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METACYCLIC p-GROUPS AND CHERN CLASSES

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1. Introduction

C. B. Thomas [9] shows that the even-dimensional subring $H^{\text{even}}(G, \mathbf{Z})$ of the integral cohomology ring $H^*(G, \mathbf{Z})$ of some split metacyclic p-group G is generated by Chern classes, and hence this group satisfies Atiyah's conjecture [1]. This result is generalized here, to a non-split metacyclic p-group by using the computational method of G. Lewis [7] together with the property of free action of G on product of two spheres. $H^{\text{even}}(G, \mathbf{Z})$ is expressed in terms of Chern classes of certain representations of G.

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2. Preliminaries

A metacyclic p-group

$$G = \langle A, B\colon A^{p^a} = 1, \quad B^{p^b} = A^{p^c}, \quad B^{-1}AB = A^k; \qquad c \geq 0,$$
$$k^{p^b} \equiv 1\ (p^a), \quad p^c(k-1) \equiv 0\ (p^a)\rangle$$

splits when $a = c$ [6]. The center of G and the commutator subgroup G^1 are generated by A^{p^d} and $A^{p^{a-d}}$ respectively, where $d = \text{minimum}\ (b, c)$. G can be given in terms of either of the following two extensions:

$$1 \to \mathbf{Z}_{p^a}\langle A\rangle \to G \xrightarrow{\pi} \mathbf{Z}_{p^b}\langle \bar{B}\rangle \to 1, \tag{1}$$

$$1 \to \mathbf{Z}_{p^d}\langle A^{p^{a-d}}\rangle \to G \to \mathbf{Z}_{p^{a-d}}\langle \bar{A}\rangle + \mathbf{Z}_{p^b}\langle \bar{B}\rangle \to 1. \tag{2}$$

Let $\lambda\colon A \to e^{2\pi i/p^a} = \xi$ and $\pi^!\lambda'\colon B \to e^{2\pi i/p^b}$, $A \to 1$ be two 1-dimensional representations of the subgroup $\mathbf{Z}_{p^a}\langle A\rangle$ and the group G respectively. G acts on the product of two spheres $S^{2p^b-1} \times S^{2p^b-1}$ by $i_!\lambda \oplus p^b(\pi^!\lambda')$ where $i_!\lambda$ is induced representation of λ and $p^b(\pi^!\lambda')$ is the direct sum of p^b copies of $\pi^!\lambda'$. We know $1 \otimes 1$, $B \otimes 1, \ldots, B^{p^b-1} \otimes 1$ form a basis for the induced module associated with $i_!\lambda$. Then, by [3, p. 75],

$$i_!\gamma(A) = \begin{bmatrix} \xi & & & 0 \\ & \xi^k & & \\ & & \ddots & \\ 0 & & & \xi^{k^{p^b-1}} \end{bmatrix} \quad \text{and} \quad i_!\lambda(B) = \begin{bmatrix} 00 & \cdots & 0\xi^{p^c} \\ 10 & \cdots & 00 \\ \vdots\vdots & & \vdots\vdots \\ 00 & \cdots & 10 \end{bmatrix}.$$

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The characteristic value of $i_! \lambda(B)$ never equals 1. Thus we have:

PROPOSITION 1. *The group G acts freely on the product of two spheres $S^{2p^b-1} \times S^{2p^b-1}$.* ∎

G acts on the sphere S^{2p^b-1}, by $i_! \lambda$, with A acting freely. We have

$$S^{2p^b-1} = S^1 * \cdots * S^1 \quad (p^b\text{-fold join}).$$

Consider the complex

$$C(S^{2p^b-1}) = \{C_0 \leftarrow C_1 \cdots \leftarrow C_{p^b-1} \leftarrow \cdots \leftarrow C_{2p^b-1}\}.$$

By [7, 6.2], C_i is a free G-module except for C_0, C_1, C_{p^b-1} and C_{2p^b-1}, where $C_0 \cong \mathbf{Z}G$, $C_1 \cong \mathbf{Z}G \oplus F$,

$$C_{p^b-1} \cong \mathbf{Z}G/\langle B^{p^{a-d}} \rangle \oplus F \quad \text{and} \quad C_{2p^b-1} \cong \mathbf{Z}G/\langle B^{p^{a-d}} \rangle \oplus F$$

for some free G-module F. Consider the sequence

$$0 \leftarrow \mathbf{Z} \leftarrow C_0 \leftarrow \cdots \leftarrow C_{2p^b-1} \leftarrow \mathbf{Z} \leftarrow 0.$$

