ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 2, Summer 1983

ORDER CONTINUOUS LINEAR FORMS

BY

BURKHARD KÜHN

Abstract. Characterizations of (sequentially) order continuous linear forms on vector lattices are given in terms of their behaviour relative to families of orthogonal elements. As a consequence, the non existence of real measurable cardinals can be characterized by the property that the sequentially order continuous and the order continuous linear forms on order complete vector lattices coincide. This gives rise to a counter example to a conjecture of [1].

We use the terminology of [3]. Two elements x, y of a vector lattice E are said to be orthogonal if $\inf\{|x|, |y|\} = 0$. The band of those elements of E which are orthogonal to a subset A of E is denoted by $A^{\perp}$, i.e.,

$$A^{\perp} = \{x \in E : \inf\{|x|, |y|\} = 0 \text{ for all } y \in A\}.$$

A vector lattice E has the principal projection property if every principal band ($B = \{x\}^{\perp\perp}$) is a projection band. If every subset M of E which possesses a supremum contains a countable subset A such that $\sup A = \sup M$, then the vector lattice E is said to be order separable. A bounded linear form f on E is (sequentially) order continuous if every net (sequence) $(x_\alpha)_{\alpha \in \mathsf{A}}$ which decreases to zero satisfies $\lim_{\alpha \in \mathsf{A}} f(x_\alpha) = 0$.

THEOREM. *Let E be a vector lattice with the principal projection property and f an order bounded linear form on E.*

(a) *A necessary and sufficient condition for f to be sequentially order continuous is that for every orthogonal sequence $(x_n)_{n \in \mathbf{N}}$ in E_+ for which* $\sup_{n \in \mathbf{N}} x_n = x_o$ *exists we have* $f(x_o) = \sum_{n=1}^{\infty} f(x_n)$.

(b) *A necessary and sufficient condition for f to be order continuous is that for every set A of pairwise orthogonal elements in E_+ for which* sup $A = x_o$ *exists we have* $f(x_o) = \sum_{x \in A} f(x)$.

Proof. The conditions are clearly necessary. Conversely let us suppose that f satisfies our condition of either (a) or (b) above. Then the same will be true of f^+ and hence also of $|f| = 2f^+ - f$. This follows by an easy

Received December 29, 1980.

argument from the relation $f^+(x) = \sup_{o \leqslant y \leqslant x} f(y)$ for all $x \in E_+$. Hence we may and do suppose in addition that f is positive. We now prove the sufficiency of the conditions in (a) and (b) separately.

(a) Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in E such that $x_n \downarrow o$. For any given $\varepsilon \in (o, 1)$ we denote by P_n the band projection associated with the principal band generated by $(x_n - \varepsilon x_1)^+$. We let $y_n = P_n(x_1)$. By means of the relation

$$(x_n - \varepsilon x_1) \leqslant (x_n - \varepsilon x_1)^+ = P_n(x_n - \varepsilon x_1)$$

the following properties of y_n hold:

(i) $\varepsilon^{-1}x_n \geqslant P_n(x_1)$ and hence $y_n \downarrow o$,
(ii) $x_n \leqslant y_n + \varepsilon x_1$.

By construction, the sequence $(y_n - y_{n+1})_{n\in\mathbf{N}}$ is orthogonal and, by (i), satisfies the identity

$$y_1 = y_1 - \inf_{n\in\mathbf{N}} y_{n+1} = \sup_{n\in\mathbf{N}} (y_1 - y_{n+1})$$

$$= \sup_{n\in\mathbf{N}} \sum_{i=1}^{n} (y_i - y_{i+1}) = \sup_{n\in\mathbf{N}} (y_n - y_{n+1}).$$

We now use the condition of (a) to deduce that $f(y_1) - f(y_{n+1}) = \Sigma_{i=1}^n f(y_i - y_{i+1})$ converges to $f(y_1)$, i.e., $\lim_{n\to\infty} f(y_n) = o$. Thus we obtain from (ii) the relation

$$o \leqslant f(x_n) \leqslant f(y_n) + \varepsilon f(x_1) \leqslant \varepsilon + \varepsilon f(x_1)$$

for n sufficiently large. Since ε was arbitrary, $\lim_{n\to\infty} f(x_n) = o$, so that f is sequentially order continuous.

(b) By part (a) we know that f is sequentially order continuous. Hence $N(f) = \{x \in E : f(|x|) = o\}$ is a σ-ideal.

