L}T$ of tempered $\mathscr{A}$-distributions which vanish in a neighbourhood of the origin in such a way that $\mathscr{L}T$ is left monogenic in $\mathscr{R}^{m+1}\backslash\mathscr{R}^m$ and that it extends the complex Laplace transform

$$\mathscr{L}T(z) = \begin{cases} \displaystyle\int_{-\infty}^{0} e^{-itz} T_t \, dt, & \text{if im } z > 0 \\ \displaystyle-\int_{0}^{+\infty} e^{-itz} T_t \, dt, & \text{if im } z < 0. \end{cases}$$

Moreover, by making use of the theory of distributional boundary values of monogenic functions (see [11]), we generalize the result which states that the boundary values $\lim_{\varepsilon\to 0+} \mathscr{L}T(x \pm i\varepsilon)$ exist in $\mathscr{S}'(\mathscr{R})$ and that

$$BV\mathscr{L}T = \lim_{\varepsilon\to 0+} (\mathscr{L}T(x+i\varepsilon) - \mathscr{L}T(x-i\varepsilon)) = \mathscr{F}T.$$

Using this Laplace transform we extend the following boundary value result to higher dimensions.

Let $\mathscr{H}_R(\mathscr{C}\backslash\mathscr{R})$ and $\mathscr{H}_R(\mathscr{C})$ be the spaces of holomorphic functions in respectively $\mathscr{C}\backslash\mathscr{R}$ and $\mathscr{C}$ which satisfy respectively estimates of the form

$$|f(z)| \le C\left(1 + \frac{1}{|y|}\right)^k (1 + |z|)^l e^{R|y|}, \quad k, l \in \mathscr{N}, C > 0$$

and

$$|f(z)| \le C(1 + |z|)^l e^{R|y|}, \; l \in \mathscr{N}, C > 0.$$

Then the boundary value mapping $BV\colon \mathscr{H}_R(\mathscr{C}\backslash\mathscr{R}) \to \mathscr{S}'(\mathscr{R})$ is surjective, bounded and open. Moreover the kernel of this mapping coincides with $\mathscr{H}_R(\mathscr{C})$.

An extension of these results to the case of holomorphic functions in tubular radial domains has been obtained by Carmichael in [3].

In a forthcoming paper [13] we shall study Laplace transforms and boundary values in $\mathscr{D}'_{L_{2,(1)}}(\mathscr{R}^m; \mathscr{A})$, as has been done in the case of several complex variables in [4], [5] and [14], for example.

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Preliminaries

Throughout this paper, $\mathscr{A}$ denotes the Clifford algebra constructed over a real quadratic vector space V. A basis for $\mathscr{A}$ is given by

$$\{e_A;\, A \subseteq N,\, N = \{1, \ldots, n\}\}$$

where $e_i = e_{\{i\}}$ $(i = 1, \ldots, n)$, $e_0 = e_\phi$ is the identity of $\mathscr{A}$, $e_i^2 = -e_0$ $(i = 1, \ldots, n)$ and $e_i e_j + e_j e_i = 0$ $(i \neq j,\ i, j = 1, \ldots, n)$, and where

$$e_A = e_{\alpha_1} \cdots e_{\alpha_h} \quad \text{when } A = \{\alpha_1, \ldots, \alpha_h\} \quad \text{and} \quad 1 \leq \alpha_1 \cdots < \alpha_h \leq n.$$

We define an involution in $\mathscr{A}$ as follows: let $a = \sum_A a_A e_A$; then we put $\bar{a} = \sum_A a_A \bar{e}_A$, where

$$\bar{e}_A = \bar{e}_{\alpha_h} \cdots \bar{e}_{\alpha_1}, \quad \bar{e}_j = -e_j\ (j = 1, \ldots, n) \quad \text{and} \quad \bar{e}_0 = e_0.$$

If $a = \sum_A a_A e_A$ is an arbitrary element of $\mathscr{A}$, then its norm $|a|_0$ is defined by

$$|a|_0^2 = 2^n \sum_A a_A^2.$$

Now let $\Omega \subseteq \mathscr{R}^{m+1}$ be open $(1 \leq m \leq n)$ and let

$$D = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i}$$

be the hypercomplex differential operator generalizing the classical Cauchy-Riemann operator (see [7]). Then $M_1(\Omega; \mathscr{A})$ and $M_1^{(r)}(\Omega; \mathscr{A})$ stand for the spaces of functions $f \in C_1(\Omega; \mathscr{A})$ satisfying respectively

$$Df = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i} f = 0 \quad \text{and} \quad fD = \sum_{i=0}^{m} \frac{\partial}{\partial x_i} fe_i = 0$$

in Ω.

By $\mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ $(\mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A}))$ we denote the left (right) $\mathscr{A}$-module of rapidly decreasing $\mathscr{A}$-valued C_∞-functions while $\mathscr{D}_{(l)}(\mathscr{R}^m; \mathscr{A})$ $(\mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A}))$ is its submodule of $\mathscr{A}$-valued C_∞-functions with compact support.

The space of the bounded left $\mathscr{A}$-linear functionals on $\mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ is denoted by $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$; it is called the right $\mathscr{A}$-module of all tempered $\mathscr{A}$-distributions.

The action of $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ on $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ is denoted by $\langle T, \phi \rangle$. Note that for any $a \in \mathscr{A}$, $\langle T, a\phi \rangle = a \langle T, \phi \rangle$.

Distributions in $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ have the same behaviour as ordinary tempered distributions, i.e., for any $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ there exists a continuous $\mathscr{A}$-valued function g which is of polynomial growth in $\mathscr{R}^m$ and $\mathbf{l} \in \mathscr{R}^m$ such that for any $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$,

$$\langle T, \phi \rangle = (-1)^{|\mathbf{l}|} \int_{\mathscr{R}^m} \partial_{\mathbf{t}}^{\mathbf{l}}(\phi(\mathbf{t})) g(\mathbf{t})\, d\mathbf{t},$$

where

$$\partial_{\mathbf{t}}^{\mathbf{l}} = \frac{\partial^{l_1}}{\partial t_1^{l_1}} \cdots \frac{\partial^{l_m}}{\partial t_m^{l_m}} \quad \text{and} \quad |\mathbf{l}| = \sum_{i=1}^{m} \mathbf{l}_i.$$

$\mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ is the submodule of distributions in $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ with compact support.

Analogous definitions may be given for $\mathscr{S}'_{(r)}(\mathscr{R}^m; \mathscr{A})$ and $\mathscr{E}'_{(r)}(\mathscr{R}^m; \mathscr{A})$. For $T \in \mathscr{S}'_{(r)}(\mathscr{R}^m; \mathscr{A})$ and $\phi \in \mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$, $\langle T, \phi \rangle$ denotes the action of T on ϕ. Furthermore, for any $a \in \mathscr{A}$, $\langle T, \phi a \rangle = \langle T, \phi \rangle a$.

In the sequel, arbitrary elements of $\mathscr{R}^m$ and $\mathscr{R}^{m+1}$ will be denoted by

$$\mathbf{t} = (t_1, \ldots, t_m) \quad \text{and} \quad x = (x_0, \mathbf{x}) = \mathbf{x} + x_0 = (x_0, x_1, \ldots, x_m)$$

while $|\mathbf{t}|^2 = \sum_{j=1}^m t_i^2$ and $|x|^2 = \sum_{i=0}^m x_i^2$ stand for their respective Euclidean norms. Moreover, if $\mathbf{l} \in \mathscr{N}^m$ and $\alpha \in \mathscr{N}^{m+1}$, we put

$$\partial_{\mathbf{t}}^{\mathbf{l}} = \partial_{t_1}^{l_1} \cdots \partial_{t_m}^{l_m} \quad \text{and} \quad \partial_x^\alpha = \partial_{x_0}^{\alpha_0} \cdots \partial_{x_m}^{\alpha_m}.$$

Let us recall that the exponential function $E(\mathbf{t}, x)$ introduced in [12] is defined in the following way. For $(\mathbf{t}, \mathbf{x}) \in \mathscr{R}^m \times \mathscr{R}^m$ we put

$$E(\mathbf{t}, \mathbf{x}) = e^{t_1 x_1 e_1} \cdots e^{t_m x_m e_m} \quad \text{where } e^{t_j x_j e_j} = \cos t_j x_j + e_j \sin t_j x_j.$$

