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ON EXTREME POINTS

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This note contains a proof of the following:

THEOREM. *Let E be a non-reflexive real Banach space. There exist closed bounded convex sets A, C_1, C_2 in E with the following properties:*

(a) *The point 0 is an exposed point of A.*

(b) *The point 0 is not an extreme point of B, the weak* closure of A in the second dual E^{**}. If E is not weakly sequentially complete, 0 is in fact the average of two exposed points of B.*

(c) *The point 0 is not in the convex hull of $C_1 \cup C_2$, but it is an exposed point of the closed convex hull of $C_1 \cup C_2$.*

Recall [1, V.1. (8)] that a point x of a convex set A is *exposed* if there is a continuous linear functional f such that $f(x) < f(y)$ for all $y \in A$, $y \neq x$. In such a case we say that f (or $-f$) *exposes* $x \in A$.

Proof. Case 1. Suppose that E is not weakly sequentially complete. Then there is a sequence $\{z_n\}$ in E which is weak* convergent in E^{**} to an element $\tilde{x}$ not in E. We choose now two linear functionals $g, h \in E^*$ as follows: first, $g \neq 0$ and $g(\tilde{x}) = 0$; pick $a \in E$ such that $g(a) = 1$ and choose h such that $h(\tilde{x}) = 1$, $h(a) = 0$.

Observe that $h(z_n) \to h(\tilde{x}) = 1$ and therefore by ignoring a finite number of terms we can (and will) assume that $h(z_n) \geq \frac{1}{2}$ for all $n \geq 1$.

Define

$$\alpha_n = |g(z_n)| + 1/n$$

$$\beta_n = (h(z_n) + 1/n)^{-1}$$

$$x_n = \beta_n(z_n + \alpha_n a)$$

$$y_n = \beta_n(-z_n + \alpha_n a).$$

It is easy to see that for each $n \geq 1$,

(1) $$g(x_n) > 0, \quad g(y_n > 0),$$

(2) $$\tfrac{1}{3} \leq h(x_n) < 1, \quad -1 < h(y_n) < -\tfrac{1}{3},$$

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and

$$\text{(3)} \qquad \text{weak* } \lim x_n = \tilde{x}, \quad \text{weak* } \lim y_n = -\tilde{x}.$$

Now, define the set A as the closed convex hull in E of

$$\{x_1, x_2, \ldots, y_1, y_2, \ldots\},$$

and denote the weak* closure of A in E^{**} by B. Since $\frac{1}{2}(x_n + y_n) \to 0$ in norm, it is clear that $0 \in A$.

Since, by definition, B is the weak* closed convex hull of the weak* compact set $K = \{\tilde{x}, -\tilde{x}, x_1, x_2, \ldots, y_1, y_2, \ldots\}$, the extreme points of B all belong to K by [1, V.1. Theorem 3]. Let $B_1 = \{\tilde{y} \in B;\ g(\tilde{y}) = 0\}$. Then B_1 is also weak* compact and convex, and all its extreme points are extreme points of B. However, by (1) the only points in K where g vanishes are $\tilde{x}$ and $-\tilde{x}$ and so by the Krein-Milman theorem,

$$B_1 = \{v\tilde{x};\ -1 \le v \le 1\}.$$

Hence $\{x \in A;\ g(x) = 0\} = B_1 \cap A = \{0\}$ and since $g \ge 0$ on A by (1), it follows that $0 \in A$ is exposed by g. This proves (a).

Now, using (2) instead of (1) and h instead of g, the same argument shows that h exposes, both $\tilde{x} \in B$ and $-\tilde{x} \in B$. Hence, $0 = \frac{1}{2}(\tilde{x} + (-\tilde{x}))$ is the average of exposed points of B, as claimed in (b).