By applying Tate cohomology to the exact sequences of the image-kernels X, Y, V, W, U at C_0, C_1, C_{p^b-2}, C_{p^b-1}, C_{2p^b-1} respectively, the following sequences are exact for odd i:

$$\begin{aligned} 0 &\to H^i(G, V) \to H^{i+1}(G, W) \to H^{i+1}(\langle B^{p^{a-d}} \rangle, \mathbf{Z}) \\ &\to H^{i+1}(G, V) \to H^{i+2}(G, W) \to 0, \\ 0 &\to H^i(G, U) \to H^{i+1}(G, \mathbf{Z}) \to H^{i+1}(\langle B^{p^{a-d}} \rangle, \mathbf{Z}) \\ &\to H^{i+1}(G, V) \to H^{i+2}(G, W) \to 0, \end{aligned}$$

and

$$\begin{aligned} H^{i+1}(G, \mathbf{Z}) &\cong H^{i+1}(G, X),\ H^{i+1}(G, \mathbf{Z}) \\ &\cong H^{i+2}(G, X),\ H^i(G, X) \\ &\cong H^{i+1}(G, Y),\ H^{i+1}(G, X) \\ &\cong H^{i+2}(G, Y) \quad \text{for odd } i \end{aligned}$$

By dimension shifting,

$$\begin{aligned} &H^i(G, Y) \cong H^{i+p^b-3}(G, V) \text{ and} \\ &H^i(G, W) \cong H^{i+p^b-1}(G, U) \quad \text{for all } i. \end{aligned}$$

Similarly, there are exact sequences for even i. Then,

$$\begin{aligned} |H^{i+2}(G, \mathbf{Z})| &\le |H^{i+1}(G, U)| \\ &= |H^{i-p^b+2}(G, W)| \\ &\le p^d |H^{i-p^b+1}(G, V)| \end{aligned}$$

$$= p^d\,|H^{i-2p^b+4}(G, Y)|$$
$$\leq p^d\,|H^{i-2p^b+3}(G, X)|$$
$$\leq p^d\,|H^{i-2p^b+2}(G, \mathbf{Z})|.$$

Thus the following lemma holds:

LEMMA 2. $|H^{j+2p^b}(G, \mathbf{Z})| \leq p^d\,|H^j(G, \mathbf{Z})|$ *for all* j. ▮

PROPOSITION 3. $H^*(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, \mathbf{Z}) = P[\alpha, \beta] \otimes E[\delta]$ *where*

$$\deg \alpha = \deg \beta = 2,$$

$\deg \delta = 3$ *and* $p^{a-d}\alpha = p^b\beta = p^{a-d}\delta = 0$ *with the relation* $\delta^2 = 0$.

Proof. The spectral sequence

$$E_2^{i,j} = H^i(\mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^{a-d}}, \mathbf{Z}))$$

of the split group extension $1 \to \mathbf{Z}_{p^{a-d}} \to \mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b} \to \mathbf{Z}_{p^b} \to 1$ is convergent to

$$H^{i+j}(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, \mathbf{Z}).$$

We have

$$E_2^{0,*} = H^*(\mathbf{Z}_{p^{a-d}}, \mathbf{Z}) = P[\alpha] \quad \text{where } \deg \alpha = 2,\ p^{a-d}\alpha = 0;$$
$$E_2^{*,0} = H^*(\mathbf{Z}_{p^b}, \mathbf{Z}) = P[\beta] \quad \text{where } \deg \beta = 2,\ p^b\beta = 0;$$
$$E_2^{1,2} = H^1(\mathbf{Z}_{p^b}, H(\mathbf{Z}_{p^{a-d}}, \mathbf{Z})) = H^1(\mathbf{Z}_{p^b}, \mathbf{Z}_{p^{a-d}}) = \mathbf{Z}_{p^b}\,\delta$$

