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EMBEDDINGS OF $S^n \times M$ IN $S^{n+2} \times M$ FORM A GROUP

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Introduction

This paper describes group structures for a large class of codimension two embedding problems. The classic example of algebraic structure in an embedding problem is furnished by the knot cobordism groups of [9], [11], [13]. Our general study uses homology surgery theory, first developed and applied to the codimension two placement problem in [7].

Let M be an arbitrary k-dimensional compact manifold. This paper classifies standard M-knots, i.e., embeddings $f: S^n \times M \to S^{n+2} \times M$ which are homotopic, rel boundary, to the standard inclusion. Using a definition of cobordism based on concordance of embeddings, we prove that the set $G_n^t(M)$ of cobordism classes of such M-knots forms an abelian group in a natural way, provided $n \geq 2$ and $n + k \geq 4$. This was known previously for M simply connected [7] and for a certain class of non simply connected M [16]. Herein we treat the general case by devising a variant of surgery theory which studies the normal cobordism problem for simply split simple homotopy equivalences [5], [8]. The desired group structure is obtained by exhibiting $G_n^t(M)$ as a subgroup of a relative homology surgery group in this theory. For all M, we interpret this group structure geometrically. When M is a point, $G_n^t(M)$ coincides with the knot cobordism groups of [11], [13], wherein the group operation is defined by taking connected sum of knots.

Two embeddings $f, g: S^n \times M \to S^{n+2} \times M$ are called cobordant if f is concordant to $\phi f \psi$, where ϕ and ψ are certain allowable automorphisms of $S^{n+2} \times M$ and $S^n \times M$ respectively. The set of cobordism equivalence classes is denoted $G_n^t(M)$; see Section 1 for a precise definition, as well as the reason for including the superscript "t" in the notation. Our results are valid for M a smooth (resp. piecewise linear, topological) manifold, provided we restrict attention to smooth (resp. piecewise linear locally flat, topological locally flat) embeddings and concordances, and require ϕ and ψ to be diffeomorphisms (resp. piecewise linear homeomorphisms, homeomorphisms). For simplicity, discussions and results are stated for the smooth case.

The groups $G_n^t(M)$ do not, in general, satisfy the fourfold periodicity proved

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in [13] for the knot cobordism groups C_n, which correspond to G_n^t (point) in the piecewise linear or topological cases. To remedy this, we defined in [16] a larger cobordism set $G_n(M)$, based on embeddings in $S^{n+2} \times M$ of manifolds simple homotopy equivalent to $S^n \times M$. In this paper, we construct a family of (abelian) relative surgery obstruction groups $\Gamma_*^{ss}(\Psi)$, where Ψ is a commutative square functorial in $\pi_1(M)$. Our main technical result is:

THEOREM 3. *For $n \geq 2$, $n + k \geq 4$, there is a bijective surgery obstruction map θ: $G_n(M) \to \Gamma_{n+k+3}^{ss}(\Psi)$.*

Since the groups $\Gamma_*^{ss}(\Psi)$ satisfy fourfold periodicity, we obtain $G_n(M) \simeq G_{n+4}(M)$. The isomorphism is constructed geometrically as follows:

THEOREM 5. *For $n \geq 2$ and $n + k \geq 4$, there are geometrically defined isomorphisms*

$$G_n(M) \xrightarrow{\cdot \times CP^2} G_n(M \times CP^2) \xrightarrow{\simeq} G_n(M \times I^4) \xrightarrow{\simeq} G_{n+4}(M)$$
$$(I = [0, 1]).$$

For related constructions in the simply connected case, see [7]. In particular, this result provides a geometric interpretation of the periodocity of knot cobordism groups; cf. [7], [2], [12].

The relative homology surgery groups $\Gamma_*^{ss}(\Psi)$ in turn depend on absolute surgery groups $L_*^{ss}(\pi)$, with $\pi = \pi_1(M)$. These groups contain the obstruction to finding a normal cobordism from a given normal map with target $M \times S^1$ to a simple homotopy equivalence which is simply split along $M \times$ pt, where pt is a basepoint of S^1 [8], [5]. We exhibit a splitting

$$L_n^{ss}(\pi \times Z) \simeq i_* L_n(\pi) \oplus L_{n-1}(\pi),$$

for $n \geq 6$. Note that both Wall groups which appear on the right are the groups $L^s(\pi)$, which study simple homotopy equivalences; cf. [17]. The groups L_*^{ss} are related to the groups L_*^{st} of [10], which study the "super-simple" homotopy equivalences defined in [6].

In order to interpret geometrically the group structure in $G_n^{(t)}(M)$, we consider, as in [16], the case that M has non-empty boundary. (Here, and throughout this paper, the superscript "(t)" indicates that we are discussing either the fake or standard cobordism groups.) In this situation, an M-knot is required by definition to coincide on the boundary with the standard inclusion. As a result, $G_{n-1}^{(t)}(M \times I)$ admits a groups operation, defined by "stacking" embeddings along part of the boundary. For $n \geq 2$, this group structure coincides with the algebraically defined one of Theorem 3 above. Furthermore, there is a natural map

$$j_M: G_{n-1}^{(t)}(M \times I) \to G_n^{(t)}(M),$$

obtained by viewing $S^{n-1} \times M \times I$ as a neighborhood of the equator in $S^n \times M$. For M^k any compact manifold, we prove:

THEOREM 1. *If $n + k \geq 4$, the map $G_{n-1}^{(t)}(M \times I) \to G_n^{(t)}(M)$ is an isomorphism for $n \geq 3$, and an epimorphism for $n = 2$.*

The commutativity of the stacking operation may be explained, as in [16, Section 13], by studying the two stacking operations in $G_{n-2}^{(t)}(M \times I \times I)$.

We obtain as a result the following "partial unkotting theorem" for M-knots.

THEOREM 2. *Let $f: S^n \times M \to S^{n+2} \times M$ be a standard M-knot, with $n \geq 2$ and $n + k \geq 4$. Let N be any (arbitrarily small) neighborhood of $S^0 \times M$ in $S^{n+2} \times M$. Then there exist diffeomorphisms $\phi: S^{n+2} \times M \to S^{n+2} \times M$ and $\psi: S^n \times M \to S^n \times M$, and an M-knot $g: S^n \times M \to S^{n+2} \times M$ such that:*

(i) *g coincides with the standard inclusion outside N.*

(ii) $g = \phi f \psi$.