Finally define the closed convex sets

$$C_1 = A \cap \{\tfrac{1}{3} \le h\}, \quad C_2 = A \cap \{-\tfrac{1}{3} \ge h\}.$$

According to (a), $g(x) > 0$ for all $x \in A$, $x \ne 0$. Then $g > 0$ on C_1 and C_2 since neither one contains 0 by definition. Hence $g > 0$ also on the convex hull of $C_1 \cup C_2$, which consequently can not contain 0. On the other hand, since $C_1 \cup C_2 \supset \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ it follows that A is the closed convex hull of $C_1 \cup C_2$ in E, and the last part of (c) then follows from (a).

Case 2. Now, suppose that E is weakly sequentially complete. According to [2, Consequence I to Main Theorem], E contains a subspace isomorphic to l^1 and therefore it suffices to describe such sets A, C_1 and C_2 in l^1.

Let $\{e_n\}$ be the canonical basis for l^1. Define

$$x_n = \frac{1}{n+1} e_1 + e_{n+1}, \quad y_n = \frac{1}{n+1} e_1 - e_{n+1},$$

and denote by g, h the functionals defined by $(1, 0, 0, \ldots)$ and $(0, 1, 1, \ldots)$ in the identification $(l^1)^* = l^\infty$.

As before, we let A be the closed convex hull of the set

$$\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$$

in l^1, and B the weak* closure of A in $(l^1)^{**}$. Since

$$\tfrac{1}{2}(x_n + y_n) = (1/(n+1))e_1$$

converges in norm to 0, we have $0 \in A$.

In order to show that $0 \in A$ is an exposed point, consider the function f defined for $x = \sum \alpha_i e_i$ by $f(x) = -\alpha_1 + \sum_{i \geq 2} |\alpha_i|/i$. It is easy to see that f is continuous and sub-additive, and that $f \leq 0$ on A (because $f(x_n) = f(y_n) = 0$ for all $n = 1, 2, \ldots$). Now, suppose that $u = \sum \alpha_i e_i \in A$ satisfies $g(u) = \alpha_1 \leq 0$. Then $-\alpha_1 = |\alpha_1|$ and therefore

$$0 \geq f(u) = \sum_{i \geq 1} |\alpha_i|/i,$$

which implies $\alpha_i = 0$ for all $i = 1, 2, \ldots$, or $u = 0$. Hence g exposes $0 \in A$ as claimed.

Now, let $\tilde{x} \in (l^1)^{**}$ be a cluster point of $\{x_n\}$. Since $x_n + y_n \to 0$ in norm, $-\tilde{x}$ is a cluster point of $\{y_n\}$ and since $h(\tilde{x}) = \lim h(x_n) = 1$, we get $\tilde{x} \neq 0$, and $0 = \frac{1}{2}(\tilde{x} + (-\tilde{x}))$ is not an extreme point of B.

Finally, define C_1 (resp. C_2) as the closed convex hull of $\{x_n\}$ (resp. $\{y_n\}$). Since $h = 1$ on C_1 and $h = -1$ on C_2, we conclude from (a) that $g > 0$ on $C_1 \cup C_2$. Then 0 is not in the convex hull of $C_1 \cup C_2$. But since A is the closed convex hull of $C_1 \cup C_2$ in E, the second part of (c) follows again.

This completes the proof of the theorem.

Remark 1. In contrast with this result, a *strongly* exposed point of a closed convex set is also strongly exposed for its weak* closure in the second dual, and a strongly exposed point of the closed convex hull of $C_1 \cup C_2$ (C_1, C_2 as above) is necessarily in C_1 or C_2. (Recall that a point x of a convex set A is strongly exposed if there is a continuous linear functional f such that if $y_n \in A$ satisfy $f(y_n) \to f(x)$, then $y_n \to x$.)

Remark 2. If E is a nonreflexive complex Banach space, the conclusions of the theorem hold for the canonical real structures of E and E^{**}.

We close this note by observing that a suitable modification of the above proof yields the following:

Let E be a real non-reflexive Banach space. There exist a closed bounded convex set D in E and a point $d \in E$ with the following properties:

(i) *d is not in the convex hull of $D \cup -D$.*
(ii) *d is an exposed point of the closed convex hull of $D \cup -D$.*
(iii) *d is not an extreme point of the weak* closure of $D \cup -D$ in E^{**}.*

References

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