The exponential function $E(\mathbf{t}, x)$, $(\mathbf{t}, x) \in \mathscr{R}^m \times \mathscr{R}^{m+1}$, is the unique function which for any fixed $\mathbf{t} \in \mathscr{R}^m$ is left monogenic for $x \in \mathscr{R}^{m+1}$ and which for $x_0 = 0$ equals $E(\mathbf{t}, \mathbf{x})$ (see [12]). For $m = 1$,

$$E(\mathbf{t}, \mathbf{x}) = e^{t_1 x_1 e_1} = \cos t_1 x_1 + e_1 \sin t_1 x_1$$

and

$$E(\mathbf{t}, x) = e^{t_1 z_1 e_1} = e^{t_1 (x_1 - x_0 e_1) e_1}.$$

Hence, if we put $-e_1 = i$, $t_1 = t$, $x_1 = x$ and $x_0 = y$, we obtain

$$E(\mathbf{t}, \mathbf{x}) = \cos tx - i \sin tx = e^{-itx} \quad (t, x) \in \mathscr{R} \times \mathscr{R}$$

and

$$E(\mathbf{t}, x) = e^{-t(x+iy)i} = e^{-itz} \quad (t, z) \in \mathscr{R} \times \mathscr{C}.$$

Let us recall that the exponential function may be written in the following way:

$$(1) \qquad E(\mathbf{t}, x) = \sum_{(s_1, \ldots, s_m) \in \{0, 1\}^m} H_{s_1}(t_1 x_1) \cdots H_{s_m}(t_m x_m) L_{s_1 \cdots s_m}(\mathbf{t}, x_0)$$

where, for $j = 1, \ldots, m$,

$$H_{s_j}(t_j x_j) = \begin{cases} \cos t_j x_j, & s_j = 0 \\ \sin t_j x_j, & s_j = 1, \end{cases}$$

and where $L_{s_1, \ldots, s_m}(\mathbf{t}, x_0)$ is analytic for $(\mathbf{t}, x_0) \in \mathscr{R}^m \times \mathscr{R}$. Moreover, for any $\mathbf{t} \in \mathscr{R}^m$, $\mathbf{l} = (l_1, \ldots, l_m) \in \mathscr{N}^m$ and $(s_1, \ldots, s_m) \in \{0, 1\}^m$ fixed, there exists a positive constant $C_{s_1, \ldots, s_m, \mathbf{t}, \mathbf{l}}$ and $k \in \mathscr{N}$ such that

$$|\partial_{\mathbf{t}}^{\mathbf{l}} L_{s_1, \ldots, s_m}(\mathbf{t}, x_0)|_0 \le C_{s_1, \ldots, s_m, \mathbf{t}, \mathbf{l}} (1 + x_0^2)^k e^{|\mathbf{t}|\,|x_0|}.$$

On the other hand, $E(\mathbf{t}, x)$ may be decomposed into

$$(2) \qquad E(\mathbf{t}, x) = E_2(\mathbf{t}, x) - E_1(\mathbf{t}, x)$$

where

$$E_2(\mathbf{t}, x) = B(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}$$

and

$$E_1(\mathbf{t}, x) = A(\mathbf{t}, \mathbf{x})e^{|\mathbf{t}|x_0}$$

are analytic for $(\mathbf{t}, x) \in (\mathscr{R}^m \backslash \{0\}) \times \mathscr{R}^{m+1}$ and monogenic in $x \in \mathscr{R}^{m+1}$. In the complex case this decomposition is given by

$$e^{-itz} = \frac{1}{2}\left(1 + \frac{t}{|t|}\right)(\cos tx - i \sin tx)e^{|t|y}$$
$$+ \frac{1}{2}\left(1 - \frac{t}{|t|}\right)(\cos tx - i \sin tx)e^{-|t|y}$$

so that, in this case,

$$B(t, x) = \frac{1}{2}\left(1 - \frac{t}{|t|}\right)(\cos tx - i \sin tx)$$

and

$$A(t, x) = -\frac{1}{2}\left(1 + \frac{t}{|t|}\right)(\cos tx - i \sin tx).$$

In the general case it easily follows from the construction of $E(\mathbf{t}, x)$ in [12] that $A(\mathbf{t}, \mathbf{x})$ and $B(\mathbf{t}, \mathbf{x})$ are of the following form:

$$B(\mathbf{t}, \mathbf{x}) = \sum_{(s_1, \ldots, s_m) \in \{0, 1\}^m} H_{s_1}(t_1 x_1) \cdots H_{s_m}(t_m x_m) B_{s_1 \cdots s_m}(\mathbf{t}),$$

(3)

$$A(\mathbf{t}, \mathbf{x}) = \sum_{(s_1, \ldots, s_m) \in \{0, 1\}^m} H_{s_1}(t_1 x_1) \cdots H_{s_m}(t_m x_m) A_{s_1 \cdots s_m}(\mathbf{t})$$

where $B_{s_1 \cdots s_m}(\mathbf{t})$ and $A_{s_1 \cdots s_m}(\mathbf{t})$ can be written as $\mathscr{A}$-linear combinations of 1, $t_1/|\mathbf{t}|, \ldots, t_m/|\mathbf{t}|$.

As we shall see in section 4, the decomposition (2) plays an essential role in the definition of the Laplace transform. The function

$$\tilde{\mathrm{E}}(\mathbf{t}, \mathbf{x} + x_0) = \frac{1}{(2\pi)^m} \bar{E}(\mathbf{t}, \mathbf{x} - x_0)$$

is called the conjugate exponential function.

For every fixed $\mathbf{t} \in \mathscr{R}^m$, $\tilde{E}(\mathbf{t}, x)$ is right monogenic in $\mathscr{R}^{m+1}$ and its restriction to the hyperplane $x_0 = 0$ equals

$$\tilde{E}(\mathbf{t}, \mathbf{x}) = \frac{1}{(2\pi)^m} \bar{\mathrm{E}}(\mathbf{t}, \mathbf{x}) = \frac{1}{(2\pi)^m} e^{-t_m x_m e_m} \cdots e^{-t_1 x_1 e_1}.$$

If we put

$$\tilde{B}(\mathbf{t}, \mathbf{x}) = \frac{-1}{(2\pi)^m} \bar{A}(\mathbf{t}, \mathbf{x}) \quad \text{and} \quad \tilde{A}(\mathbf{t}, \mathbf{x}) = \frac{-1}{(2\pi)^m} \bar{B}(\mathbf{t}, \mathbf{x}),$$

it is easy to see that the conjugate exponential function admits the decomposition

$$\tilde{E}(\mathbf{t}, x) = \tilde{E}_2(\mathbf{t}, x) - \tilde{E}_1(\mathbf{t}, x)$$

where

$$\tilde{E}_2(\mathbf{t}, x) = \tilde{B}(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0} \quad \text{and} \quad \tilde{E}_1(\mathbf{t}, x) = \tilde{A}(\mathbf{t}, \mathbf{x})e^{|\mathbf{t}|x_0}$$

are analytic for $(\mathbf{t}, x) \in (\mathscr{R}^m\backslash\{0\}) \times \mathscr{R}^{m+1}$ and right monogenic in $x \in \mathscr{R}^{m+1}$.

Of course we may also obtain relations of the form (3) for the functions $\tilde{A}(\mathbf{t}, \mathbf{x})$ and $\tilde{B}(\mathbf{t}, \mathbf{x})$.

1. The Fourier transforms in $\mathscr{S}_{\{l\}}^{\{r\}}$ and $\mathscr{S}'^{\{r\}}_{\{l\}}$

In this section we study the hypercomplex version of the classical Fourier transform in $\mathscr{S}(\mathscr{R}^m)$ and $\mathscr{S}'(\mathscr{R}^m)$.

Let $\phi \in \mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$ and let $j \in \{1, \ldots, m\}$; then we define

$$\mathscr{F}_j\phi(t_1, \ldots, t_{j-1}, x_j, t_{j+1}, \ldots, t_m) = \int_{-\infty}^{+\infty} e^{t_j x_j e_j}\phi(\mathbf{t})\, dt_j$$

and

$$\mathscr{F}\phi(\mathbf{x}) = \int_{\mathscr{R}^m} E(\mathbf{t}, \mathbf{x})\phi(\mathbf{t})\, d\mathbf{t} = \mathscr{F}_1 \circ \cdots \circ \mathscr{F}_m\phi(\mathbf{x}).$$

$\mathscr{F}\phi$ is called the Fourier transform of ϕ. As in the classical theory we obtain:

THEOREM 1. *$\mathscr{F}_j$ is a topological automorphism of $\mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$. Moreover*

$$\mathscr{F}_j^{-1}\phi(t_1, \ldots, t_{j-1}, x_j, t_{j+1}, \ldots, t_m) = \frac{1}{2\pi}\int_{-\infty}^{-\infty} e^{-t_j x_j e_j}\phi(\mathbf{t})\, dt_j.$$