where $\deg \delta = 3$. Since $\deg \delta$ is odd, $\delta^2 = 0$. Thus

$$E_2^{*,0} = \sum_1^\infty \mathbf{Z}\beta^i, \quad E_2^{0,*} = \sum_1^\infty \mathbf{Z}\alpha^i, \quad E_2^{*,2} = \sum_1^\infty (\mathbf{Z}\beta^i\alpha + \mathbf{Z}\beta^i\delta);$$

and

$$\beta\colon E_2^{i,j} \to E_2^{i+2,j} \quad (i, j \geq 0), \qquad \alpha\colon E_2^{i,j} \to E_2^{i,j+2} \quad (i \geq 0, j > 0)$$

are isomorphisms by periodicity [2.XII, §6]. Since the extension is split, $E_2 = E_\infty$ and α, β, δ survive to E_∞ [10]. Therefore

$$H^*(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, \mathbf{Z}) = P[\alpha, \beta] \oplus E[\delta]. \qquad ▮$$

3. Integral cohomology rings

Consider the spectral sequence of extension (1):

$$E_2^{i,j} = H^i(\mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^a}, \mathbf{Z})).$$

We have $H^*(\mathbf{Z}_{p^a}, \mathbf{Z}) = P[\alpha]$ where $\deg \alpha = 2$ and $p^a\alpha = 0$. α is a maximal generator corresponding to $A \to 1/p^a$. The action of $\mathbf{Z}_{p^b}\langle t\rangle$ on $H^*(\mathbf{Z}_{p^a}, \mathbf{Z})$

induced by B is given by $t\alpha = k\alpha$. We have

$$E_2^{0,*} = H^*(\mathbf{Z}_{p^a}, \mathbf{Z})^{\mathbf{Z}_{p^b}\langle t\rangle},$$
$$E_2^{*,0} = H^*(\mathbf{Z}_{p^b}, \mathbf{Z}) = P[\beta] \quad \text{where } \deg \beta = 2,\ p^b\beta = 0$$

and

$$E_2^{1,2p^b} = H^1(\mathbf{Z}_{p^b}, H^{2p^b}(\mathbf{Z}_{p^a}, \mathbf{Z})) = \mathbf{Z}_\eta \quad \text{where } \deg \eta = 2p^b + 1,\ p^b\eta = 0.$$

PROPOSITION 4. *$H^{2i}(G, Z) \cong \mathbf{Z}_{p^{a-d}} + \mathbf{Z}_{p^d}$ and $H^{2i+1}(G, \mathbf{Z}) \cong 0$ for $1 \le i < p^b$.*

Proof. We have

$$H^2(G, \mathbf{Z}) \cong \text{Hom}\,(G/G^1, Q/\mathbf{Z}) \cong \mathbf{Z}_{p^{a-d}}\langle\alpha_1\rangle + \mathbf{Z}_{p^b}\langle\beta\rangle$$

where α_1 and β are maximal generators of $H^2(G, \mathbf{Z})$ corresponding to α_1: $\bar{A} \to 1/p^{a-d}$, $\bar{B} \to 0$ and β: $\bar{A} \to 0$, $\bar{B} \to 1/p^b$ respectively. Also α_1 and β correspond to $p^d\alpha$ and β in E_2 term. We have

$$\text{Res Cor}\,(\alpha) = N(\alpha) = (1 + t + \cdots + t^{p^b-1})\alpha = (k^{p^b-1})/(k-1),\ \alpha = p^d\alpha = \alpha_1$$
[2,XII, §8].

Then $\alpha_1 = \text{Cor}\,\alpha$ and $\text{Cor}\,(\text{Res}\,\beta \cdot \alpha) = \beta\,\text{Cor}\,\alpha = 0$. Thus, $\alpha_1\beta = 0$. By the Res-Cor sequence [7, §2] the following sequence is exact:

$$0 \to H^2(H, \mathbf{Z})_t \xrightarrow{\phi} T^3 \xrightarrow{\varepsilon} H^3(H, \mathbf{Z})^t \to 0$$

where $H = \langle A\rangle$ is a normal subgroup of G. Since $H^3(H, \mathbf{Z}) \cong 0$, $T^3 = H^2(H, \mathbf{Z})_t$. We have

$$|T^3| = |H^2(H, \mathbf{Z})_t| = |H^2(H, \mathbf{Z})/(t-1)H^2(H, \mathbf{Z})| = p^{a-d}.$$

