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THE PRODUCT OF TWO OR MORE NEIGHBORING INTEGERS IS NEVER A POWER

BY

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1. Introduction

In 1975 Erdös and Selfridge [1] proved the elegant result: *the product of two or more consecutive positive integers is never a power*. Earlier, Rigge [6] and, independently, Erdös [7] had shown that such a product is never a square.

In this paper we prove a generalisation, namely: *the product of two or more neighboring integers is never a power.*

Here we say that distinct integers are *neighboring* if they belong to a short interval $(N, N + c\log\log\log N)$ for some absolute constant c (and, of course, if $N \leqslant 16$ we interpret $\log\log\log N$ as 1). Our principal result is false for infinitely many N if the interval is lengthened to

$$\exp(12(\log N \log\log N)^{1/2}),$$

and is false for all N if the interval is as long as $cN^{1/2-\varepsilon}$ for certain positive constants c, ε.

Actually, we consider a more general situation. Our products of neighboring integers allow for repetition, so our statement becomes that *the product of two or more neighboring integers, allowing repetition, is never, other than trivially, a power*. Moreover, we deal with "almost powers" rather than "perfect powers." This is to say: we consider quantities of the shape ab^m with the integer a "small" relative to b^m and we actually show that the quantities we consider are never (other than trivially) "almost powers."

Finally we remark on what constitutes "triviality." We consider finite products

$$\Pi n_i^{m_i},$$

with the n_i in the given "short interval," and consider such products "not trivially a possible almost *m-th* power" if $\gcd(m,m_i) = 1$ for each i. It would seem more natural to ask of the m_i that not all the m_i be multiples of m. But this raises considerable difficulties: though Tijdeman [8] has shown that two sufficiently large *consecutive* integers cannot both be powers,

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it has not been established that large neighboring integers cannot both be powers; this, and more, is required to effect the desired relaxation of the condition on the m_i.

2. Lemmas

Notation. For $a,m \in \mathbb{N}$ with $m \geqslant 2$ we denote the set of integers of the form ax^m, $x \in \mathbb{N}$, by $a\mathbb{N}^m$. As usual, $\omega(n)$ is the number of distinct primes dividing $n \in \mathbb{N}$, $P(n)$ is the greatest prime dividing $n > 1$, while $P(1) = 1$. The greatest common divisor of n and m is denoted by $\gcd(n,m)$ and the least common multiple of $n_1, \ldots, n_f$ by $\operatorname{lcm}[n_1, \ldots, n_f]$. We write $|S|$ for the number of elements of a set S.

LEMMA 1. *Let $a,m,f \in \mathbb{N}$ with $m \geqslant 2$ and $f \geqslant 2$. Let $n_1, \cdots, n_f$ be distinct positive integers in an interval of length K with the property that $\Pi_{i=1}^f n_i^{m_i} \in a\mathbb{N}^m$ for certain $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m_i,m) = 1$ for $1 \leqslant i \leqslant f$. Write $n_i = a_i x_i^m$, $a = a_0 x_0^m$ with $a_i, x_i \in \mathbb{N}$ and a_i m-free for $0 \leqslant i \leqslant f$. Then:*

(1) *$a_1, \cdots, a_f$ are composed of the primes dividing a_0 and the primes not exceeding K.*

(2) $a_i \leqslant \exp\left(m\left(CK + \sum_{p|a_0} \log p\right)\right)$ *for $i = 1, \cdots, f$, where C is some absolute constant.*

(3) $a_i \leqslant \exp\left(m\left((f-1)\log K + \sum_{p|a_0} \log p\right)\right)$ *for $i = 1, \cdots, f$.*

(4) *There exist two (three if $f \geqslant 3$) a_i's with*

$$a_i \leqslant \exp\left(Cf^{-1}\left(mK + K\log K + m\sum_{p|a_0}\log p\right)\right),$$

where C is some absolute constant.

(5) *There exist two (three if $f \geqslant 3$) a_i's with*

$$a_i \leqslant \exp\left(Cm\left(K^{1/2}(\log K)^{1/2} + \sum_{p|a_0}\log p\right)\right),$$

where C is some absolute constant.