Proof. In view of the equation $e^{t_j x_j e_j} = \cos t_j x_j + (\sin t_j x_j)e_j$, for any real valued $\phi \in \mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$ the classical Fourier inversion formula can be written as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t_j x_j e_j}\mathscr{F}_j\phi\, dx_j = \mathscr{F}_j\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t_j x_j e_j}\phi\, dt_j\right) = \phi(\mathbf{t}).$$

As $\mathscr{F}_j$ and $\mathscr{F}_j^{-1}$ are right $\mathscr{A}$-linear, the above equality holds for any $\phi \in \mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$. Furthermore $\mathscr{F}_j$ and $\mathscr{F}_j^{-1}$ are continuous. ∎

As $\mathcal{F} = \mathcal{F}_1 \circ \cdots \circ \mathcal{F}_m$, $\mathcal{F}$ is a topological automorphism of $\mathcal{S}_{(r)}(\mathcal{R}^m; \mathcal{A})$ and

$$\mathcal{F}^{-1}\phi(\mathbf{x}) = \frac{1}{(2\pi)^m}\int_{\mathcal{R}^m} e^{-t_m x_m e_m} \cdots e^{-t_1 x_1 e_1}\phi(\mathbf{t})\, d\mathbf{t}.$$

In an analogous way, for any $\phi \in \mathcal{S}_{(1)}(\mathcal{R}^m; \mathcal{A})$, we put

$$\phi\mathcal{F}_j(t_1, \ldots, t_{j-1}, x_j, t_{j+1}, \ldots, t_m) = \int_{-\infty}^{+\infty} \phi(\mathbf{t})e^{t_j x_j e_j}\, dt_j, \quad j = 1, \ldots, m,$$

and

$$\phi\mathcal{F}(\mathbf{x}) = \phi(\mathcal{F}_m \circ \cdots \circ \mathcal{F}_1)(\mathbf{x}) = \int_{\mathcal{R}^m} \phi(\mathbf{t})E(\mathbf{t}, \mathbf{x})\, d\mathbf{t}.$$

So $\mathcal{F}$ is a topological automorphism of the left $\mathcal{A}$-module $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$.

Now consider a distribution $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ and define its Fourier transform $\mathcal{F}T$ by

$$\langle \mathcal{F}T, \phi\rangle = \langle T, \phi\mathcal{F}\rangle, \quad \phi \in \mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A}).$$

As $\mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ is the dual module of $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$, $\mathcal{F}$ is a topological automorphism of both $\mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})_{,s}$ and $\mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})_{,b}$, where s and b stand for the weak and strong topology. For $T \in \mathcal{S}'_{(r)}(\mathcal{R}^m; \mathcal{A})$ we put for any $\phi \in \mathcal{S}_{(r)}(\mathcal{R}^m; \mathcal{A})$,

$$\langle T\mathcal{F}, \phi\rangle = \langle T, \mathcal{F}\phi\rangle.$$

Remark. We want to mention some important calculation formulae for the Fourier transform introduced above. Hereby we make use of the reflection operators S_i, $i = 1, \ldots, m$, given by

$$S_i f(x_1, \ldots, x_i, \ldots, x_m) = f(x_1, \ldots, -x_i, \ldots, x_m)$$

where f stands for a function or an $\mathcal{A}$-distribution in $\mathcal{R}^m$.

(1) $\mathcal{F}\left(\frac{\partial}{\partial t_i} f\right)(\mathbf{x}) = -x_i e_i S_1 \cdots S_{i-1}\mathcal{F}(f)(\mathbf{x})$

(2) $\mathcal{F}(e_i f) = e_i S_1 \cdots S_{i-1} S_{i+1} \cdots S_m \mathcal{F}(f)$

(3) $\mathcal{F}\left(e_i \frac{\partial}{\partial t_i} f\right)(\mathbf{x}) = x_i S_{i+1} \cdots S_m \mathcal{F}(f)(\mathbf{x})$

(4) $\mathcal{F}(t_i f)(\mathbf{x}) = -e_i \frac{\partial}{\partial x_i} S_1 \cdots S_{i-1}\mathcal{F}(f)(\mathbf{x})$

(5) $\mathcal{F}(t_i e_i f)(\mathbf{x}) = \frac{\partial}{\partial x_i} S_{i+1} \cdots S_m \mathcal{F}(f)(\mathbf{x})$

(6) $\mathcal{F}(f(\mathbf{t} + \mathbf{a}))(\mathbf{x}) = e^{-a_1 x_1 e_1} e^{-a_2 x_2 e_2 S_1} \cdots e^{-a_m x_m e_m S_1 \cdots S_{m-1}} \mathcal{F}(f(\mathbf{t}))(\mathbf{x})$

where for $\lambda \in \mathcal{R}$, $e^{\lambda e_i S_1 \cdots S_{i-1}} = (\cos\lambda) + e_i(\sin\lambda)S_1 \cdots S_{i-1}$.

2. The space $\mathscr{L}_l(m; \mathscr{A})$

Take $\phi \in \mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$ and define

$$\mathscr{F}\phi(x) = \int_{\mathscr{R}^m} E(\mathbf{t}, x)\phi(\mathbf{t})\, dt.$$

Then clearly $\mathscr{F}\phi$ is left monogenic in $\mathscr{R}^{m+1}$. Moreover $\mathscr{F}\phi(x)$ is the unique left monogenic extension of the Fourier transform

$$\mathscr{F}\phi(\mathbf{x}) = \int_{\mathscr{R}^m} E(\mathbf{t}, \mathbf{x})\phi(\mathbf{t})\, dt.$$

Hence we obtain the inversion formula

$$\int_{\mathscr{R}^m} [\tilde{E}(\mathbf{t}, x)\ \mathscr{F}\phi(x)]_{x_0=0}\, d\mathbf{x} = \phi(\mathbf{t}).$$

We call $\mathscr{L}_l(m; \mathscr{A})$ the space of all functions $\mathscr{F}\phi$ where $\phi \in \mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$. Obviously $\mathscr{L}_l(m; \mathscr{A})$ is a right submodule of $M_1(\mathscr{R}^{m+1}; \mathscr{A})$. In the following theorem we prove an estimate for elements belonging to $\mathscr{L}_l(m; \mathscr{A})$.

THEOREM 2. *Let $f \in M_1(\mathscr{R}^{m+1}; \mathscr{A})$ be of the form*

$$f = \mathscr{F}\phi \quad \text{for some } \phi \in \mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$$

with supp $\phi \subseteq \bar{B}(0, R)$. *Then for any $\mathbf{l} \in \mathscr{R}^m$ and $\varepsilon > 0$, a positive constant $C_{\varepsilon,\mathbf{l}}$ may be found such that*

$$|x_1^{l_1} \cdots x_m^{l_m} f(x)|_0 \leq C_{\varepsilon,\mathbf{l}}\, e^{(\varepsilon+R)|x_0|}$$

for every $x \in \mathscr{R}^{m+1}$.

Proof. As $f = \mathscr{F}\phi$ for some $\phi \in \mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$ with supp $\phi \subseteq \bar{B}(0, R)$, for any $\mathbf{l} = (l_1, \ldots, l_m) \in \mathscr{N}^m$,

$$\begin{aligned}
|x_1^{l_1} \cdots x_m^{l_m} f(x)|_0 &\leq \left| \int_{\mathscr{R}^m} x_1^{l_1} \cdots x_m^{l_m} E(\mathbf{t}, x)\phi(\mathbf{t})\, dt \right|_0 \\
&\leq \sum_{(s_1, \ldots, s_m) \in \{0,1\}^m} \left| \int_{\mathscr{R}^m} x_1^{l_1} H_{s_1}(t_1 x_1) \cdots x_m^{l_m} H_{s_m}(t_m x_m) \right. \\
&\qquad \left. \times L_{s_1 \cdots s_m}(\mathbf{t}, x_0)\phi(\mathbf{t})\, dt \right|_0 \\
&= \sum_{(s_1, \ldots, s_m) \in \{0,1\}^m} \left| \int_{\mathscr{R}^m} \partial_{t_1}^{l_1} H_{s_1+l_1}(t_1 x_1) \cdots \partial_{t_m}^{l_m} H_{s_m+l_m}(t_m x_m) \right. \\
&\qquad \left. \times L_{s_1 \cdots s_m}(\mathbf{t}, x_0)\phi(\mathbf{t})\, dt \right|_0 .
\end{aligned}$$

A partial integration yields

$$\left| \int_{\mathscr{R}^m} \partial_{t_1}^{l_1} H_{s_1+l_1}(t_1 x_1) \cdots \partial_{t_m}^{l_m} H_{s_m+l_m}(t_m x_m) L_{s_1 \cdots s_m}(\mathbf{t}, x_0)\phi(\mathbf{t})\, d\mathbf{t} \right|_0$$

$$\leq \int_{\mathscr{R}^m} |\partial_{t_1}^{l_1} \cdots \partial_{t_m}^{l_m}(L_{s_1 \cdots s_m}(\mathbf{t}, x_0)\phi(\mathbf{t}))|_0\, d\mathbf{t}.$$