The sequence

$$0 \to H^3(G, \mathbf{Z}) \xrightarrow{\rho} T^3 \xrightarrow{\tau} H^2(G, \mathbf{Z}) \xrightarrow{\cup\beta} H^4(G, \mathbf{Z})^{\cdot}$$

is exact and $\text{Cor}_2 = \tau\phi$, $\text{Res}_2 = \varepsilon\rho$ [7]. Since $\alpha_1\beta = 0$, $|\text{Ker} \cup \beta| = p^{a-d}$. We have $|I_m\tau| = |\text{Ker} \cup \beta| = p^{a-d} = |T^3|$. Therefore $|H^3(G, \mathbf{Z})| = 1$, and hence $H^3(\text{G}, \mathbf{Z}) \cong 0$.

Since ϕ is an isomorphism and $\text{Res}\,\beta = 0$, we have $|I_m\,\text{Res}| = |I_m\tau| = p^{a-d}$. The following two sequences are exact [7, proposition 2.1]:

$$H^2(H, \mathbf{Z}) \xrightarrow{\text{Cor}} H^2(G, \mathbf{Z}) \to H^3(G, X) \to 0,$$

$$0 \to H^3(G, X) \to H^4(G, \mathbf{Z}) \xrightarrow{\text{Res}} H^4(H, \mathbf{Z}),$$

where $X = \text{Ker}\,\{\mathbf{Z}\langle\bar{B}\rangle \to \mathbf{Z}\}$. Thus, $|H^3(G, X)| = p^b$ and $|H^4(G, \mathbf{Z})| = p^{a-d} \times p^b$. Therefore

$$H^4(G, \mathbf{Z}) = \mathbf{Z}_{p^{a-d}}\langle\alpha_2\rangle + \mathbf{Z}_{p^b}\langle\beta^2\rangle.$$

Similarly, $\alpha_2 = \mathrm{Cor}\ \alpha^2 = p^d\alpha^2$ and $\alpha_2\beta = 0$. Then, by induction,

$$H^{2i}(G, \mathbf{Z}) \cong \mathbf{Z}_{p^{a-d}}\langle\alpha_i\rangle + \mathbf{Z}_{p^b}\langle\beta^i\rangle \quad \text{and} \quad H^{2i+1}(G, \mathbf{Z}) \cong 0$$

for $1 \leq i < p^b$ where $\alpha_i = \mathrm{Cor}\ \alpha^i = p^d\alpha^i$ and $\alpha_i\beta = 0$. Moreover $\alpha_i\alpha_j = 0$ for all i,j because if $\alpha_i\alpha_j = e\beta^{i+j}$ then $\beta\alpha_i\alpha_j = e\beta^{i+j+1} = 0$ and $e = 0$. ∎

Now, consider the spectral sequence of extension (2):

$$E_2^{i,j} = H^i(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^d}, \mathbf{Z})).$$

The action of $\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}$ on $H^j(\mathbf{Z}_{p^d}, \mathbf{Z})$ is trivial since A^{p^d} generates the centre of G. Then, by Proposition 3,

$$E_2^{*,0} = H^*(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, \mathbf{Z}) = P[\alpha, \beta] \otimes E[\delta]$$

where $\deg \alpha = \deg \beta = 2$, $\deg \delta = 3$ and $p^{a-d}\alpha = p^b\beta = p^{a-d}\delta = 0$; and

$$E_2^{0,*} = H^*(\mathbf{Z}_{p^d}, \mathbf{Z}) = P[\gamma]$$

where $\deg \gamma = 2$ and $p^d\gamma = 0$. By comparing the two spectral sequences, $\alpha^i \leftrightarrow \alpha_i$ and

$$H^{2i}(G, \mathbf{Z}) \cong \mathbf{Z}_{p^{a-d}}\langle\alpha^i\rangle + \mathbf{Z}_{p^b}\langle\beta^i\rangle$$

with the relation $\alpha^i\beta^i = 0$ for $1 \leq i < p^b$. By Küneth's formula,

$$E_2^{*,2j} = H^*(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, \mathbf{Z}_{p^d}) \cong H^*(\mathbf{Z}_{p^{a-d}}, \mathbf{Z}_{p^d}) \otimes H^*(\mathbf{Z}_{p^b}, \mathbf{Z}_{p^d}), \quad j > 0.$$