Proof. Let, for $n \in \mathbb{N}$, $\Pi_p\, p^{v_p(n)} = n$ be the prime factorization of n. Then we have, for every prime p,

$$\sum_{i=1}^{f} m_i v_p(a_i) \equiv v_p(a_0) \pmod m.$$

Hence if $p \nmid a_0$ and $p \mid a_i$ for some $1 \leqslant i \leqslant f$ then $p \mid a_j$ for some $1 \leqslant j \leqslant f, j \neq i$. It follows from $p \mid \gcd(a_i, a_j)$ that $p \mid \gcd(n_i, n_j)$ which divides $n_i - n_j (\neq 0)$. Hence $p \leqslant |n_i - n_j| \leqslant K$. This proves (1). Since a_i is m-free it follows from (1) that, for $1 \leqslant i \leqslant f$,

$$a_i = \prod_p p^{v_p(a_i)} \leqslant \left(\prod_{p \leqslant K} p \cdot \prod_{p \mid a_0} p\right)^{m-1} \leqslant \left(C_0^K \prod_{p \mid a_0} p\right)^{m-1},$$

which implies (2), with $C = \log C_0$, and $C_0 = 3$ for example. To prove (3) we note that, in view of the foregoing, a_i divides

$$\prod_{p \mid a_0} p^{v_p(a_i)} \prod_{\substack{j=1 \\ j \neq i}}^{f} \prod_{\substack{p \mid a_j \\ p \nmid a_0}} p^{v_p(a_i)}$$

which divides

$$\left(\prod_{p \mid a_0} p\right)^{m-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{f} \gcd(a_i, a_j)^{m-1}\right).$$

As observed already, $\gcd(a_i, a_j) \leqslant K$ for $i \neq j$ which gives (3). To prove (4) we consider

$$\prod_{i=1}^{f} a_i = \prod_p p^{\alpha_{\bar{p}}} = \prod_{p \leqslant K} p^{\alpha_p} \prod_{\substack{p > K \\ p \mid a_0}} p^{v_p} \text{ where } \alpha_p = \sum_{i=1}^{f} v_p(a_i)$$

where $0 \leqslant v_p \leqslant m - 1$. Note that

$$\begin{aligned}
\sum_{i=1}^{f} v_p(a_i) &= \sum_{j=1}^{m-1} |\{1 \leqslant i \leqslant f \mid p^j \text{ divides } a_i\}| \\
&\leqslant \sum_{j=1}^{m-1} |\{1 \leqslant i \leqslant f \mid p^j \text{ divides } n_i\}| \\
&\leqslant \sum_{j=1}^{m-1} (1 + [Kp^{-j}]) \\
&\leqslant m - 1 + \sum_{j=1}^{\infty} [Kp^{-j}] \\
&= m - 1 + v_p(K!).
\end{aligned}$$

Hence

$$\prod_{i=1}^{f} a_i \leqslant \prod_{p \leqslant K} p^{m-1}(K!) \prod_{p \mid a_0} p^{m-1} \leqslant \exp(C_0 K m + K \log K + m \sum_{p \mid a_0} \log p).$$

If $f \geqslant 3$ there exist at least three a_i's with $a_i \leqslant (\Pi_{i=1}^{f} a_i)^{1/(f-2)}$, hence with

$$a_i \leqslant \exp(3f^{-1}(C_0 Km + K \log K + m \sum_{p|a_0} \log p)).$$

If $f = 2$ then we have, by (3), $a_i \leqslant \exp(m(\log K + \Sigma_{p|a_0} \log p))$ for $i = 1,2$. This proves (4) (with $C = 3C_0$, e.g., with $C = 3 \log 3$).

To prove (5) we distinguish two cases. If

$$f \geqslant 2 + [K^{1/2}(\log K)^{-1/2}] =: \lambda$$

then we have $3 \leqslant \lambda \leqslant f$ and

$$\begin{aligned} \prod_{i=1}^{\lambda} a_i &\leqslant \operatorname{lcm}[a_1, \cdots, a_\lambda] \prod_{1 \leqslant i < j \leqslant \lambda} \gcd(a_i, a_j) \\ &\leqslant \prod_{\substack{p|a_0 \\ p>K}} p^{m-1} \cdot \prod_{p \leqslant K} p^{m-1} \cdot K^{\lambda(\lambda-1)/2} \\ &\leqslant \exp((m-1)(C_0 K + \sum_{p|a_0} \log p) + \lambda(\lambda-1)/2 \cdot \log K). \end{aligned}$$

Since there exist at least three a_i's among $a_1, \cdots, a_\lambda$ with

$$a_i \leqslant \left(\prod_{i=1}^{\lambda} a_i \right)^{1/(\lambda-2)}$$

it follows that there exist at least three a_i's with

$$\begin{aligned} a_i &\leqslant \exp(3/2 \cdot (\lambda - 1)\log K + 3(m-1)\lambda^{-1}(C_0 K + \sum_{p|a_0} \log p)) \\ &\leqslant \exp(3C_0\, mK^{1/2}(\log K)^{1/2} + m \sum_{p|a_0} \log p). \end{aligned}$$