Thus every standard cobordism class contains a representative which coincides, away from two copies of M, with the standard inclusion. Theorems 1 and 2 generalize the result of [7] that the natural map $\# i_0: C_{n+k} \to G_n^t(M)$, defined by taking the connected sum of a classical knot with the standard inclususion, is a bijection for M a closed, simply connected, piecewise linear or topological manifold.

Section 1

We recall the definition of $G_n^t(M)$, where M is a compact manifold with possibly nonempty boundary. A parametrized knot in M, or more briefly a standard M-knot, is an embedding $f: S^n \times M \to S^{n+2} \times M$ which is homotopic, rel ∂, to the standard inclusion i_0. Two M-knots f and g are *conjugate* provided there exist diffeomorphisms

$$\phi: S^{n+2} \times M \to S^{n+2} \times M \quad \text{and} \quad \psi: S^n \times M \to S^n \times M$$

such that:

(i) ϕ and ψ are the identity on the boundary.

(ii) There exist homotopies rel ∂, $\pi_M \phi \sim \pi_M$ and $\pi_M \psi \sim \psi$, where π_M denotes projection to M.

The M-knots f_0 and f_1 are *concordant* provided there exists a smooth embedding $F: S^n \times M \times I \to S^{n+2} \times M \times I$, such that:

(i) $F(x, i) = (f_i(x), i)$, $i = 0, 1$; $x \in S^n \times M$.

(ii) F coincides with i_0 on the boundary.

Finally, f and g are *cobordant* provided they are conjugate to concordant M-knots. The set of cobordism equivalence classes so obtained is denoted $G_n^t(M)$. Technically speaking, an M-knot comes equipped with a framing $\bar{f}: S^n \times M \times D^2 \to S^{n+2} \times M$; we sometimes omit reference to the framing in order to simplify the exposition. See [16, Section 1] for complete information.

In order to realize the entire surgery obstruction group which we propose to define, we must study fake M-knots. Such a knot is defined by a triple (f, X, ξ), where:

(i) $\xi: S^n \times M \to X$ is a simple homotopy equivalence of manifolds which has zero normal invariant and restricts to a diffeomorphism on the boundary.

(ii) $f: X \to S^{n+2} \times M$ is an embedding such that $f \circ \xi$ is homotopic rel ∂ to the standard inclusion.

The definition of the cobordism relation is given in [16, Section 8]; the resulting set of equivalence classes is denoted $G_n(M)$. Let $T_0 = S^n \times M \times D^2$ and $W_0 = D^{n+1} \times M \times S^1$ denote the corresponding tube and complement of the standard embedding i_0. Then there is an associated *characteristic map* $\hat{F}: S^{n+2} \times M \to S^{n+2} \times M$ such that:

(C1) $\pi_M \hat{F} \sim \pi_M$ rel ∂.

(C2) $\hat{F} | T = (\bar{f})^{-1}: T \to T_0$.

(C3) The *complementary map* $F = \hat{F} | W: W \to W_0$ is a simple homology equivalence with coefficients $Z[\pi_1(M)]$.

See [16, Section 2] for details.

The last condition motivated [7] to construct a surgery theory for studying homology equivalent manifolds. We now indicate briefly the results of [7] which we need.

Let $\pi = \pi_1(M)$, and let $\Pi: Z[\pi \times Z] \to Z[\pi] = \Lambda$ be induced by projection. Set $d = n + k + 2$, the dimension of W_0.

First, there exists a surgery group $\Gamma_d(\Pi)$ for $d \geq 5$ which contains the obstruction $\sigma(G, B)$ to finding a normal cobordism rel ∂ from a given normal map (G, B), $G: W \to W_0$ to a simple Λ homology equivalence. Of course we assume that $G | \partial W$ is a simple Λ-homology equivalence to begin with [7, 1.7 and 2.1].

Next suppose given a normal map (F, B), $F: W \to W_0$ with F a simple Λ-homology equivalence of pairs, together with a surgery group element $\gamma \in \Gamma_{d+1}(\Pi)$. If $d \geq 5$, there is a normal cobordism (H, C), $H: Z \to W_0$ from (F, B) to a normal map which we shall denote

$$(\gamma \cdot F, \gamma \cdot B), \gamma \cdot F: \quad \gamma \cdot W \to W_0,$$

also a simple homology equivalence of pairs, such that $\sigma(H, C) = \gamma$ [7, 1.8 and 2.2].

This last result permits the construction of new M-knots by surgery, starting from a given M-knot f with complementary map $F: W \to W_0$, as follows. Since F is the restriction of the simple homotopy equivalence $\hat{F}$, F is covered by a canonical bundle map which we shall henceforth not mention. Given $\gamma \in \Gamma_{d+1}(\Pi)$, construct $\gamma \cdot F: \gamma \cdot W \to W_0$ as above. The manifold $\gamma \cdot W$ will be the complement of the new knot. Define

$$\gamma \cdot \hat{F} = (\bar{f})^{-1} \cup \gamma \cdot W: \quad T \cup \gamma \cdot W \to T_0 \cup W_0 = S^{n+2} \times M;$$

note that the domain of this map is obtained by pasting the original tube to the new complement. Then $\gamma \cdot \hat{F}$ is homotopic rel ∂ to a diffeomorphism g, provided γ is in $\Gamma_{d+1}(\Pi)$, the kernel of the natural map $\Gamma_{d+1}(\Pi) \to L_{d+1}(\pi)$. Define

$$\gamma \cdot f = (g \mid T) \circ \bar{f}\colon T_0 \to S^{n+2} \times M.$$

This yields a new M-knot, whose cobordism class depends only on that of f. Hence there is an induced action of $\tilde{\Gamma}_{d+1}(\Pi)$ on $G_n^t(M)$. Similar remarks apply to $G_n(M)$; see [16, Sections 4, 9] for details.

An important invariant of a cobordism class $x \in G_n^{(t)}(M)$ is its "Seifert surface obstruction" $\rho(x) \in L_{n+k+1}(\pi)$. This is defined to be the Wall surgery obstruction of the restriction of the map $F\colon W \to W_0 = D^{n+1} \times M \times S^1$ to the transverse inverse image of $D^{n+1} \times M \times \text{pt}$. We shall show later that the natural map $i\colon G_n^t(M) \to G_n(M)$ is injective, and that $x \in G_n(M)$ is in the image of i if and only if $\rho(x)$ acts *trivially* on the simple homotopy triangulations of $D^n \times M$. This is the reason for the "t" in the notation "$G_n^t(M)$".