This induces a horizontal multiplication

$$o\colon E_2^{i,2j} \times E_2^{l,2j} \to E_2^{i+l,2j}, \quad j > 0 \ [7, 6.3].$$

For $\gamma \varepsilon E_2^{0,2}$, $\gamma\colon E_2^{i,j} \to E_2^{i,j+2}$ is a monomorphism for $j \geq 0$, and an isomorphism for $j > 0$, by periodicity [2]. By the double cosets formula of generalization of corestriction [4, Theorem 3], $\mathrm{Res}_{\langle A\rangle}\ \mathcal{N}(\gamma) = \gamma^{p^b}$. Then γ^p survives to E_∞ [7, corollary II] and

$$\gamma^p\colon E_2^{i,j} \to E_2^{i,j+2p^b}$$

is an isomorphism for $j > 0$. If $\mu, \upsilon \in E_2^{1,2}$ are two independent generators then $\chi = \mu \circ \nu \in E_2^{2,2}$ is a new generator, We have $E_2 = E_3$ since the odd rows are zero. Then the additive structure of E_2 is given as follows:

LEMMA 5.

$$E_2^{2n,0} = \mathbf{Z}\alpha^n + \mathbf{Z}\beta^n,$$

$$E_2^{2n,2} = \mathbf{Z}\chi\alpha^{n-1} + \mathbf{Z}\chi\beta^{n-1} + \mathbf{Z}\gamma\alpha^n + \mathbf{Z}\gamma\beta^n,$$

$$E_2^{2n+1,0} = \mathbf{Z}\delta\gamma^{n-1} + \mathbf{Z}\delta\beta^{n-1},$$

$$E_2^{2n+1,2} = \mathbf{Z}\mu\alpha^n + \mathbf{Z}\upsilon\beta^n, \quad E_2^{*,2m+1} = 0,$$

where $0 < m < p^b$ *and* $E_2^{1,2p^b} = \mathbf{Z}\eta$. ∎

The other terms are given by periodicity: $E_2^{*,2} \overset{\gamma}{=} E_2^{*,4} \overset{\gamma}{=} \cdots$. Furthermore,

$$J_*\colon H^i(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^b}, \mathbf{Z})) \to H^i(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^b}, H^j(\mathbf{Z}_{p^d}, \mathbf{Z}_{p^d}))$$

is induced by the projection $J\colon \mathbf{Z} \to \mathbf{Z}_{p^d}$. J_* is a monomorphism for $j \geq 0$, and an isomorphism for j even and greater than zero. We have

$$H^*(\mathbf{Z}_{p^{a-d}} \times \mathbf{Z}_{p^d}, \mathbf{Z}_{p^d}) = E[U_1, U_2] \otimes P[V_1, V_2] \quad \text{with } \Delta U_i = V_i,\ i = 1, 2,$$

where Δ is the Bockstein homomorphism. Also, $J_*(\alpha) = V_1$, $J_*(\beta) = V_2$, $J_*(\delta) = V_1 U_2 - V_2 U_1$, $J_*(\mu) = J_*(\gamma)U_2$, $J_*(\nu) = J_*(\gamma)U_1$, $J_*(\chi) = J_*(\gamma)U_1 U_2$ and J_* preserves the product. We have

$$J_*(\delta\mu) = (V_1 U_2 - V_2 U_1)J_*(\gamma)U_2 = -V_2 U_1 U_2 J_*(\gamma) = -J_*(\beta x);$$

i.e., $\delta\mu = -\beta x$. Similarly we have the following:

LEMMA 6.

$$\delta\mu = -\beta\chi, \quad \delta\nu = -\alpha\chi;$$

$$\delta\chi = 0, \quad \mu\chi = \nu\chi = 0, \quad \mu^2 = \nu^2 = \chi^2 = 0. \quad ∎$$