Consequently, as the support of ϕ is contained in $\bar{B}(0, R)$, we find—using Leibniz's formula and the above mentioned estimates for the derivatives of $L_{s_1 \cdots s_m}$—that

$$|x_1^{l_1} \cdots x_m^{l_m} f(x)|_0 \leq C(1 + x_0^2)^k e^{R|x_0|},$$

for some $C > 0$ and $k \in \mathscr{N}$. Hence given $\varepsilon > 0$, a constant $C_{\varepsilon, \mathbf{l}} > 0$ may be found such that

$$|x_1^{l_1} \cdots x_m^{l_m} f(x)|_0 \leq C_{\varepsilon, \mathbf{l}}\, e^{(R+\varepsilon)|x_0|}. \quad \blacksquare$$

Note that in virtue of Cauchy's representation theorem (see [7]), for any $\varepsilon > 0$, $\mathbf{l} \in \mathscr{N}^m$ and $\alpha \in \mathscr{N}^{m+1}$ there exists $C_{\varepsilon, \alpha, \mathbf{l}} > 0$ such that

$$|x_1^{l_1} \cdots x_m^{l_m}\, \partial_x^\alpha f(x)|_0 \leq C_{\varepsilon, \alpha, \mathbf{l}}\, e^{(R+\varepsilon)|x_0|}.$$

In the following theorem we prove that the element of $\mathscr{Z}_l(m; \mathscr{A})$ are completely determined by such estimates.

THEOREM 3. *Let $f \in M_1(\mathscr{R}^{m+1}; \mathscr{A})$ be such that for a certain $R > 0$ and for any $\mathbf{l} \in \mathscr{N}^m$, $\varepsilon > 0$ and $\alpha \in \mathscr{N}^{m+1}$, there exists $C_{\varepsilon, \alpha, \mathbf{l}} > 0$ such that*

$$|x_1^{l_1} \cdots x_m^{l_m}\, \partial_x^\alpha f(x)|_0 \leq C_{\varepsilon, \alpha, \mathbf{l}}\, e^{(R+\varepsilon)|x_0|}.$$

Then there exists $\phi \in \mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$ with supp $\phi \subseteq \bar{B}(0, R)$ *such that $T = \mathscr{F}\phi$.*

Proof. In view of the stated estimates $f(\mathbf{x}) = f(x)|_{x_0=0}$ belongs to $\mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$. Hence $f(\mathbf{x}) = \mathscr{F}\phi(\mathbf{x})$ for some $\phi \in \mathscr{S}_{(r)}(\mathscr{R}^m; \mathscr{A})$. We now prove that the support of ϕ is contained in $B(0, R + \varepsilon)$ for any $\varepsilon > 0$.

Choose $\varepsilon > 0$ and $\mathbf{t} \in \mathscr{R}^m \backslash \bar{B}(0, R + \varepsilon)$ arbitrarily. As $\phi = \mathscr{F}^{-1} f$, we obtain that

$$\phi(\mathbf{t}) = \int_{\mathscr{R}^m} \tilde{B}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x} - \int_{\mathscr{R}^m} \tilde{A}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x}.$$

As $\tilde{A}$ and $\tilde{B}$ are bounded for $\mathbf{x} \in \mathscr{R}^m$ and as for a certain constant $C > 0$,

$$|f(x)|_0 \leq \frac{C e^{(R+\varepsilon)|x_0|}}{1 + |\mathbf{x}|^{m+1}},$$

we obtain, by applying Cauchy's theorem (see [7]), that, for any $\delta > 0$,

$$\int_{\mathscr{R}^m} \tilde{A}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = \int_{\mathscr{R}^m} \tilde{A}(\mathbf{t}, \mathbf{x}) e^{-|\mathbf{t}|\delta} f(\mathbf{x} - \delta)\, d\mathbf{x}.$$

Hence, in view of the definition of $\tilde{A}(\mathbf{t}, \mathbf{x})$ and the relations (3), for any $\delta > 0$,

$$\left| \int_{\mathscr{R}^m} \tilde{A}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \right|_0 \leq C' e^{(R+\varepsilon-|\mathbf{t}|)\delta}$$

and, as $R + \varepsilon - |\mathbf{t}| < 0$, by taking the limit for $\delta \to +\infty$,

$$\int_{\mathscr{R}^m} \tilde{A}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = 0.$$

Analogously,

$$\int_{\mathscr{R}^m} \tilde{B}(\mathbf{t}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x} = 0$$

so that we have proved that $\phi(\mathbf{t}) = 0$.

Hence the support of ϕ is contained in $\bar{B}(0, R)$ and $\mathscr{F}\phi$ is a left monogenic extension of $f|_{\mathscr{R}^m} = \mathscr{F}\phi|_{\mathscr{R}^m}$. As this extension is unique (see [12]), $f = \mathscr{F}\phi$ in $\mathscr{R}^{m+1}$. ∎

Now we are able to construct a natural topology on $\mathscr{Z}_l(m; \mathscr{A})$. Let $k, s \in \mathscr{N}$ and call $\mathscr{Z}_{l,k,s}$ the right Fréchet $\mathscr{A}$-module consisting of those left monogenic functions in $\mathscr{R}^{m+1}$ such that, given $\mathbf{l} \in \mathscr{N}^m$ and $\alpha \in \mathscr{N}^{m+1}$, a constant $C_{\alpha,\mathbf{l}} > 0$ may be found such that

$$|x_1^{l_1} \cdots x_m^{l_m} \partial_x^\alpha f(x)|_0 \leq C_{\alpha,\mathbf{l}} e^{(k+1/s)|x_0|}.$$

Then a locally convex topology may be defined on $\mathscr{Z}_l(m; \mathscr{A})$ by putting

$$\mathscr{Z}_l(m; \mathscr{A}) = \lim_{k \in \mathscr{N}} \text{ind} \lim_{s \in \mathscr{N}} \text{proj}\, \mathscr{Z}_{l,k,s}.$$

Note that $\mathscr{Z}_l(m; \mathscr{A})$ is an inductive limit of right Fréchet $\mathscr{A}$-modules.

We now state the topological result:

THEOREM 4. *$\mathscr{D}_{(r)}(\mathscr{R}^m; \mathscr{A})$ is topologically isomorphic to $\mathscr{Z}_l(m; \mathscr{A})$.*

Proof. Making use of Theorem 1 and Theorem 3, one can prove this theorem analogously to the case of one complex variable (see [1] for example). ∎

Remarks. (i) Denote by $\mathscr{Z}'_l(m; \mathscr{A})$ the left $\mathscr{A}$-module of bounded right $\mathscr{A}$-linear functionals on $\mathscr{Z}_l(m; \mathscr{A})$. In view of the previous theorem $\mathscr{Z}'_l(m; \mathscr{A})$ and $\mathscr{D}'_{(r)}(\mathscr{R}^m; \mathscr{A})$ are topologically isomorphic spaces.

(ii) If $\phi \in \mathscr{D}_{(l)}(\mathscr{R}^m; \mathscr{A})$ then we may define

$$\phi\mathscr{F}^{-1}(x) = \int_{\mathscr{R}^m} \phi(\mathbf{t}) \tilde{E}(\mathbf{t}, x)\, d\mathbf{t}.$$

Then $\phi\mathscr{F}^{-1} \in M_1^{(r)}(\mathscr{R}^m; \mathscr{A})$.

Let $\mathscr{Z}_r(m; \mathscr{A})$ be the set of the functions $\phi\mathscr{F}^{-1}$, $\phi \in \mathscr{D}_{(l)}(\mathscr{R}^m; \mathscr{A})$; then this space may be characterized in the same way as $\mathscr{Z}_l(m; \mathscr{A})$. Moreover a locally convex topology may be defined on it such that $\mathscr{Z}_r(m; \mathscr{A})$ is topologically isomorphic to $\mathscr{D}_{(l)}(\mathscr{R}^m; \mathscr{A})$. Its dual module $\mathscr{Z}'_r(m; \mathscr{A})$ is then topologically isomorphic to $\mathscr{D}'_{(l)}(\mathscr{R}^m; \mathscr{A})$.