Section 2

This section determines the isotropy subgroup of the trivial cobordism class under the action of $\tilde{\Gamma}_{d+1}(\Pi)$. Let

$$k_*\colon L_{d+1}(\pi \times Z) \to \Gamma_{d+1}(\Pi)$$

be the natural map, and let

$$\cdot \times S^1\colon L_d(\pi) \to L_{d+1}(\pi \times Z)$$

be induced by crossing normal maps with a circle [17]. The composite $k_*(\cdot \times S^1)$ is easily seen to take values in $\tilde{\Gamma}_{d+1}(\Pi)$, which vanishes if d is even [16, p. 18].

PROPOSITION 1. *Let $\alpha \in L_d(\pi)$, $d \geq 6$. Then $k_*(\alpha \times S^1) \cdot x = x$ for all $x \in G_n^{(t)}(M)$.*

Remark. The lack of this result in [16] forced the author to assume that $k_*(\cdot \times S^1)$ is the zero homomorphism. For reasons that will become clear later, it was in fact necessary to assume that the composite

$$L^n(\pi) \xrightarrow{\cdot \times S^1} L(\pi \times Z) \xrightarrow{k_*} \tilde{\Gamma}(\Pi)$$

is zero. This is the circle perfect condition on $\pi = \pi_1(M)$ [16, Section 17].

Proof. For convenience, we consider only the case $x \in G_n^t(M)$. Minor variations of the proof yield the result in the fake case; see [16, Sections 8–10] for necessary information.

Let $F\colon W \to W_0$ be the complementary map of a knot in the cobordism class x. By [7, 13.7], we may assume that F induces an isomorphism on π_1. We shall use the diffeomorphism $\bar{f}\colon T_0 \to T$ to identify T_0 and T. Set $\partial_+ W = \partial W \cap \partial T$; note that this is identified with $S^n \times M \times S^1 = \partial_+ W_0$.

Let $\gamma = k_*(\alpha \times S^1)$. We construct $\gamma \cdot F$ by doing surgery on id_W to obtain a simple homotopy equivalence $(\alpha \times S^1) \cdot \mathrm{id}_W$; then

$$\gamma \cdot F = F \circ ((\alpha \times S^1) \cdot \mathrm{id}_W).$$

Since $L_*(\pi_1(\partial_+ W)) \to L_*(\pi_1(W))$ is an epimorphism for all n, the surgery may be performed on a collar neighborhood $\partial W \times I$ of ∂W in W. Write $W = \partial_+ W \times I \bigcup_{\partial_+ W \times 0} \bar{W}$.

If $d \geq 6$, let $h\colon X \to S^n \times M \times I$ be the simple homotopy equivalence obtained by using α to do surgery rel ∂ on $\mathrm{id}_{S^n \times M \times I}$, as in [18, 5.8 and 6.5]. If $d \geq 7$, the s-cobordism theorem yields a diffeomorphism $\xi\colon S^n \times M \times I \to X$ such that

$$h \circ \xi\colon S^n \times M \times I \to S^n \times M \times I$$

is a homotopy from $\mathrm{id}_{S^n \times M \times 0}$ to a diffeomorphism

$$\psi\colon S^n \times M \times 1 \to S^n \times M \times 1.$$

Write $\theta = h \times S^1$ and $\gamma = k_*(\alpha \times S^1)$. We have realized our desired surgery obstruction by the map of collars

$$\theta\colon X \times S^1 \to \partial_+ W_0 \times I.$$

Pasting back the tubes and complements, we see that $\gamma \cdot \hat{F}$ is the composite

$$T \bigcup_{S^n \times M \times 1} (X \times S^1) \bigcup_{S^n \times M \times 0} \bar{W} \xrightarrow{(\psi \times D^2) \cup \theta \cup \mathrm{id}}$$

$$T \cup (\partial_+ W \times I) \cup \bar{W} \xrightarrow{\hat{F}} S^{n+2} \times M.$$

This follows from naturality of surgery obstructions. Then $\gamma \cdot f$ is, by definition, $(g \mid T) \circ \bar{f}$, where $\bar{f}\colon T_0 \to T$ is the given framed knot and g is a diffeomorphism homotopic rel ∂ to $\gamma \cdot \hat{F}$.

To see that $\gamma \cdot f$ is cobordant to f, construct the diffeomorphism

$$\Phi = (\psi^{-1} \times D^2) \cup (\xi^{-1} \times S^1) \cup \mathrm{id}\colon$$

$$T \cup (X \times S^1) \cup \bar{W} \to T \cup (\partial_+ W \times I) \cup \bar{W} = S^{n+2} \times M$$

Then $\gamma \cdot f$ may be rewritten as the composite

$$T_0 \xrightarrow{(\Phi \mid T) \circ \bar{f}} S^{n+2} \times M \xrightarrow{g \circ \Phi^{-1}} S^{n+2} \times M.$$

Now observe that $(\Phi \mid T) \circ \bar{f} = \bar{f} \circ (\psi^{-1} \times D^2)$ as a result of our identification of T_0 and T. Hence $\gamma \cdot f$ is cobordant to f as desired.

The s-cobordism theorem used in the above argument fails when $d = 6$. The proposition may be proved in this case by using a modified definition of $G_n^{(t)}(M)$ in the case $n + k = 4$; see [16, pp. 43, 59]. We leave the details to the reader. ■

Conversely, [16, 6.2 and 10.6.2] show that if $d \geq 5$ and $\gamma \in \tilde{\Gamma}_{d+1}(\Pi)$ acts trivially on $x_0 \in G_n^{(t)}(M)$, then $\gamma = k_*(\alpha \times S^1)$ for some $\alpha \in L_d(\pi)$. It follows that $k_*(L_d(\pi) \times S^1) \subset \tilde{\Gamma}_{d+1}(\Pi)$ is the isotropy subgroup of x_0 for $d = n + k + 2 \geq 6$.