We choose a maximal family $\{u_\alpha : \alpha \in \mathsf{A}\}$ of pairwise orthogonal positive elements of $N(f)$. Clearly, $N(f)^{\perp\perp} = \{u_\alpha : \alpha \in \mathsf{A}\}^{\perp\perp}$. For any positive x in the band $N(f)^{\perp\perp}$ we let $x_\alpha = \sup_{n\in\mathbf{N}} x \wedge nu_\alpha$ for each $\alpha \in \mathsf{A}$. We can represent x as $x = \sup_{\alpha\in\mathsf{A}} x_\alpha$, where the x_α are pairwise orthogonal elements in $N(f)$. Our condition implies $f(x) = \Sigma_{\alpha\in\mathsf{A}} f(x_\alpha) = o$. Hence $N(f)$ is a band and our proof is complete by virtue of the following lemma. ■

LEMMA. *Let E denote an archimedean vector lattice and let f denote a positive sequentially order continuous linear form on E. If the absolute kernel $N(f) = \{x \in E : f(|x|) = 0\}$ is a band, then f is order continuous.*

Proof. The archimedean vector lattice E possesses a Dedekind completion $\hat{E}$. In the sequel we regard E as a vector sublattice of $\hat{E}$. Every positive functional $g \in E^b$ has a positive linear extension to $\hat{E}$ given by

$$\hat{g}(u) = \sup_{\substack{o \leq x \leq u^+ \\ x \in E}} g(x) - \sup_{\substack{o \leq y \leq u^- \\ y \in E}} g(y).$$

Moreover, g and $\hat{g}$ are simultaneously order continuous. This is a consequence of the following property of archimedean vector lattices. If $\{y_\beta\}_{\beta \in \mathrm{B}}$ denotes the decreasing net of all upper bounds of a given net $\{x_\alpha\}_{\alpha \in \mathrm{A}}$ in E such that $x_\alpha \uparrow x$, then $(y_\beta - x_\alpha) \downarrow 0$, $\alpha \in \mathrm{A}$, $\beta \in \mathrm{B}$ [2, Theorem 22.5]. At the same time we see that the extension is unique if we restrict attention to the order continuous functionals.

Now suppose that $N(f)$ is a band. We shall show that

$$N(\hat{f}) = \{ \hat{x} \in E : \hat{f}(|\hat{x}|) = 0\}$$

is a band in $\hat{E}$. We consider a positive net $\{\hat{x}_\alpha\}_{\alpha \in \mathrm{A}}$ in $N(\hat{f})$ such that $\hat{x}_\alpha \uparrow \hat{x}$ and let $\varepsilon > o$ be given. By definition there exists a positive element $x \in E$ such that the inequalities

$$o \leq x \leq \hat{x} \quad \text{and} \quad \hat{f}(\hat{x}) \leq f(x) + \varepsilon$$

are satisfied. As a result of the relation $\hat{x}_\alpha = \sup[o, x_\alpha] \cap E$ we can express x as $x = \sup[o, x] \cap N(f)$. Since $N(f)$ is a band, it follows that

$$o \leq \hat{f}(\hat{x}) \leq f(x) + \varepsilon = \varepsilon.$$

Thus $\hat{f}(\hat{x}) = o$, so that $N(\hat{f})$ is likewise a band.

The linear form $\hat{f}$ is strictly positive on $N(\hat{f})^\perp$ and hence $N(\hat{f})^\perp$ is order separable [3, Prop. II.4.9]. Using the definitions of $\hat{E}$ and $\hat{f}$ it is easy to see that $\hat{f}$ is sequentially order continuous and hence order continuous on the band $N(\hat{f})^\perp$. Finally as a consequence of $\hat{E} = N(\hat{f}) + N(\hat{f})^\perp$, the linear forms $\hat{f}$ and f are order continuous. ■

Remarks. 1. Similar characterizations of order continuous and sequentially order continuous lattice semi-norms can be given in terms of their behaviour relations to families of orthogonal elements.

2. Theorem (a) is more generally true for order bounded maps T from E to arbitrary order complete vector lattices F. Theorem (b) can be extended to such linear maps T under the additional assumption that the order continuous linear forms on F separate points.

3. The following example shows that the condition "E has the principal projection property" in the theorem above cannot be omitted: Let $E = C([0, 1])$. Every Dirac measure δ_x for $x \in [0, 1]$ on E satisfies the orthogonality condition of our Theorem. However, it is well known that Dirac measures are not sequentially order continuous.

A cardinal I is said to be real measurable if and only if there exists a measure $\mu \neq 0$ on the power set of I with $\mu(\{\alpha\}) = 0$ for each $\alpha \in I$ (i.e., μ is a diffuse measure $\neq 0$ on $\mathcal{P}(I)$). It is known that on the basis of ZF (axioms of Zermelo-Fraenkel) and AC (axiom of choice) and generalized CH (continuum hypothesis) the existence of real measurable cardinals cannot be proved. On the contrary, much effort has been made to prove the nonexistence. In any case real measurable cardinals are, on the basis of ZF, AC, CH, much larger than all "good" cardinals; for instance they are inaccessible (see [4]–[6]).