3. The generalized Fourier transform in $\mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$

Let $T \in \mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$. Then $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ and hence $\mathscr{F}T$ is defined. On the other hand, one may consider the function

$$\tilde{T}(\mathbf{x}) = \langle T_{\mathbf{t}}, E(\mathbf{t}, \mathbf{x})\rangle,$$

and it is easily proved that for any $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$

$$\langle \mathscr{F}T, \phi\rangle = \int_{\mathscr{R}^m} \phi(\mathbf{x})\tilde{T}(\mathbf{x})\, d\mathbf{x}.$$

Hence it is natural to define the generalized Fourier transform of T by

$$\mathscr{F}T(x) = \langle T_{\mathbf{t}}, E(\mathbf{t}, x)\rangle,$$

As T is left $\mathscr{A}$-linear and bounded on $\mathscr{E}_{(l)}(\mathscr{R}^m; \mathscr{A})$ and as $E(\mathbf{t}, x)$ is analytic in $\mathscr{R}^m \times \mathscr{R}^{m+1}$, one can easily show that $\mathscr{F}T(x) \in C_1(\mathscr{R}^{m+1}; \mathscr{A})$ and that

$$D\mathscr{F}T(x) = \langle T_{\mathbf{t}}, D_x E(\mathbf{t}, x)\rangle = 0 \quad \text{in } \mathscr{R}^{m+1}.$$

Hence $\mathscr{F}T(x)$ is in fact the unique left monogenic function in $\mathscr{R}^{m+1}$ such that

$$\mathscr{F}T(x)\big|_{x_0=0} = \tilde{T}(\mathbf{x}).$$

The two following theorems may be considered as the hypercomplex analogues of the well known Paley-Wiener-Schwartz theorems (see [9] and [10] for example).

THEOREM 5. *Let* $T \in \mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ *and* $R > 0$ *such that* supp $T \subseteq \mathring{B}(0, R)$. *Then for some* $k \in \mathscr{N}$ *and* $C > 0$

$$|\mathscr{F}T(x)|_0 \leq C(1 + |x|^2)^k e^{R|x_0|}.$$

Proof. The desired inequality follows immediately from the definition of $\mathscr{F}T$. ∎

Now we prove that such estimates determine completely the Fourier transform of elements in $\mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$.

THEOREM 6. *Let* $f \in M_1(\mathscr{R}^{m+1}; \mathscr{A})$ *and* $R > 0$ *such that for some* $C > 0$ *and* $k \in \mathscr{N}$,

$$|f(x)|_0 \leq C(1 + |x|^2)^k e^{R|x_0|}.$$

Then $f = \mathscr{F}T$ *for some* $T \in \mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ *with* supp $T \subseteq \bar{B}(0, R)$.

Proof. As $f(\mathbf{x}) = f(x)|_{x_0=0} \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$, $\mathscr{F}^{-1}(f(\mathbf{x})) = T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$. Now we claim that the support of T is contained in $\bar{B}(0, R)$. Choose $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ such that supp $\phi \subseteq \mathscr{R}^m \backslash \bar{B}(0, R)$. Then anticipating the results stated in Section 4, Lemma 1 and Lemma 2, $\phi\mathscr{F}^{-1}$ is the $\mathscr{S}$-boundary value of some $h \in M_1^{(r)}(\mathscr{R}^{m+1} \backslash \mathscr{R}^m; \mathscr{A})$ satisfying

$$|x_1^{l_1} \cdots x_m^{l_m} \partial_x^\alpha h(x)|_0 \leq C_{\alpha, \mathbf{l}} e^{-R'|x_0|}$$

for every $x \in \mathscr{R}^{m+1} \backslash \mathscr{R}^m$, for some $R' > R$. Hence

$$\langle T, \phi \rangle = \langle \mathscr{F} T, \phi\mathscr{F}^{-1} \rangle = \lim_{x_0 \to 0+} \int_{\mathscr{R}^m} (h(\mathbf{x} + x_0) - h(\mathbf{x} - x_0)) f(\mathbf{x})\, d\mathbf{x}.$$

In view of Cauchy's theorem (see [7]), for any $\delta > 0$

$$\int_{\mathscr{R}^m} h(\mathbf{x} + x_0) f(\mathbf{x})\, d\mathbf{x} = \int_{\mathscr{R}^m} h(\mathbf{x} + x_0 + \delta) f(\mathbf{x} + \delta)\, d\mathbf{x}$$

and as

$$\int_{\mathscr{R}^m} h(\mathbf{x} + x_0 + \delta) f(\mathbf{x} + \delta)\, d\mathbf{x} \to 0$$

whenever $\delta \to +\infty$, (which immediately follows from the above estimates for f and h),

$$\int_{\mathscr{R}^m} h(\mathbf{x} + x_0) f(\mathbf{x})\, d\mathbf{x} = 0.$$

Analogously,

$$\int_{\mathscr{R}^m} h(\mathbf{x} - x_0) f(\mathbf{x})\, d\mathbf{x} = 0,$$

so that $\langle T, \phi \rangle = 0$. Hence T has its support contained in $B(0, R)$. Finally as $\mathscr{F} T|_{x_0=0} = f|_{x_0=0}$ and since both functions are left monogenic in $\mathscr{R}^{m+1}$ we have that $f = \mathscr{F} T$. ∎

Let $k \in \mathscr{N}$ and $R > 0$ and call $\chi_{l,k,R}$ the space of all left monogenic functions in $\mathscr{R}^{m+1}$ satisfying an estimate of the form

$$|f(x)|_0 \leq C(1 + |x|^2)^k e^{R|x_0|}.$$

Then clearly $\chi_{l,k,R}$ is a right Banach $\mathscr{A}$-module. Letting

$$\chi_l(m; \mathscr{A}) = \lim_{k, R} \text{ind}\, \chi_{l,k,R},$$

we obtain the following result.

THEOREM 7. *The mapping* $\mathscr{F}: \mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})_b \to \chi_l(m; \mathscr{A})$ *is a topological isomorphism.*

Proof. Using Theorem 1 and Theorem 6, the proof runs analogously to the case of one complex variable. ∎

4. The Laplace transform in $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$

In this section we show that the Fourier transform of an element $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ is the distributional boundary value of a special left monogenic function in $\mathscr{R}^{m+1}\backslash\mathscr{R}^m$, called the Laplace transform $\mathscr{L}T$ of T. In this way we construct a new class of representing functions for $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ (see also [11]).

Let $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ with $\phi = 0$ in $\bar{B}(0, R)$; then we know that $\phi\mathscr{F}^{-1}$ exists and belongs to $\mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$.

Now we construct a right monogenic function in $\mathscr{R}^{m+1}\backslash\mathscr{R}^m$ which admits $\phi\mathscr{F}^{-1}$ as $\mathscr{S}$-boundary value. Let

$$\mathscr{R}^{m+1}_+ = \{x \in \mathscr{R}^{m+1}: x_0 > 0\}, \mathscr{R}^{m+1}_- = \{x \in \mathscr{R}^{m+1}: x_0 < 0\}$$

and, for any $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ such that supp $\phi \subseteq \mathscr{R}^m\backslash\bar{B}(0, R)$, define

$$\phi\mathscr{L}^{-1}(x) = \begin{cases} \displaystyle\int_{\mathscr{R}^m} \phi(\mathbf{t})\tilde{B}(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, dt & \text{if } x \in \mathscr{R}^{m+1}_+, \\ \displaystyle\int_{\mathscr{R}^m} \phi(\mathbf{t})\tilde{A}(\mathbf{t}, \mathbf{x})e^{|\mathbf{t}|x_0}\, dt & \text{if } x \in \mathscr{R}^{m+1}_-. \end{cases}$$

LEMMA 1. *Let $\phi \in \mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$ be equal to zero in $\bar{B}(0, R)$. Then, for any $\alpha \in \mathscr{N}^{m+1}$, $\mathbf{l} \in \mathscr{N}^m$ and $0 < \varepsilon < R$, there exists $C_{\alpha,\varepsilon,\mathbf{l}} > 0$ such that, in $\mathscr{R}^{m+1}\backslash\mathscr{R}^m$,*

$$|x_1^{l_1} \cdots x_m^{l_m}\, \partial_x^\alpha \phi\mathscr{L}^{-1}(x)|_0 \le C_{\alpha,\varepsilon,\mathbf{l}}\, e^{(\varepsilon - R)|x_0|}.$$

Proof. In view of the relations (3), there exist analytic functions $\tilde{\beta}_{s_1 \cdots s_m}$ in $\mathscr{R}^m\backslash\{0\}$, which are of polynomial growth when $|\mathbf{t}| \to \infty$, such that

$$\partial_x^\alpha \tilde{B}(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0} = \sum_{(s_1, \ldots, s_m) \in \{0,1\}^m} H_{s_1}(t_1x_1) \cdots H_{s_m}(t_mx_m)\tilde{\beta}_{s_1 \cdots s_m}(\mathbf{t})e^{-|\mathbf{t}|x_0}.$$