Section 3

The next part of this paper is devoted to showing that $k_*(L_d(\pi) \times S^1)$ is the isotropy subgroup for all classes $x \in G_n(M)$. To accomplish this goal, we follow the idea of [16, Section 11] and try to map $G_n(M)$ to an appropriate relative surgery group. Specifically, we wish for a diagram with exact bottom row

$$\begin{array}{ccccc} L_d(\pi) & \xrightarrow{k_*(\cdot \times S^1)} & \tilde{\Gamma}_{d+1}(\Pi) & \longrightarrow & G_n(M) \\ \cap & & \cap & & \downarrow \theta \\ L_d(\pi) \oplus L_{d+1}(\pi) & \xrightarrow{(k_*(\cdot \times S^1),\, i_*)} & \Gamma_{d+1}(\Pi) & \xrightarrow{m_*} & \Gamma_{d+1}(?) \end{array}$$

in which the right-hand square commutes, i.e.,

$$\theta(\gamma \cdot x) = m_*(\gamma) + \theta(x) \quad \text{for } \gamma \in \tilde{\Gamma}_{d+1}(\Pi) \text{ and } x \in G_n(M).$$

It will then follow formally that the isotropy subgroup of $x \in G_n(M)$ is independent of x; cf. [16, Section 12].

In order to construct the group at the lower left, we in turn need a Wall surgery group for $\pi_1 = \pi \times Z$, which satisfies a splitting

$$L_{d+1}^{ss}(\pi \times Z) \approx i_* L_{d+1}(\pi) \oplus L_d(\pi) \times S^1.$$

In the usual splitting of $L_{d+1}(\pi \times Z)$ [17], we encounter the group $L_d^h(\pi) \times S^1$ because the group $Wh(\pi)$ gives rise to an obstruction to finding a simple splitting of a simple homotopy equivalence [8], [5]. We apply the methods of [18, Section 9] to construct the group $L^{ss}(\pi \times Z)$; here the superscript stands for "simply split".

Consider a Poincaré pair (Y, X), together with a codimension one Poincaré subpair (y, x) with trivial normal bundle. A simple homotopy equivalence $f: (N, M) \to (Y, X)$, where (N, M) is a manifold pair, will be called *simply split along* (y, x) if f and $f\,|\,M$ are transverse regular to y and x respectively, and if $f\,|\,(f^{-1}(y), f^{-1}(x))$ is a simple homotopy equivalence of pairs. Briefly, we call f as *ss-equivalence along* (y, x).

If $F: W \to W_0$ is the complementary map of a standard (resp. fake) M-knot, it follows from [16] that

$$\partial F: \partial_+ W \to \partial_+ W_0 = S^n \times M \times S^1$$

is the product of a diffeomorphism (resp. simple homotopy equivalence) with id_{S^1}, hence ∂F is an ss-equivalence along $S^n \times M \times \mathrm{pt}$. Furthermore, if an

M-knot is conjugate to i_0, its complementary map is an ss-equivalence along

$$(D^{n+1} \times M \times \text{pt}, S^n \times M \times \text{pt}).$$

These facts indicate the role of simple splitting in the study of $G_n^{(t)}(M)$.

Let $(f, b), f\colon (N, M) \to (Y, X)$ be a normal map. Assume given a subpair (y, x) of (Y, X) as above, with $y \subset Y$ inducing the natural inclusion $\pi \to \pi \times Z$ of fundamental groups, and that $f \mid M$ is an ss-equivalence along x. We now proceed to the construction of $L^{ss}(\pi \times Z)$, which contains the obstruction to finding a normal cobordism rel ∂ from (f, b) to a normal map (g, c), with g an ss-equivalence along (y, x).

Let K be a CW-complex with finite 2-skeleton and

$$w\colon \pi_1(K) \to \{\pm 1\}$$

a homomorphism. As in [18, Section 9] we have in mind the case $K = K(\pi, 1)$ with π a finitely presented group. We now construct a group based on unrestricted objects over $K \times S^1$, with additional data provided by a codimension one surgery problem. Specifically, let an *object* consist of data

$$\theta = (Y, X, v, N, M, \phi, F, \omega, y, x, n, m).$$

Here, the first eight entries define an unrestricted object over $K \times S^1$, as in [18, p. 86]. In particular, recall that $\phi\colon (N, M) \to (Y, X)$ is a degree one map from a manifold pair in dimension n to a Poincaré pair, and that ω is a map from Y to $K \times S^1$. To this we add the following structure: (y, x) is a codimension one subpair of (Y, X) with trivial normal bundle. The map $\omega\colon K \times S^1$ is transverse regular to $K \times 0 \subset K \times S^1$, with

$$(\omega, \omega \mid X)^{-1}(K \times 0) = (y, x).$$

Here $0 \in S^1$ is a base point, and in the future we will think of S^1 as $[0, 2\pi]/0 \sim 2\pi$. In addition, ϕ is transverse regular to $(y, x) \subset (Y, X)$, and $(n, m) = \phi^{-1}(y, x)$. Let $\phi \mid M_m\colon M_m \to X_x$ be the map obtained by splitting $\phi \mid M$ along m, as in [4], [17]. We require finally that $\phi \mid m$ and $\phi \mid M_m$ be homotopy equivalences. As usual, the fundamental classes $[N]$ and $[n]$ are part of the structure of θ; we obtain the object $-\theta$ by reversing their signs.

Next, define an object θ as above to be null equivalent (write $\theta \sim 0$) if there exist data

$$((Z, Y, Y_+), \mu, (P, N, N_+), \psi, G, \Omega, (z, y, y_+), (p, n, n_+))$$

which extend the object θ, as in [18]. Here, (Z, Y, Y_+) is a Poincaré triad of dimension $n + 1$, with $Y \cap Y_+ = X$, and (z, y, y_+) is a codimension one Poincaré subtriad with trivial normal bundle. The map $\Omega\colon Z \to K \times S^1$ is a transverse regular (to K) extension of $\omega\colon Y \to K \times S^1$, with

$$(\Omega, \Omega \mid Y, \Omega \mid Y_+)^{-1}(K) = (z, y, y_+).$$

Similarly, ψ: $(P, N, N_+) \to (Z, Y, Y_+)$ extends ϕ, is transverse regular to (z, y, y_+), and $\psi^{-1}(z, y, y_+) = (p, n, n_+)$. Finally, $\psi | n_+$ and $\psi | N_+$ must be simple homotopy equivalences. Now write $\theta_1 \sim \theta_2$ if the object $\theta_1 + \theta_2$, obtained by taking disjoint unions, is null equivalent. As in [18], we obtain an abelian group of equivalence classes under $\sim$, which we denote $L_n^{ss}(K \times S^1)$. Let $L_n^1(K)$ denote the Wall group based on unrestricted objects over K; recall that

$$L_n^1(K(\pi, 1)) \approx L_n(\pi, w)$$

provided $n \geq 5$ [18, 9.4.1].[2]