In [1] the (non)existence of real measurable cardinals is considered in connection with "residual measures". In view of a result of [1] and the corollary at the end of this paper we make the following observations. If the existence of real measurable cardinals is presupposed, then there exist compact spaces X and measures $0 \neq \mu \in M^+(X)$ which vanish on all compact nowhere dense Baire sets without being residual. In other words, there are meager subsets $M \subseteq X$ for which $\mu(M) \neq 0$. Conversely, Armstrong-Prikry conjecture that the existence of such compact spaces implies the existence of real measurable cardinals. The following example shows that this is false even for quasi-Stonian compacta.

Example. Let $X = \mathbf{R} \cup \{\infty\}$ denote the Alexandroff compactification of $(\mathbf{R}, d)$, where d denotes the discrete topology. On the one hand, if $B \neq \emptyset$ is a compact nowhere dense subset, then clearly $B = \{\infty\}$. On the other hand, the Baire algebra on X consists of sets M with the following property:

$\mathbf{C}M$ is countable if ∞ belongs to M and M is countable, if $\infty \notin M$.

Hence all measures $\mu \in M^+(X)$ vanish on compact nowhere dense Baire sets B, since $B = \emptyset$ is always satisfied. However, the measure δ_∞ is nonresidual, because $\{\infty\}$ is meager.

Finally, as a consequence of our theorem we prove in the vector lattice setting the following result, which goes back to [1].

COROLLARY. *The following statements are equivalent:*

1°. *There does not exist a real measurable cardinal.*

2°. *The sequentially order continuous and the order continuous linear forms on order complete vector lattices coincide.*

3°. *If the order continuous functionals on an order complete vector lattice E separate points, then every sequentially order continuous functional on E is already order continuous.*

Proof. 1° $\Rightarrow$ 2°. Let f denote a countably order continuous (without loss of generality) positive linear form on an order complete vector lattice E. Let us assume that f is not order continuous. Then our theorem (proof and lemma) implies the existence of an order bounded family $\{x_\alpha : \alpha \in A\}$ of positive and pairwise orthogonal elements of E with the following two properties:

$$f(x_\alpha) = 0 \quad \text{for all } \alpha \in A \quad \text{and} \quad f(\sup_{\alpha \in A} x_\alpha) \neq 0.$$

For every sequence $(A_n)_{n \in \mathbf{N}}$ of disjoint subsets of A we then have

$$f\left(\sup\left\{x_\alpha : \alpha \in \bigcup_{n\in\mathbf{N}} A_n\right\}\right) = f(\sup_{n\in\mathbf{N}}(\sup_{\alpha\in A_n} x_\alpha)) = \sum_{n=1}^{\infty} f(\sup_{\alpha\in A_n} x_\alpha),$$

so that $M \mapsto f(\sup_{\alpha\in M} x_\alpha)$ clearly defines a finite diffuse non-zero measure on the power set of A. This provides the desired contradiction to property 1°.

The implication 2° $\Rightarrow$ 3° is clearly true.

3° $\Rightarrow$ 1°. Let μ denote a finite diffuse measure on the power set of a non-empty set A. The space $\mathscr{L}^\infty(\mu)$ forms an order complete vector lattice on which μ operates in a countably order continuous manner. By our hypothesis we therefore obtain

$$\mu(A) = \sum_{\alpha\in A} \mu(\{\alpha\}) = 0.$$

∎

Remark. For a sequentially order continuous linear form f on an order complete vector lattice our method shows: if the absolute kernel of f possesses a maximal orthogonal system of non real measurable cardinality, then f is order continuous.

References

1. T. E. Armstrong and K. Prikry, *Residual measures,* Illinois J. Math., vol. 22 (1979), pp. 64–78.
2. W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces I,* North Holland, Amsterdam, 1971.
3. H. H. Schaefer, *Banach lattices and positive operators,* Springer, New York, 1974.
4. J. Silver, "The consistency of the GCH with the existence of a measurable cardinal" in *Axiomatic set theory,* Amer. Math. Soc., Providence, R.I., 1971, pp. 391–397 (Proc. Symposium Pure Math., vol. 13).
5. R. M. Solovay, "Real valued measurable cardinals" in *Axiomatic set theory,* Amer. Math. Soc., Providence, R.I., 1971, pp. 397–428 (Proc. Symposium Pure Math., vol. 13).
6. S. Ulam. *Zur Maßtheorie in der allgemeinen Mengenlehre,* Fund. Math., vol. 16 (1930), pp. 140–150.

Universität Dortmund
Dortmund, West Germany