ILLINOIS JOURNAL OF MATHEMATICS
Volume 26, Number 3, Fall 1982

DETERMINING SETS FOR MEASURES ON $\mathbf{R}^n$

BY

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1. Introduction

Let M be a class of measures on $\mathbf{R}^n$. A Borel set E is said to be a determining set for M if $\mu, \nu \in M$, and $\mu(x+E) = \nu(x+E)$ for all $x \in \mathbf{R}^n$ implies $\mu = \nu$.

Let $\Gamma = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n;\ x_i \geq 0 \text{ for all } i\}$. Then it is well known to probabilists that Γ is a determining set for the class P of all probability measures on $\mathbf{R}^n$. (For a Fourier transform theoretic proof of this, for $n = 2$, see [4].)The aim of this paper is to generalize the above result to an arbitrary Borel set E of positive Lebesgue measure contained in Γ (see Theorem 3.3). The proof of this theorem is based on Proposition 3.1 which is very similar to the results in [4]. (For a discussion of determining sets in the context of locally compact abelian groups or symmetric spaces see [2].)

2. Notation and terminology

For any unexplained notation or terminology please see [3].

Throughout this paper λ denotes the Lebesgue measure on $\mathbf{R}^n$. Let C denote the class of all (finite) complex measures and P the class of all probability measures on $\mathbf{R}^n$. If T is a tempered distribution (in the sense of Schwartz), then $\hat{T}$ denotes the Fourier transform of T (which is again a distribution) and Supp T denotes the (closed) support of T. For standard facts regarding distributions, Fourier transforms etc., see [3]. If g is a bounded Borel function on $\mathbf{R}^n$, then g defines a tempered distribution (see [3]) and $\hat{g}$ will denote the (distributional) Fourier transform of g. If μ is a finite complex measure, then $\mu * g$ is the bounded Borel function defined by

$$(\mu * g)(x) = \int_{\mathbf{R}^n} g(x-y)\, d\mu(y)$$

Finally, we note that for a complex measure or an L^1-function the usual notion of Fourier transform coincides with the notion of distributional Fourier transform.

If M is a class of measures on $\mathbf{R}^n$, a Borel set E of $\mathbf{R}^n$ is said to be a "determining set" for M if $\mu, \nu \in M$ and $\mu(x+E) = \nu(x+E)$ for all $x \in \mathbf{R}^n$ implies $\mu = \nu$.

Received September 4, 1980.

Γ will always stand for the subset of $\mathbf{R}^n$ defined by

$$\Gamma = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n;\ x_i \geq 0 \text{ for all } i\}.$$

If $E \subseteq \mathbf{R}^n$, let 1_E denote the indicator function of E; i.e., $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ if $x \notin E$.

We end this section by quoting a result that will be needed in the next section.

THEOREM 2.1. *Let f be a bounded measurable function and $K \in L^1(\mathbf{R}^n)$. If $K * f$ vanishes identically, then $\hat{K}$ vanishes on* Supp $\hat{f}$.

(*Note.* For a proof of this theorem we refer to page 232 of [1]. Note that Supp $\hat{f}$ is called "spectrum of f" in [1]. The theorem is a consequence of a theorem of Beurling on the spectrum of a bounded distribution—see page 230 of [1].)

3. The main results

As Sapagov has already observed in [4], if E is a Borel set of positive finite measure and Supp $\hat{1}_E = \mathbf{R}^n$, then E is a determining set for P. We prove that this continues to be true even if we assume $\lambda(E) = \infty$.

PROPOSITION 3.1. *Let E be a Borel set of positive Lebesgue measure in $\mathbf{R}^n$ (i.e., $0 < \lambda(E) \leq \infty$). If* Supp $\hat{1}_E = \mathbf{R}^n$, *then E is a determining set for C.*

Proof. If $\mu, \nu \in C$ and $\mu(x + E) = \nu(x + E)$ for all $x \in \mathbf{R}^n$, then $\check{\mu} * 1_E = \check{\nu} * 1_E$, where $\check{\mu}(A) = \mu(-A)$. Let f be an arbitrary L^1-function. Then we have $f * (\check{\mu} * 1_E) = f * (\check{\nu} * 1_E)$. Now, an easy Fubini argument shows $f * (\check{\mu} * 1_E) = (f * \check{\mu}) * 1_E$ and so we will have $(f * (\check{\mu} - \check{\nu})) * 1_E = 0$. But $f * (\check{\mu} - \check{\nu})$ is an L^1-function, and hence, by Theorem 2.1, $(f * (\check{\mu} - \check{\nu}))^\wedge$ vanishes on Supp $\hat{1}_E = \mathbf{R}^n$. Hence, $f * (\check{\mu} - \check{\nu}) = 0$ a.e on $\mathbf{R}^n$. However, this is true for an arbitrary f in $L^1(\mathbf{R}^n)$. Hence, $\check{\mu} - \check{\nu} = 0$, so $\check{\mu} = \check{\nu}$, $\mu = \nu$, and the proof of the proposition is complete.