PROPOSITION 2. *There is a natural split short exact sequence*

$$0 \longrightarrow L_n^1(K) \xrightarrow{i_*} L_n^{ss}(K \times S^1) \xrightarrow{s_*} L_{n-1}^1(K) \longrightarrow 0.$$

Proof. Let $\varepsilon = 1$, say, and define $i: K \to K \times S^1$ by $i(k) = (k, \varepsilon)$. Let $p: K \times S^1 \to K$ denote the projection. Given

$$\alpha = (Y, X, v, N, M, \phi, F, w) \in L_n^1(K),$$

define an object $i_\#(\alpha) \in L_n^{ss}(K \times S^1)$ by including null subobject data:

$$i_\#(\alpha) = (Y, X, v, N, M, \phi, F, i \circ w, \emptyset, \emptyset, \emptyset, \emptyset).$$

Similarly, given θ representing a class in $L_n^{ss}(K \times S^1)$, define an object $p_\#(\theta)$ over K by omitting the subobject data and replacing ω by $p \circ \omega$: $Y \to K$. It is easy to check that $i_\#$ and $p_\#$ induce well-defined homomorphisms

$$i_*: L_n^1(K) \to L_n^{ss}(K \times S^1) \quad \text{and} \quad p_*: L_n^{ss}(K) \to L_n^1(K)$$

with $p_* i_*$ the identity. Hence, i_* is injective.

The splitting map s_* sends an n-dimensional object over $K \times S^1$ to the $(n-1)$-dimensional object over K obtained by restricting maps and bundles to the subobject data. We may write

$$s_\#(\theta) = (y, x, v | y, n, m, v | n, F | N, \omega | y)$$

for θ as specified above. This induces a homomorphism

$$s_*: L_n^{ss}(K \times S^1) \to L_{n-1}^1(K).$$

Finally, crossing with a circle defines in an obvious way a homomorphism

$$\cdot \times S^1: L_{n-1}^1(K) \to L_n^{ss}(K \times S^1)$$

such that $s_*(\alpha \times S^1) = \alpha$ for $a \in L_{n-1}^1(K)$. Hence s_* is onto.

To prove exactness, let $[\theta]$ be a class in $L_n^{ss}(K \times S^1)$ such that $s_*([\theta]) = 0$. By cobordism extension [3], θ is equivalent to an object, still denoted θ, in which $\phi | n$: $(n, m) \to (y, x)$ is a simple homotopy equivalence of pairs. Split

$$\phi: (N, M) \to (Y, X) \quad \text{and} \quad \omega: Y \to K \times S^1$$

[2] Henceforth we omit reference to the orientation character and write $L_n(\pi)$ for $L_n(\pi, w)$.

along y and $K \times 0$ respectively, thereby obtaining maps

$$\phi_s\colon (N_n, 2n \cup M_m) \to (Y_y, 2y \cup X_x) \quad \text{and} \quad \omega_s\colon Y_y \to K \times [\varepsilon, 2\pi - \varepsilon].$$

Note that the boundary of Y_y, for instance, consists of two disjoint copies of y, each glued to X_x along a copy of x. By Mayer Vietoris sequences for homotopy equivalences and Whitehead Torsion [18], ϕ_s is a simple homotopy equivalence on the boundary; recall that $\phi_s | M_m$ is required to be a simple homotopy equivalence in our definition of objects. Let $\omega\colon Y_y \to K$ be the composite of ω_s followed by projection. Then the data obtained by restricting attention to the split maps ϕ_s and ω define an object β representing a class in $L_n^1(K)$.

We claim that $i_*([\beta]) = [\theta]$. Recall the 12-tuple which defines θ, and set

$$\psi = \phi \times I\colon N \times I = P \to Z = Y \times I \quad (I = [0, 1]),$$
$$\Omega = (\text{projection}) \circ (\omega \times I)\colon Y \times I \to K \times S^1 \times I \to K \times S^1.$$

From Ω and ψ, extract the information for defining an equivalence $i_\#(\beta) \sim \theta$ as follows. Let

$$\begin{aligned} z &= y \times I \\ y_+ &= y \times 1 \cup \partial y \times I \\ p &= n \times I \\ n_+ &= n \times 1 \cup \partial n \times I \\ Y_0 &= Y \times 0 \\ Y_1 &= (\omega \times 1)^{-1}(K \times [\varepsilon, 2\pi - \varepsilon]) \\ Y_+ &= \partial(Y \times I) - \overline{(Y_0 \cup Y_1)}. \end{aligned}$$

It is easy to check that these data, together with obvious unmentioned bundle data, define an equivalence between the objects over $K \times S^1$ defined by data along Y_0 and Y_1 respectively. But the data along Y_0 define the object θ, while the data along Y_1 define an object which is clearly equivalent to $i_\#(\beta)$. Hence $i_*([\beta]) = [\theta]$ as desired. ■

We now write $L_n^{ss}(\pi \times Z) = L_n^{ss}(K(\pi, 1) \times S^1)$. It follows from [18, 9.4.1] and the last result that there is a canonical splitting for $n \geq 6$:

$$L_n^{ss}(\pi \times Z) \approx i_* L_n(\pi) \oplus L_{n-1}(\pi) \times S^1.$$

To apply this result, consider an n-dimensional Poincaré pair (Y, X) and an $(n-1)$-dimensional subpair (y, x) with trivial normal bundle. Assume that the inclusion $y \subset Y$ induces the inclusion $\pi \to \pi \times Z$ of fundamental groups. Construct a map $\omega\colon Y \to K \times S^1$, transverse regular to $K \times 0$, such that

$$(\omega, \omega | X)^{-1}(K \times 0) = (y, x).$$

Let (f, b), $f\colon (N, M) \to (Y, X)$ be a normal map, transverse regular to (y, x), and as usual set $(n, m) = (f, f\,|\,M)^{-1}(y, x)$. Assume that $f\,|\,M$, $f\,|\,m$, and $f\,|\,M_m$ are all simple homotopy equivalences. It is clear that these data define an object $\theta(f, b)$ over $K \times S^1$, representing a class $\sigma(f, b) \in L_n^{ss}(\pi \times Z)$.