Hence, we obtain by partial integration that for any $\mathbf{l} \in \mathscr{N}^m$ and $0 < \varepsilon < R$

$$\begin{aligned} &|x_1^{l_1} \cdots x_m^{l_m}\, \partial_x^\alpha \phi\mathscr{L}^{-1}(x)|_0 \\ &\quad\le \sum_{(s_1, \ldots, s_m) \in \{0,1\}^m} \int_{\mathscr{R}^m\backslash B(0,R)} |\partial_{t_1}^{l_1} \cdots t_m^{l_m}(B_{s_1 \cdots s_m}\phi e^{-|\cdot|x_0})(\mathbf{t})|_0\, dt \\ &\quad\le C_{\alpha,\varepsilon,\mathbf{l}}\, e^{(\varepsilon - R)|x_0|}, \end{aligned}$$

in $\mathscr{R}^{m+1}_+$. An analogous estimate holds in $\mathscr{R}^{m+1}_-$. ∎

LEMMA 2. *The boundary values $\lim_{x_0 \to 0+} \phi\mathscr{L}^{-1}(\mathbf{x} \pm x_0)$ exist in $\mathscr{S}_{(l)}(\mathscr{R}^m; \mathscr{A})$. Moreover*

$$\phi\mathscr{F}^{-1}(\mathbf{x}) = \lim_{x_0 \to 0+} (\phi\mathscr{L}^{-1}(\mathbf{x} + x_0) - \phi\mathscr{L}^{-1}(\mathbf{x} - x_0)).$$

Proof. In view of the previous lemma, for each $\alpha \in \mathcal{N}^{m+1}$ and $\mathbf{l} \in \mathcal{N}^m$ there exists $C_{\mathbf{l},\alpha} > 0$ such that for $x_0 \in]0, 1]$,

$$|x_1^{l_1} \cdots x_m^{l_m} \partial_x^\alpha \phi \mathcal{L}^{-1}(x)|_0 \leq C_{\mathbf{l},\alpha}.$$

Hence for each $\boldsymbol{\alpha} \in \mathcal{N}^m$ and $x_{0,1}, x_{0,2} \in]0, 1]$,

$$\begin{aligned} |x_1^{l_1} \cdots x_m^{l_m} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\phi\mathcal{L}^{-1}(\mathbf{x} + x_{0,1}) - \phi\mathcal{L}^{-1}(\mathbf{x} + x_{0,2}))|_0 \\ = \left| \int_{x_{0,2}}^{x_{0,1}} x_1^{l_1} \cdots x_m^{l_m} \frac{\partial}{\partial s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\phi\mathcal{L}^{-1}(\mathbf{x} + s))\, ds \right| \\ \leq C_{\mathbf{l},(1,\boldsymbol{\alpha})} |x_{0,1} - x_{0,2}|, \end{aligned}$$

which implies that $(\phi\mathcal{L}^{-1}(\mathbf{x} + x_0))_{x_0 \in]0,1]}$ is a Cauchy-net in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$ and hence that

$$\lim_{x_0 \to 0+} \phi\mathcal{L}^{-1}(\mathbf{x} + x_0)$$

exists in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$. Analogously

$$\lim_{x_0 \to 0+} \phi\mathcal{L}^{-1}(\mathbf{x} - x_0)$$

exists in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$. Furthermore

$$\phi\mathcal{L}^{-1}(\mathbf{x} + x_0) - \phi\mathcal{L}^{-1}(\mathbf{x} - x_0) = \int_{\mathcal{R}^m} \phi(\mathbf{t})\tilde{E}(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, dt$$

converges to $\phi\mathcal{F}^{-1}(\mathbf{x})$ for $x_0 \to 0+$. ∎

A converse of Lemma 1 runs as follows.

LEMMA 3. *Let $f \in M_1^{(r)}(\mathcal{R}^{m+1}\backslash\mathcal{R}^m; \mathcal{A})$ be such that there exists $R > 0$ for which, given any $\alpha \in \mathcal{N}^{m+1}$, $\mathbf{l} \in \mathcal{N}^m$ and $0 < \varepsilon < R$, a constant $C_{\alpha,\varepsilon,\mathbf{l}} > 0$ may be found such that*

$$|x_1^{l_1} \cdots x_m^{l_m} \partial_x^\alpha f(x)|_0 \leq C_{\alpha,\varepsilon,\mathbf{l}} e^{(\varepsilon - R)|x_0|}.$$

Then $f = \phi\mathcal{L}^{-1}$ for some $\phi \in \mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$ with $\phi = 0$ in $\bar{B}(0, R)$.

Proof. From the given estimates and the proofs of Lemma 2, it follows that the boundary value $f(\mathbf{x} + 0) - f(\mathbf{x} - 0)$ of f exists in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$. Let $\phi(\mathbf{t}) = (f(\mathbf{x} + 0) - f(\mathbf{x} - 0))\mathcal{F}$; then ϕ belongs to $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$. Using Cauchy's theorem (see [7]) one easily proves that $(f(\mathbf{x} \pm x_0)\mathcal{F})(\mathbf{t}) = 0$ for $\mathbf{t} \in \mathring{B}(0, R)$ which implies that $\phi = 0$ in $\bar{B}(0, R)$. Hence, as both f and $\phi\mathcal{L}^{-1}$ have the same boundary value and satisfy estimates of the above type, using Liouville's theorem (see [6]) we obtain $f = \phi\mathcal{L}^{-1}$. ∎

In the following theorems we consider distributions in $\mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ which are equal to zero in $\mathring{B}(0, R)$.

Let $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ be zero in $\mathring{B}(0, R)$ and choose a real-valued C_∞-function $\alpha_\varepsilon(\mathbf{t})$ depending on $\varepsilon \in \,]0, R[$ such that

$$\alpha_\varepsilon(\mathbf{t}) = \begin{cases} 0 & \text{if } \mathbf{t} \in \bar{B}(0, R-\varepsilon) \\ 1 & \text{if } \mathbf{t} \in \mathcal{R}^m \backslash \mathring{B}(0, R-\varepsilon/2) \end{cases}.$$

Then for any $\phi \in \mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$,

$$\langle T, \phi \rangle = \langle T, \alpha_\varepsilon \phi \rangle.$$

Furthermore the functions

$$A_\varepsilon(\mathbf{t}, \mathbf{x}) = \alpha_\varepsilon(\mathbf{t}) A(\mathbf{t}, \mathbf{x}), \qquad B_\varepsilon(\mathbf{t}, \mathbf{x}) = \alpha_\varepsilon(\mathbf{t}) B(\mathbf{t}, \mathbf{x}),$$

and their $\mathbf{t}$-derivatives, are C_∞-functions of polynomial growth in $\mathcal{R}^m \times \mathcal{R}^m$. Let

$$E_{2,\varepsilon}(\mathbf{t}, x) = B_\varepsilon(\mathbf{t}, \mathbf{x}) e^{-|\mathbf{t}| x_0}, \quad x \in \mathcal{R}^{m+1}_+,$$

and

$$E_{1,\varepsilon}(\mathbf{t}, x) = A_\varepsilon(\mathbf{t}, \mathbf{x}) e^{|\mathbf{t}| x_0}, \quad x \in \mathcal{R}^{m+1}_-.$$

Then for x fixed, $E_{1,\varepsilon}(\mathbf{t}, x)$ and $E_{2,\varepsilon}(\mathbf{t}, x)$ belong to $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$. Now we define the Laplace transform $\mathcal{L}T$ of T by

$$\mathcal{L}T(x) = \begin{cases} \langle T_{\mathbf{t}}, E_{2,\varepsilon}(\mathbf{t}, x) \rangle, & x \in \mathcal{R}^{m+1}_+ \\ \langle T_{\mathbf{t}}, E_{1,\varepsilon}(\mathbf{t}, x) \rangle, & x \in \mathcal{R}^{m+1}_- \end{cases}.$$

One may prove that $\mathcal{L}T$ is left monogenic in $\mathcal{R}^{m+1} \backslash \mathcal{R}^m$ and that its definition does not depend on ε.

In the following theorem an estimate for $\mathcal{L}T$ is given.