The next proposition is probably well known in the folklore—the proof we give here is due to H. Helson.

PROPOSITION 3.2. *If E is a Borel set of $\mathbf{R}^n$ contained in Γ with $0 < \lambda(E) \leq \infty$, then* Supp $\hat{1}_E = \mathbf{R}^n$.

Proof. If $0 \neq f \in L^1(\mathbf{R}^n)$ and Supp $f \subseteq \Gamma$, then it follows from the definition that $\hat{f}$ can be extended to a bounded function g in the region

$$H = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n;\ \text{Im } z_i \leq 0 \text{ for all } i\}.$$

g will be analytic in

$$H_0 = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n;\ \text{Im } z_i < 0 \text{ for all } i\}$$

and continuous in H. Thus $\hat{f}$ is the "boundary value" of a bounded analytic function in H_0 and consequently $\hat{f}$ cannot vanish identically on an open subset of $\mathbf{R}^n$; i.e., Supp $\hat{f} = \mathbf{R}^n$. To prove the proposition, we prove the slightly more general result that if $0 \neq h$ is a bounded Borel function with Supp $h \subseteq \Gamma$, then Supp $\hat{h} = \mathbf{R}^n$. To see this, let us assume $\hat{h}$ vanishes on a nonempty open set U of $\mathbf{R}^n$; i.e., (Supp $\hat{h}$) $\cap U = \phi$. Let $x_0 \in U$. Choose ε sufficiently small such that the open ball of radius 2ε with centre at x_0 is contained in U. Let $0 \neq h_1$ be a function in $L^1(\mathbf{R}^n)$ such that $\hat{h}_1$ is a C^∞ function and Supp $\hat{h}_1$ is contained in the ball of radius ε around 0. Then

$$\text{Supp } (hh_1)^\wedge = \text{Supp } (\hat{h} * \hat{h}_1) \subseteq \text{Supp } \hat{h} + \text{Supp } \hat{h}_1.$$

So, if $U' = \{x; \|x - x_0\| < \varepsilon\}$ then Supp $(hh_1)^\wedge \cap U' = \phi$. However, hh_1 is an L^1-function with Supp $hh_1 \subseteq \Gamma$, and, by the first part of our proof, hh_1 must be zero almost everywhere on $\mathbf{R}^n$. Since $\hat{h}_1$ is a C^∞-function of compact support, h_1 is the restriction of an entire function to $\mathbf{R}^n$, and hence $h_1(x) \neq 0$ a.e. x. Thus, h is zero almost everywhere which gives us a contradiction, and the proof of our proposition is complete.

Propositions 3.1 and 3.2 together imply the following theorem:

THEOREM 3.3. *Let E be a Borel subset of Γ. If $0 < \lambda(E) \leq \infty$, then E is a determining set for C.*

Remarks. (1) The method of proof of Proposition 3.2 can be modified to prove the following slightly more general result: If T is a tempered distribution, $T \neq 0$ and Supp $T \subseteq \Gamma$, then Supp $\hat{T} = \mathbf{R}^n$.

(2) Theorem 3.3 can be generalised slightly. Thus, for $n = 2$, we can replace Γ by the region between two half lines, where the angle between the half lines is strictly less than π. (Γ would correspond to the case of $\pi/2$.)

(3) Proposition 3.1 is much in the same spirit as the following result proved in [4]: If Supp $\hat{1}_E$ contains a nonempty open set then E is a determining set for the class P_c of probability measures of compact support.

(4) For a discussion of determining sets for measures of polynomial growth and its connection with the Wiener-Tauberian theorem see [5].

Acknowledgements. We thank Professor H. Helson for several helpful discussions. In particular the proof of Proposition 3.2 is due to him. We also thank R. L. Karandikar and M. G. Nadkarni for useful conversations.

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