PROPOSITION 3. *Assume that $n \geq 6$, Y_y and y are connected, and (f, b) is a normal map as above. Then $\sigma(f, b) = 0$ if and only if (f, b) is normally cobordant* rel ∂ *to an ss-equivalence along (y, x).*

Proof. If (f, b) is normally cobordant rel ∂ to an *ss*-equivalence, it follows immediately from the definitions that $\theta(f, b) \sim 0$. Conversely, assume $\theta = \theta(f, b) \sim 0$ and $n \geq 6$. By Proposition 2, $s(\theta) \sim 0$ in $L_{n-1}^{1}(K) \simeq L_{n-1}(\pi)$ [18, 9.4.1]. Since y is connected, the restriction of (f, b) to (n, m) is normally cobordant rel ∂ to a simple homotopy equivalence of pairs. By a cobordism extension argument, we may perform a normal cobordism of (f, b), thereby obtaining an equivalent object, still denoted $\theta(f, b)$, such that $(f, b)\,|\,(n, m)$ is a simple homotopy equivalence of pairs. Split (f, b) along (n, m); it follows similarly that a further normal cobordism will yield an equivalent object $\theta(f, b)$ whose restriction to $(N_n, \partial(N_n))$ is also a simple homotopy equivalence of pairs. That (f, b) is a simple homotopy equivalence, and hence an *ss* equivalence along (y, x), follows as usual from Mayer-Vietoris sequences. ■

Section 4

We are now ready to define the relative homology surgery group which realizes $G_n(M)$. Let $\pi = \pi_1(M)$, let

$$\Pi\colon Z[\pi \times Z] \to Z[\pi]$$

be the group ring homomorphism induced by projection, and $\Psi\colon \mathrm{id}_{Z[\pi \times Z]} \to \Pi$ the commutative square

$$\begin{array}{ccc} Z[\pi \times Z] & \xrightarrow{\ \mathrm{id}\ } & Z[\pi \times Z] \\ \Big\downarrow \mathrm{id} & & \Big\downarrow \Pi \\ Z[\pi \times Z] & \xrightarrow{\ \Pi\ } & Z[\pi]. \end{array}$$

Now recall that the construction of relative surgery groups in [18, Section 9] and [7, Section 3] is based on the definition of surgery groups in terms of unrestricted objects. Hence we may use our definition of $L_n^{ss}(\pi \times Z)$ and that of [7] for $\Gamma_n(\Pi)$, to produce a relative group, denoted $\Gamma_n^{ss}(\Psi)$, which fits into a sequence

$$\Gamma_{n+1}^{ss}(\Psi) \to L_n^{ss}(\pi \times Z) \to \Gamma_n(\Pi) \to \Gamma_n^{ss}(\Psi)$$

which is exact for $n \geq 6$; cf. [7, Section 3].

The group $\Gamma_n^{ss}(\Psi)$ solves the following surgery problem. Fix an n-dimensional Poincaré triad (Y, X_-, X_+) together with a Poincaré subpair $(x, \partial x)$ of $(X_+, \partial X_+)$ with trivial normal bundle. Assume that the inclusions $x \subset X_+ \subset Y$ induce the homomorphisms $\pi \to \pi \times Z = \pi \times Z$ on fundamental groups (and that these three Poincaré complexes are connected). Then the following is proved precisely as in [18, Section 9] and [7, Section 3] by using Proposition 3 above:

PROPOSITION 4. *Given data as above, let*

$$(F, B),\ F\colon (N, M_-, M_+) \to (Y, X_-, X_+)$$

be a normal map which is transverse regular to $(x, \partial x)$, *with preimage* $(m, \partial m)$. *Assume that* $F | M_-$ *is a simple homology equivalence over* $Z[\pi]$ *and that* $F | \partial M_- = \partial M_+$ *is an ss-equivalence along* ∂x. *Then there is a relative surgery obstruction* $\sigma(F, B) \in \Gamma_n^{ss}(\Psi)$ *which vanishes* (*for* $n \geq 7$) *if and only if* (F, B) *is normally cobordant* rel M_- *to a normal map*

$$(G, C),\ G\colon (Q, P_-, P_+) \to (Y, X_-, X_+)$$

such that G *is a simple homology equivalence over* $Z[\pi]$ *and* $G | P_+$ *is an ss-equivalence along* $(x, \partial x)$. ■

We are now prepared to define the relative surgery obstruction map

$$\theta\colon G_n(M) \to \Gamma_{d+1}(\Psi), \quad \text{where } d = n + k + 2,$$

and k is the dimension of M. Let $F\colon W \to W_0 = D^{n+1} \times M \times S^1$ be the complementary map of an M-knot. As observed in [7, Section 13], F is the restriction of the homotopy equivalence $\hat{F}$, hence is covered by a canonical bundle map (which we will henceforth not mention). In the notation of Proposition 4, and following the argument of [16, Section 11], set

$$\begin{aligned} Y &= W_0 \times I \\ X_- &= W_0 \times 0 \\ X_+ &= W_0 \times 1 \cup \partial W_0 \times I \\ x &= D^{n+1} \times M \times \mathrm{pt} \times 1 \cup \partial(D^{n+1} \times M) \times \mathrm{pt} \times I \end{aligned}$$

where pt denotes a base point of S^1. Decompose $W \times I$ similarly; it is easy to see that the normal map $F\colon W \times I \to W_0 \times I$ satisfies the hypotheses of Proposition 4. Hence the surgery obstruction $\sigma(F \times I) \in \Gamma_{d+1}(\Psi)$ is defined and solves the surgery problem described provided $d \geq 6$. It follows as in [16, Section 11] that σ takes the same value on cobordant knots, hence defines a map $\theta\colon G_n(M) \to \Gamma_{d+1}^{ss}(\Psi)$. Furthermore, the diagram

$$\begin{array}{ccccc} L_d(\pi) & \xrightarrow{k_*(\cdot \times S^1)} & \tilde{\Gamma}_{d+1}(\Pi) & \xrightarrow{k_*(\cdot \times S^1)} & G_n(M) \\ \downarrow{\scriptstyle \cdot \times S^1} & & \downarrow{\scriptstyle \cap} & & \downarrow{\scriptstyle \theta} \\ L_{d+1}^{ss}(\pi \times Z) & \xrightarrow{k_*} & \Gamma_{d+1}(\Pi) & \longrightarrow & \Gamma_{d+1}^{ss}(\Psi) \end{array}$$

commutes [16, p. 69]. The bottom row is exact provided $d \geq 5$. Then Proposition 1 and [16, 6.2 and 10.6.2], applied to the above diagram, yield the fundamental technical result which we have been seeking:

PROPOSITION 5. *Assume* $d \geq 6$, $x \in G_n^{(t)}(M)$, *and* $\gamma \in \tilde{\Gamma}_{d+1}(\Pi)$. *Then* $\gamma \cdot x = x$ *if and only if* $\gamma = k_*(\alpha \times S^1)$ *for some* $\alpha \in L_d(\pi)$.

Let $\rho\colon G_n(M) \to L_{d-1}(\pi)$ be the composite

$$G_n(M) \xrightarrow{\sigma} L_d^{ss}(\pi \times Z) \xrightarrow{s_*} L_{d-1}(\pi)$$

where σ measures the surgery obstruction of the complementary map. In the case that M is a point, ρ measures the index or Arf invariant of the Seifert surface. Combining [16, 10.5.1 and 10.7.1] and Proposition 1, we obtain:

PROPOSITION 6. *Let* $d \geq 6$, $n \geq 2$. *Then the sequence*

$$L_d(\pi) \xrightarrow{k_*(\cdot \times S1)} \tilde{\Gamma}_{d+1}(\Pi) \xrightarrow{\cdot} G_n(M) \xrightarrow{\rho} L_{d-1}(\pi) \xrightarrow{k_*(\cdot \times S1)} \tilde{\Gamma}_d(\Pi)$$

is exact. If $d \geq 6$, $n = 0$ *or* 1, *the sequence is exact at* $\tilde{\Gamma}_{d+1}(\Pi)$ *and* $L_{d-1}(\pi)$, *and* $\rho(\gamma \cdot x) = \rho(x)$ *for* $\gamma \in \tilde{\Gamma}_{d+1}(\Pi)$, $x \in G_n(M)$.

Of course, we mean that the sequence is exact in the strong sense that $\gamma \cdot x = x$ iff $\gamma = k_*(\alpha \times S^1)$ for some $\alpha \in L_d(\pi)$ and $\rho(x) = \rho(y)$ iff $x = \gamma \cdot y$.

The path to our final results is clear. It is easy to see that the natural splittings

$$L_{d+1}^{ss}(\pi \times Z) \approx L_{d+1}(\pi) \oplus L_d(\pi) \times S^1 \quad \text{and} \quad \Gamma_{d+1}(\Pi) \approx L_{d+1}(\pi) \oplus \tilde{\Gamma}_{d+1}(\Pi)$$

are compatible with the natural map

$$k_*\colon L_{d+1}^{ss}(\pi \times Z) \to \Gamma_{d+1}(\Pi).$$

It follows that there is an exact sequence

$$L_{d+1}^{ss}(\pi \times Z) \xrightarrow{k_*} \Gamma_{d+1}(\Pi) \xrightarrow{(\cdot,0)} G_n(M) \xrightarrow{\sigma} L_d^{ss}(\pi \times Z) \xrightarrow{k_*} \Gamma_d(\Pi)$$

given the hypotheses of Proposition 6. Then the surgery obstruction map $\theta\colon G_n(M) \to \Gamma_{d+1}^{ss}(\Psi)$ defined above induces a map from this sequence to the exact relative surgery sequence for $\Gamma_{d+1}^{ss}(\Psi)$. See [16, Section 11] for the (nontrivial) proof that the appropriate diagrams commute; the crucial result [16, 11.2] is based on the definition of surgery groups in [18, Section 9] and carries over to our case. The Five Lemma immediately yields:

THEOREM 3. *Assume* $n + k \geq 4$. *Then* $\theta\colon G_n(M^k) \to \Gamma_{n+k+3}^{ss}(\Psi)$ *is a bijection for* $n \geq 2$ *and a surjection for* $n \geq 0$.

Section 5

The theorems of the introduction follow immediately from Theorem 3 and the definition of $j_M\colon G_{n-1}^{(t)}(M \times I) \to G_n^{(t)}(M)$. Recall that an $(M \times I)$-knot $f\colon S^{n-1} \times M \times I \to S^{n+1} \times M \times I$ coincides with the standard inclusion i_0 on the boundary. Embed $S^{n-1} \times M \times I$ as a tubular neighborhood of $S^{n-1} \times M \subset S^n \times M$; similarly for $S^{n+1} \times M \times I$. It follows that f extends to an M-knot

$$j_M(f) \colon S^n \times M \to S^{n+2} \times M$$

which coincides with i_0 outside $S^{n-1} \times M \times I$. It is easy to check that this induces a well-defined map

$$j_M \colon G_{n-1}^{(t)}(M \times I) \to G_n^{(t)}(M) \quad [16, \text{Section } 13].$$

Then, naturality of surgery obstructions and Theorem 3 imply that $j_M \colon G_{n-1}(M \times I) \to G_n(M)$ is a surjection for $n + k \geq 4$ and a bijection if, in addition, $n \geq 2$. This proves the fake M-knot assertion of Theorem 1 of the introduction.

As noted in the introduction, $G_{n-1}^{(t)}(M \times I)$ admits a natural group structure, defined by "stacking" of embeddings. Here the essential fact is that an $M \times I$-knot is required to coincide with the standard inclusion on $M \times \partial I$; see [16, Section 13] for a precise definition. Furthermore, naturality of surgery obstructions implies that the composite

$$G_{n-1}^{(t)}(M \times I) \to G_n^{(t)}(M) \xrightarrow{\theta} \Gamma_{n+k+3}(\Psi)$$

is a homomorphism for $n \geq 1$. For $n \geq 2$, this provides a geometric interpretation for the group structure induced on $G_n(M)$ by the bijection θ.