THEOREM 8. *Let $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ be zero in $\mathring{B}(0, R)$ and let $\varepsilon \in \,]0, R[$. Then there exist $k, r \in \mathcal{N}$ and $C_\varepsilon > 0$ such that, in $\mathcal{R}^{m+1} \backslash \mathcal{R}^m$,*

$$|\mathcal{L}T(x)|_0 \leq C_\varepsilon \left(1 + \frac{1}{|x_0|^k}\right)(1 + |x|^2)^r e^{-(R-\varepsilon)|x_0|}.$$

Proof. As $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ there exist a continuous function g, $\mathbf{l} \in \mathcal{N}^m$ and $s \in \mathcal{N}$ such that $|g(\mathbf{t})|_0 \leq C(1 + |\mathbf{t}|^2)^s$ and $T = \partial^{\mathbf{l}}_{\mathbf{t}} g$. Consequently for any $\phi \in \mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$,

$$\langle T, \phi \rangle = (-1)^{|\mathbf{l}|} \langle g, \partial^{\mathbf{l}}_{\mathbf{t}}(g(\mathbf{t})) \rangle$$

whereby, in $\mathcal{R}^{m+1}_+$,

$$\mathcal{L}T(x) = (-1)^{|\mathbf{l}|} \int_{\mathcal{R}^m \backslash B(0, R-\varepsilon)} \partial^{\mathbf{l}}_{\mathbf{t}}(B_\varepsilon(\mathbf{t}, \mathbf{x}) e^{-|\mathbf{t}| x_0}) \phi(\mathbf{t}) \, d\mathbf{t}.$$

Using Leibniz's formula and the equations (3) we have that for some $C' > 0$, $r \in \mathcal{N}$,

$$|\partial_{\mathbf{t}}^{\mathbf{l}}(B_\varepsilon(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0})|_0 \leq C'(1 + |x|^2)^r e^{-|\mathbf{t}|x_0}.$$

Hence, letting $C'' = CC'$,

$$|\mathcal{L}T(x)|_0 \leq C''(1 + |x|^2)^r \int_{\mathcal{R}^m \setminus B(0, R-\varepsilon)} (1 + |\mathbf{t}|^2)^s e^{-|\mathbf{t}|x_0}\, d\mathbf{t}$$

and, as

$$\int_{\mathcal{R}^m \setminus B(0, R-\varepsilon)} (1 + |\mathbf{t}|^2)^s e^{-|\mathbf{t}|x_0}\, d\mathbf{t} \leq Cte\left(1 + \frac{1}{|x_0|^{2s+m}}\right) e^{(\varepsilon - R)|x_0|},$$

we may find $C_\varepsilon > 0$ such that

$$|\mathcal{L}T(x)|_0 \leq C_\varepsilon\left(1 + \frac{1}{|x_0|^{2s+m}}\right)(1 + |x|^2)^r e^{(\varepsilon - R)|x_0|}.$$

An analogous estimate may be obtained for $x \in \mathcal{R}_-^{m+1}$. ∎

Now we prove that $\mathcal{F}T$ is the boundary value of $\mathcal{L}T$.

THEOREM 9. *Let $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ be zero in $\mathring{B}(0, R)$. Then for any $\phi \in \mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$,*

$$\langle \mathcal{F}T, \phi \rangle = \lim_{x_0 \to 0+} \int_{\mathcal{R}^m} \phi(\mathbf{x})(\mathcal{L}T(\mathbf{x} + x_0) - \mathcal{L}T(\mathbf{x} - x_0))\, d\mathbf{x}.$$

Proof. For $x_0 > 0$ fixed, $\phi(\mathbf{x})\mathcal{L}T(\mathbf{x} + x_0)$ belongs to $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$ and hence, using an approximation by Riemann sums,

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})\mathcal{L}T(\mathbf{x} + x_0)\, d\mathbf{x}$$

$$= \lim_{N \to \infty} \left\langle T_{\mathbf{t}}, \sum_{\nu=0}^{N} \phi(\mathbf{x}_{\nu, N})B_\varepsilon(\mathbf{t}, \mathbf{x}_{\nu, N})e^{-|\mathbf{t}|x_0}\mu(K_{\nu, N})\right\rangle$$

where μ denotes Lebesgue measure.

Observe that $\sum_{\nu=0}^{N} \phi(\mathbf{x}_{\nu, N})B_\varepsilon(\mathbf{t}, \mathbf{x}_{\nu, N})\mu(K_{\nu, N})$ and all its $\mathbf{t}$-derivatives are uniformly bounded with respect to N and $\mathbf{t}$ and that this sequence converges in C_∞ to $\int_{\mathcal{R}^m} \phi(\mathbf{x})B_\varepsilon(\mathbf{t}, \mathbf{x})\, d\mathbf{x}$. Hence

$$\sum_{\nu=0}^{N} \phi(\mathbf{x}_{\nu, N})B_\varepsilon(\mathbf{t}, \mathbf{x}_{\nu, N})e^{-|\mathbf{t}|x_0}\mu(K_{\nu, N})$$

converges to

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})B_\varepsilon(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, d\mathbf{x}$$

in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$, which implies that

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})\mathcal{L}T(\mathbf{x}+x_0) = \left\langle T_{\mathbf{t}}, \int_{\mathcal{R}^m} \phi(\mathbf{x})B_\varepsilon(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, d\mathbf{x}\right\rangle.$$

Furthermore,

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})B_\varepsilon(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, d\mathbf{x}$$

converges to

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})B_\varepsilon(\mathbf{t}, \mathbf{x})\, d\mathbf{x}$$

in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$ if $x_0 \to 0+$. Analogously,

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})A_\varepsilon(\mathbf{t}, \mathbf{x})e^{-|\mathbf{t}|x_0}\, d\mathbf{x}$$

converges to

$$\int_{\mathcal{R}^m} \phi(\mathbf{x})A_\varepsilon(\mathbf{t}, \mathbf{x})\, d\mathbf{x}$$

in $\mathcal{S}_{(l)}(\mathcal{R}^m; \mathcal{A})$ if $x_0 \to 0+$. Hence, as $E(\mathbf{t}, \mathbf{x})\alpha_\varepsilon(\mathbf{t}) = B_\varepsilon(\mathbf{t}, \mathbf{x}) - A_\varepsilon(\mathbf{t}, \mathbf{x})$,

$$\lim_{x_0 \to 0+} \int_{\mathcal{R}^m} \phi(\mathbf{x})(\mathcal{L}T(\mathbf{x}+x_0) - \mathcal{L}T(\mathbf{x}-x_0))\, d\mathbf{x}$$

$$= \left\langle T, \alpha_\varepsilon(\mathbf{t}) \int_{\mathcal{R}^m} \phi(\mathbf{x})E(\mathbf{t}, \mathbf{x})\, d\mathbf{x}\right\rangle = \langle \mathcal{F}T, \phi\rangle. \quad \blacksquare$$

Now we state the converse of Theorem 9.

THEOREM 10. *Let $f \in M_1(\mathcal{R}^{m+1}\backslash\mathcal{R}^m; \mathcal{A})$ be such that there exists $R > 0$ for which, given any $0 < \varepsilon < R$, $k, r \in \mathcal{N}$, $C_\varepsilon > 0$ may be found such that, in $\mathcal{R}^{m+1}\backslash\mathcal{R}^m$,*

$$|f(x)|_0 \leq C_\varepsilon\left(1 + \frac{1}{|x_0|^k}\right)(1 + |x|^2)^r e^{(\varepsilon - R)|x_0|}.$$

Then there exists $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$ such that $T = 0$ in $\mathring{B}(0, R)$ and $f = \mathcal{L}T$.

Proof. As

$$|f(x)|_0 \leq C\left(1 + \frac{1}{|x_0|^k}\right)(1 + |\mathbf{x}|^2)^r \quad \text{for } x_0 \in]-1, 0[\cup]0, 1[,$$

f admits an $\mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$-boundary value for $x_0 \to 0\pm$ (see [11]). Hence $f(\mathbf{x}+x_0) - f(\mathbf{x}-x_0)$ tends to $\mathcal{F}T$ for a certain $T \in \mathcal{S}'_{(l)}(\mathcal{R}^m; \mathcal{A})$. We prove that $T_{|\mathring{B}(0,R)} = 0$. Choose a testfunction ϕ with support in $\mathring{B}(0, R)$ and take $\varepsilon > 0$ sufficiently small such that supp $\phi \subseteq B(0, R - 2\varepsilon)$. As $\phi\mathscr{F}^{-1} \in \mathscr{Z}_r(m; \mathscr{A})$, there exists $C > 0$ such that

$$(1 + |x|^2)^{r+m+1}\phi\mathscr{F}^{-1}(x)|_0 \leq Ce^{(R - 3/2\varepsilon)|x_0|}.$$

Hence

$$\langle T, \phi\rangle = \lim_{x_0 \to 0+} \int_{\mathscr{R}^m} \phi\mathscr{F}^{-1}(\mathbf{x})(f(\mathbf{x} + x_0) - f(\mathbf{x} - x_0))\, d\mathbf{x},$$

and in view of the given estimate and Cauchy's theorem (see [7]), for any $\delta > 0$,

$$\left|\int_{\mathscr{R}^m} \phi\mathscr{F}^{-1}(\mathbf{x})f(\mathbf{x} + x_0)\, d\mathbf{x}\right|_0 = \left|\int_{\mathscr{R}^m} \phi\mathscr{F}^{-1}(\mathbf{x} + \delta)f(\mathbf{x} + x_0 + \delta)\, d\mathbf{x}\right|_0$$

$$\leq C'\left(\int_{\mathscr{R}^m} \frac{(1 + |x + \delta|^2)^r}{(1 + |\mathbf{x} + \delta|^2)^{r+m+1}}\, d\mathbf{x}\right)e^{-1/2\varepsilon(x_0+\delta)}.$$

Consequently, letting $\delta \to \infty$, for any $x_0 > 0$,

$$\int_{\mathscr{R}^m} \phi\mathscr{F}^{-1}(\mathbf{x})f(x_0 + \mathbf{x})\, d\mathbf{x} = 0.$$

Analogously, for any $x_0 > 0$,

$$\int_{\mathscr{R}^m} \phi\mathscr{F}^{-1}(\mathbf{x})f(\mathbf{x} - x_0)\, d\mathbf{x} = 0$$

so that $T = 0$ in $\mathring{B}(0, R)$.