Since $H^2(G, \mathbf{Z}) \cong \mathbf{Z}_\alpha + \mathbf{Z}_\beta$, $d_3(\alpha) = d_3(\beta) = 0$ and $d_3 = \delta$ (That is, $d_3(\gamma) = s\delta$ for $s \not\equiv 0\ (p^{a-d})$). We have

$$d_3(\mu) = \alpha^2 + \beta^2 = d_3(\nu)$$

since $H^3(G, \mathbf{Z}) \cong 0$;

$$d_3(\chi) \doteq \alpha^3 + \beta^3 \quad \text{for } H^4(G, \mathbf{Z}) \cong \mathbf{Z}_{\alpha^2} + \mathbf{Z}_{\beta^2};$$

$$E_4^{2n,0} = \mathbf{Z}_{\alpha^n} + \mathbf{Z}_{\beta^n}$$

because $\operatorname{Ker} d_3 = \{\alpha^n, \beta^n\}$; and $I_m d_3 = 0$ since $\alpha_i\alpha_j = 0$ and $\alpha_i\beta = 0$. Furthermore, $E_4^{1,2m} = 0$ for $m \neq 1\ (p^b)$, $m > 1$, because $\operatorname{Ker} d_3 = 0$ for $E_3^{1,2m} = \mathbf{Z}_{\mu\gamma^{m-1}} + \mathbf{Z}_{\nu\gamma^{m-1}}$. Similarly, by Lemmas 5 and 6, the additive structure of E_4 is given as follows:

LEMMA 7.

$$E_4^{2n,0} = \mathbf{Z}_{\alpha^n} + \mathbf{Z}_{\beta^n};$$

$$E_4^{1,2m} = 0 \quad \text{for } m \not\equiv 1\ (p^b),\ m > 1;$$

$$E_4^{2n,2m} = 0, \quad m > 0,\ m \not\equiv 0\ (p^b);$$

$$E_4^{2n+1,2m} = 0, \quad m \not\equiv -1\ (p^b),\ m \neq 1,\ n > 0;$$

$$E_4^{2n+1,2} = 0; \quad E_4^{2n+1,2(p^b-1)} = 0; \quad E_4^{1,2p^b} = \mathbf{Z}_\eta. \quad ∎$$

The other terms are given by periodicity:

$$E_4^{*,j} \overset{\gamma^p}{=} E_4^{*,j+2p^b} = \cdots, \quad j > 0.$$

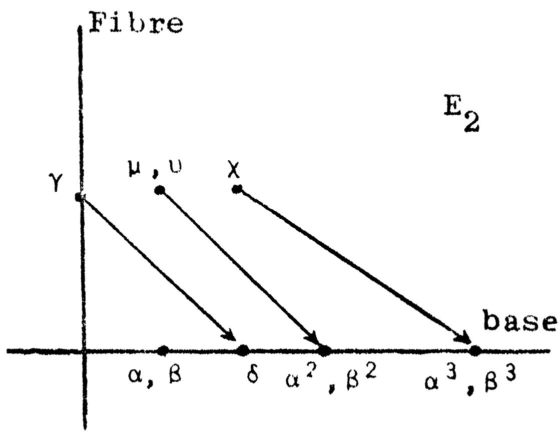
Thus, $E_4 = E_\infty$ in dimensions less than or equal to $2p^b + 1$.

LEMMA 8. $|H^{2p^b}(G, \mathbf{Z})| = p^{a+b}$.

Proof. By Proposition 4, $H^{2p^b-1}(G, \mathbf{Z}) = 0$. By Lemma 1, G acts freely on the product of the two spheres $S^{2p^b-1} \times S^{2p^b-1}$. Then, by [8, Corollary 2.7] the following sequence is exact:

$$0 \to H^{2p^b}(G, \mathbf{Z}) \to \mathbf{Z}_{p^{a+b}} \times \mathbf{Z}_{p^{a+b}} \to H^{2p^b}(G, \mathbf{Z}) \to 0.$$

Therefore, $|H^{2p^b}(G, \mathbf{Z})| = p^{a+b}$. ∎



By Res-Cor sequences, the following two sequences are exact:

$$0 \to H^{2p^b}(G, X) \to H^{2p^b}(H, \mathbf{Z}) \overset{\text{Cor}}{\to} H^{2p^b}(G, \mathbf{Z}) \to H^{2p^b+1}(G, X) \to 0,$$

$$0 \to H^{2p^b-1}(G, X) \to H^{2p^b}(G, \mathbf{Z}) \overset{\text{Res}}{\to} H^{2p^b}(H, \mathbf{Z}) \to H^{2p^b}(G, X) \to 0.$$