We now turn to the computation of $G_n^t(M)$. Define $L_{n+k+1}^t(\pi, M)$ to be the subgroup of $L_{n+k+1}(\pi)$ which acts trivially on the class of $\mathrm{id}_{D^n \times M}$ in $\mathcal{S}(D^n \times M)$, the set of simple homotopy triangulations of $D^n \times M$ rel ∂. The next result follows from Proposition 1 and [16, 5.1 and 7.1].

Proposition 6′. *Let $d = n + k + 2 \geq 6$. If $n \geq 2$, the sequence*

$$L_d(\pi) \xrightarrow{k_*(\cdot \times S^1)} \tilde{\Gamma}_{d+1}(\Pi) \xrightarrow{\cdot} G_n^t(M) \xrightarrow{\rho} L_{d-1}^t(\pi, M) \xrightarrow{k_*(\cdot \times S^1)} \tilde{\Gamma}_d(\Pi)$$

is exact. If $n = 0$ or 1, the sequence is exact at $\tilde{\Gamma}_{d+1}(\Pi)$ and $L_{d-1}^t(\pi, M)$, and $\rho(\gamma \cdot x) = x$ for $\gamma \in \tilde{\Gamma}_{d+1}(\Pi)$, $x \in G_n(M)$. ■

Now consider the natural map $j_M \colon G_{n-1}^t(M \times I) \to G_n^t(M)$. To prove the assertion of Theorem 1 that this is a bijection for $n \geq 2$ and a surjection for $n \geq 0$, note that

$$L_{n+k+1}^t(\pi, M) = L_{(n-1)+(k+1)+1}^t(\pi, M \times I).$$

The map j_M therefore induces a map of the sequences for $G_{n-1}^t(M \times I)$ and $G_n^t(M)$ given by Proposition 6′. The Five Lemma yields the desired result. As before, stacking of embeddings defines a group structure on $G_{n-1}^t(M \times I)$; the map j_M induces a group structure on $G_n^t(M)$ for $n \geq 2$. Note that iteration of j_M induces a surjection

$$G_0^t(M \times I^n) \to G_n^t(M);$$

together with the definition of cobordism, this proves Theorem 2 of the introduction.

Let $\hat{L}_i(\pi, M) = L_i(\pi)/L_i(\pi, M)$. Then a comparison of the exact sequences of Propositions 6 and 6′ yields:

THEOREM 4. *Assume $n \geq 2$, $n + k \geq 4$. There is an exact sequence of abelian groups*

$$0 \to G_n^t(M) \to G_n(M) \to \hat{L}_{n+k+1}(\pi, M) \to 0.$$

By the argument of [7, 3.6], the groups $\Gamma_{d+1}^{ss}(\Psi)$ satisfy fourfold periodicity for $d \geq 6$. By Theorem 3, there is a group isomorphism $G_n(M) \approx G_{n+4}(M)$ for $n \geq 2$. As shown in [16, 16.2], this isomorphism may be realized geometrically by combining the following three isomorphisms:

(i) $G_n(M) \to G_n(M \times CP^2)$, obtained by crossing an M-knot with id_{CP^2},

(ii) $G_n(M \times I^4) \to G_n(M \times CP^2)$, induced by the inclusion of a 4-disc in CP^2, and

(iii) $G_n(M \times I^4) \to G_{n+4}(M)$, the fourfold iteration of the map j_M.

This yields Theorem 5 of the introduction. A similar argument in [7] in the special case that M is a point provided the first geometric proof of the periodicity of knot cobordism. For other explanations of knot periodicity, see [12], [2].

Our next result states necessary and sufficient criteria for unknotting M-knots up to cobordism. First we need a definition. A map of manifolds f: $(M, \partial M) \to (N, \partial N)$ is a *collared diffeomorphism* if it is obtained by gluing a level preserving homotopy $\partial M \times I \to \partial N \times I$ to a diffeomorphism $\bar{M} \to \bar{N}$; here as previously $\bar{M}$ is the closure in M of the complement of the collar neighborhood $\partial M \times I$ of ∂M. Assume as usual that $n \geq 2$, $n + k \geq 4$.

THEOREM 6. *Let F: $W \to W_0$ be the complementary map of a standard (resp. fake) M-knot f. Then f is cobordant to the standard embedding i_0 if and only if F is $Z[\pi_1(M)]$-homology s-cobordant,* rel *boundary, to a collared diffeomorphism (resp. to a map) which is an ss-equivalence along*

$$(D^{n+1} \times M \times \mathrm{pt}, S^n \times M \times \mathrm{pt}).$$

Proof. It follows easily from [16, 3.1] that the complementary map of a standard M-knot conjugate to i_0 is both a collared diffeomorphism and an *ss*-equivalence of the desired type. A similar but easier argument shows that the complementary map of a fake M-knot conjugate to i_0 is an *ss*-equivalence. By the argument of [16, 3.1], concordant (fake or standard) M-knots have $Z[\pi_1(M)]$-homology *s*-cobordant complementary maps. This proves the "only if" part.

Conversely, assume that the complementary map of a knot in the cobordism class x has the desired property. By Proposition 4, the relative surgery obstruction $\theta(x)$ (resp. $\theta i(x)$) of the fake (resp. standard) cobordism class x vanishes. Here, i: $G_n^t(M) \to G_n(M)$ is the natural map. Since θ and i are both injective (Theorems 3 and 4) it follows that $x = x_0$, the trivial cobordism class. ∎

Finally, we state an easy corollary of Propositions 6 and 6′, and the vanishing of $\tilde{\Gamma}_*(\Pi)$ in odd dimensions.

THEOREM 7. *Assume that $n \geq 2$ and $n + k \geq 4$ is even. Then $G_n(M)$ and $G_n^t(M)$ are subgroups of $L_{n+k+1}(\pi)$.* ■

This generalizes the vanishing of the even-dimensional knot cobordism groups C_n. It follows that even-dimensional M-knot cobordism groups are finitely generated if M is compact and $\pi_1(M)$ is finite [18], [1]. In contrast, C_n is not finitely generated for n odd [11], [15]. In fact, Levine has shown that (for $n \geq 3$) C_n is an infinite direct sum of infinitely many copies of Z, Z_2, and Z_4 [13]. Since the natural map $\# i_0\colon C_{n+k} \to G_n^{(t)}(M)$ is a monomorphism [16, 16.3], it follows that $G_n^{(t)}(M)$ is never finitely generated when $n + k$ is odd.

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