Consider $\mathscr{L}T$; then as $\mathscr{L}T$ and f have the same distributional boundary value, $\mathscr{L}T - f$ is left monogenic in $\mathscr{R}^{m+1}$ (see [11]). Furthermore, as $\mathscr{L}T - f \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$, it is a polynomial (see [11]) and, as both $\mathscr{L}T$ and f satisfy an estimate of the given form, $\mathscr{L}T(\mathbf{x} + x_0) - f(\mathbf{x} + x_0) \to 0$ if $|x_0| \to \infty$. Hence $f = \mathscr{L}T$. ∎

Now, let

$$T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A}), \quad R > 0, 0 < \varepsilon < R$$

be given. Then $T = T_1 + T_2$ where $T_1 \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$, $T_1 = 0$ in $\mathring{B}(0, R - \varepsilon)$ and $T_2 \in \mathscr{E}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ has its support in $\mathring{B}(0, R - \varepsilon/2)$.

As we know from the foregoing, $\mathscr{F}T_1$ is the $\mathscr{S}'$-boundary value of a certain $f \in M_1(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})$ satisfying

$$|f(x)|_0 \leq C\left(1 + \frac{1}{|x_0|^r}\right)(1 + |x|^2)^k e^{(\delta+\varepsilon-R)|x_0|},$$

and $\mathscr{F}T_2(\mathbf{x})$ is the $\mathscr{S}'$-boundary value of $g \in M_1(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})$ defined by

$$g(x) = \begin{cases} \tfrac{1}{2}\mathscr{F}T_2(x) & \text{if } x \in \mathscr{R}^{m+1}_+ \\ -\tfrac{1}{2}\mathscr{F}T_2(x) & \text{if } x \in \mathscr{R}^{m+1}_- \end{cases}$$

and which satisfies

$$|g(x)|_0 \leq C(1 + |x|^2)^k e^{R|x_0|}.$$

Hence $\mathscr{F}T$ is the boundary value of $\mathscr{L}T = f + g$ satisfying an inequality of the form

$$(*) \qquad |\mathscr{L}T(x)|_0 \leq C\left(1 + \frac{1}{|x_0|^r}\right)(1 + |x|^2)^k e^{R|x_0|}.$$

Conversely when $h \in M_1(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})$ satisfies an estimate of the type $(*)$, then the boundary value of h exists in $\mathscr{S}'$ and hence equals $\mathscr{F}T$ for some $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ (see [11]). In view of [11] one easily shows that $h - \mathscr{L}T$ is an entire monogenic function satisfying

$$(**) \qquad |h - \mathscr{L}T(x)|_0 \leq C(1 + |x|^2)e^{R|x_0|}.$$

Now let $R > 0$ be given and call $M_{1,R}(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})$ the space of monogenic functions in $\mathscr{R}^{m+1}\backslash\mathscr{R}^m$ satisfying $(*)$, and $M_{1,R}(\mathscr{R}^{m+1}; \mathscr{A})$ the space of entire monogenic functions satisfying $(**)$. Then, in view of the above considerations and [11], one obtains the following theorem.

THEOREM 11. (a) *$\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ and $M_{1,R}(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})/M_{1,R}(\mathscr{R}^{m+1}, \mathscr{A})$ are isomorphic right $\mathscr{A}$-modules.*

(b) *The boundary value mapping from $M_{1,R}(\mathscr{R}^{m+1}\backslash\mathscr{R}^m; \mathscr{A})$ to $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$ is bounded and open.*

Remarks. The previous theory leads to the following decomposition.

Let $R > 0$ and let $T \in \mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$, $T = 0$ in $\mathring{B}(0, R)$. Then we define

$$\mathscr{L}_+ T(x) = \begin{cases} \mathscr{L}T(x) & \text{if } x \in \mathscr{R}^{m+1}_+ \\ 0 & \text{if } x \in \mathscr{R}^{m+1}_-, \end{cases}$$

and

$$\mathscr{L}_- T(x) = \begin{cases} 0 & \text{if } x \in \mathscr{R}^{m+1}_+ \\ \mathscr{L}T & \text{if } x \in \mathscr{R}^{m+1}_-. \end{cases}$$

As both $\mathscr{L}_+ T$ and $\mathscr{L}_- T$ satisfy the estimate of Theorem 10, there exist unique $P_+ T$ and $P_- T$ in $\mathscr{S}'_{(l)}(\mathscr{R}^m; \mathscr{A})$, being equal to zero in $\mathring{B}(0, R)$, such that

$$\mathscr{L}_+ T = \mathscr{L}P_+ T \quad \text{and} \quad \mathscr{L}_- T = \mathscr{L}P_- T.$$

Furthermore $P_+^2 T = P_+ T$, $P_-^2 T = P_- T$, $(P_+ + P_-)T = T$ and $P_+ P_- T = P_- P_+ T = 0$. We illustrate the decomposition of such a T in the cases where $m = 1$ and $m = 2$.

If $m = 1$, then one may easily check that

$$P_\mp T = \frac{1}{2}\left(1 \pm \frac{t}{|t|}\right)T_t.$$

Note that $P_\mp$ is the restriction operator to $\mathscr{R}_\pm$.

For $m = 2$, let $\mathbf{t} = (t_1, t_2) \in \mathscr{R}^2$ and let θ be the angle between the positive t_1-axis and the oriented half line joining the origin with $\mathbf{t}$. Then one obtains

$$P_{\mp} T_{\mathbf{t}} = \tfrac{1}{2}(1 \pm \cos\theta) T_{(t_1, t_2)} \pm \sin\frac{\theta}{2} T_{(-t_1, t_2)}.$$

In complex analysis one can define the Laplace transform as follows. Let f be a continuous function of polynomial growth in $\mathscr{R}^2$ for example, and let

$$P_{\pm 1, \pm 1} f = f \Big|_{\{\mathbf{t} \in \mathscr{R}^2:\ \mp t_1 > 0,\ \mp t_2 > 0\}}.$$

Then for $(\sigma_1, \sigma_2) \in \{1, -1\}^2$ one can define

$$\mathscr{L}_{\sigma_1, \sigma_2} f(z_1, z_2) = \int_{\mathscr{R}^2} e^{-i(t_1 z_1 + t_2 z_2)} P_{\sigma_1, \sigma_2} f(\mathbf{t})\, d\mathbf{t},$$

which is a holomorphic function in $\{(z_1, z_2) \in \mathscr{C}^2 \colon \operatorname{sgn} y_i = \sigma_i\}$. So the Laplace transform, which is defined by

$$\mathscr{L} f(z_1, z_2) = \mathscr{L}_{\sigma_1, \sigma_2} f(z_1, z_2) \quad \text{if } \operatorname{sgn} y_i = \sigma_i,$$

is holomorphic in $(\mathscr{C} \backslash \mathscr{R})^2$ and can be divided in four parts,

$$\mathscr{L} f = \mathscr{L} P_{1,1} f + \mathscr{L} P_{-1,1} f + \mathscr{L} P_{1,-1} f + \mathscr{L} P_{-1,-1} f,$$

which correspond to the Laplace transforms of the restriction of f to the "octants". (Hence one could say that this Laplace transform is of "Carthesian nature".) In hypercomplex function theory, the Laplace transform always consists of two parts,

$$\mathscr{L} f = \mathscr{L} P_{+} f + \mathscr{L} P_{-} f,$$

where $P_{\pm}$ are the above introduced "orientation operators" in the Euclidean space, which have a rather "spherical nature". Only in the case where $m = 1$ do both ways of thinking coincide.

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