We have $|\text{Im Res}| = p^{a-d}$. Then

$$0 \to \mathbf{Z}_{p^{a-d}} \to H^{2p^b}(H, \mathbf{Z}) \to H^{2p^b}(G, \mathbf{Z}) \to 0$$

is exact. Thus, $|H^{2p^b}(G, X)| = p^d$. If Cor is zero then $H^{2p^b}(H, \mathbf{Z}) = \mathbf{Z}_{p^b}$ which is a contradiction. Therefore Cor $(\alpha^{p^d}) \neq 0$. Let $\xi = \mathcal{N}(\alpha)$. $\text{Res}_A \mathcal{N}(\alpha) = \alpha^{p^b}$ [4, Theorem 3] and Cor Res $\mathcal{N}(\alpha) = p^b \mathcal{N}(\alpha) = $ Cor $(\alpha^{p^b}) \neq 0$. Then $\mathcal{N}(\alpha)$ has additive order p^a and $\mathcal{N}(\alpha) \in H^{2p^b}(G, Z)$. We have $\alpha^{p^c} \neq 0$ in $H^*(G, \mathbf{Z})$ because if $\alpha^{p^c} = 0$ then Cor $(\alpha^{p^c}) = 0$. Then α^{p^c} must be given in terms of powers of β. We have $\alpha^{p^c} = p^{b-a}\beta^{p^c}$; i.e., $p^c\alpha^{p^c} = p^{b+c-a}\beta^{p^c}$. By Lemmas 2 and 8,

$$|H^{2p^b+2}(G, \mathbf{Z})| \leq p^d |H^2(G, \mathbf{Z})| = p^{a+b}.$$

Then,

$$|H^{2p^b+2}(G, \mathbf{Z})| = |H^{2p^b}(G, \mathbf{Z})| = p^{a+b}.$$

THEOREM 9. *We have the integral cohomology ring*

$$H^*(G, Z) = Z[\beta; \alpha_1, \ldots, \alpha_{p^b-1}; \xi, \eta]$$

where $\deg \beta = 2$, $\deg \alpha_i = 2i$, $\deg \xi = 2p^b$, $\deg \eta = 2p^b + 1$ *and* $p^b\beta = p^{a-d}\alpha_i = p^a\xi = p^b\eta = 0$, *with the relations* $\alpha_i\beta = 0, \alpha_i\alpha_j = 0, \alpha_i\eta = 0, \eta^2 = 0$ *for all i, j and* $p^c\alpha^{p^c} = p^{c+b-a}\beta^{p^c}$. *Furthermore,* α *is a maximal generator in* $H^2(\langle A\rangle, Z)$, $\alpha_i = \mathrm{Cor}\ (\alpha^i)$, $1 \le i < p^b$, *and* $d = \min\ (b, c)$. ▮

Now, it is possible to deduce the integral cohomology ring of a split metacyclic p-group as a special case when $a = c$. The relation $p^c\alpha^{p^c} = p^{c+b-a}\beta^{p^c}$ will be satisfied. This provides an explanation of the fact that not all the terms on the base of the spectral sequence of extension (2) survive to E_∞.

$H^{\mathrm{even}}(G, Z)$ is generated by $\beta; \alpha_1, \ldots, \alpha_{2p^b-1}; \xi$. $\beta = c_1(\hat{\beta})$ is the first Chern class of the 1-dimensional representation given by $\hat{\beta}(B) = 1/p^b$. $\hat{\beta}$ corresponds to the maximal generator of $H^2(\langle B\rangle, Z)$. We have

$$\mathrm{Res}\ (c_i(i_!\hat{\alpha})) = c_i(p^b\hat{\alpha}) = \binom{p^b}{i}\alpha^i$$

where $\alpha = c_1(\hat{\alpha})$ and $\hat{\alpha}(A) = 1/p^a$; $\hat{\alpha}$ corresponds to the maximal generator of $H^2(\langle A\rangle, Z)$. The binomial coefficient

$$\binom{p^b}{i}$$

is divisible by p^b, but by no higher power of p^b. So $c_i(i_!(\hat{\alpha}))$ generates the same summand as α_i, $1 \le i < p^b$. By [5, Theorem 4], $\xi = c_{p^b}(i_!\hat{\alpha})$. By [1, Appendix], we have:

THEOREM 10. $H^{\mathrm{even}}(G, Z)$ *is generated by Chern classes and hence G satisfies Atiyah's conjecture.* ▮

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