If $f \leqslant 1 + [K^{1/2}(\log K)^{-1/2}]$ then we have, by (3),

$$a_i \leqslant \exp(m(K^{1/2}(\log K)^{1/2} + \sum_{p|a_0} \log p)) \quad \text{for } 1 \leqslant i \leqslant f$$

so that (5) holds in both cases with $C = 3C_0$ (e.g., $C = 3 \log 3$).■

LEMMA 2. *Suppose the interval* $[N, N+K]$, *where* $1 \leqslant K \leqslant N^{2/3}$, *contains two distinct integers* n_i *of the form* $n_i = a_i x_i^m$ *where* $a_i, x_i \in \mathbb{N}$ *for* $i = 1,2$ *with* $m \geqslant 3$. *Let* $p_1, \cdots, p_t$ *be the distinct primes dividing* $a_1 a_2$ *and put*

$$a = p_1 \cdots p_t, \quad A = \max\{p_1, \cdots, p_t, 3\}.$$

Then

$$(1) \qquad \begin{cases} m \leqslant a^{C_1} & \text{if } (x_1, x_2) \neq (1,1) \\ \log N \leqslant a^{C_1} & \text{if } (x_1, x_2) = (1,1), \end{cases}$$

$$(2) \qquad A^{C_2 m^3} \log 3K \log\log 3K > \log N,$$

where C_1 and C_2 are absolute constants.

Proof. See [2, Proposition 1].

LEMMA 3. *Suppose the interval $[N,N + K]$, where $N \geqslant 1$, $K \geqslant 1$, contains three distinct integers n_i of the form $n_i = a_i x_i^2$, where $a_i, x_i \in \mathbb{N}$. Put $A = \max\{a_1,a_2,a_3,3\}$. Then*

$$CA^{3+\varepsilon} \log 3K \log\log 3K > \log N,$$

where $\varepsilon > 0$ is arbitrary and $C = C(\varepsilon) > 0$.

Proof. See [2, Proposition 2].

The proofs of Lemmas 2 and 3 are based on a lower bound for linear forms in logarithms of rational numbers.

3. Products of neighboring integers and almost powers: The general case

THEOREM 1. *There exist positive numbers c_1, c_2 and c_3 with the following property. For $N \geqslant 16$ write $K(N) = c_1 \log\log\log(N)$. For $f \geqslant 2$, let $n_1, \cdots, n_f$ be f distinct integers in an interval of the form $I_N = [N,N + K(N)]$. Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m,m_i) = 1$ for $i = 1, \cdots, f$. Write $\Pi_{i=1}^{f} n_i^{m_i} = ab$, where $a \in \mathbb{N}$, $b \in \mathbb{N}^m$. Then*

$$a > (\log\log N)^{c_2}(>1) \quad \textit{and} \quad P(a) > c_3 \log\log\log N$$

except if $f = 2$, $m = 2$, $a \notin \mathbb{N}^2$. In fact, if $a \notin \mathbb{N}^2$ then $n_1 n_2 \in a\mathbb{N}^2$ with $n_1 \neq n_2$ in I_N occurs for infinitely many intervals I_N, $N \in \mathbb{N}$.

Proof of Theorem 1. For $f \geqslant 2$, suppose $n_1, \cdots, n_f$ are f distinct integers in an interval $[N,N + K]$, where $N \geqslant 16$ and $K \geqslant 1$. Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m,m_i) = 1$. Suppose

$$\prod_{i=1}^{f} n_i^{m_i} \in a\mathbb{N}^m \quad \text{where } a \in \mathbb{N}.$$

Write $n_i = a_i x_i^m$, $a = a_0 x_0^m$ with $a_i, x_i \in \mathbb{N}$ and a_i m-free for $0 \leqslant i \leqslant f$. First we consider the cases where $f = m = 2$. If $a \in \mathbb{N}^2$ then it follows from $n_1^{m_1} n_2^{m_2} \in a\mathbb{N}^2$, m_1, m_2 odd, that $a_1 = a_2$; hence $[N,N + K]$ contains two distinct integers of the form $a_1 x_1^2, a_1 x_2^2$, which implies that $K > N^{1/2}$. Hence if $K \leqslant N^{1/2}$ and $f = m = 2$ then $a \notin \mathbb{N}^2$. It is well known that for every a $\notin \mathbb{N}^2$ there exist infinitely many $x_1, x_2 \in \mathbb{N}$ with $x_1^2 - a x_2^2 = 1$; hence $n_1 = x_1^2$, $n_2 = a x_2^2$ satisfy $n_1 n_2 \in a\mathbb{N}^2$ and $n_1 - n_2 = 1$. We have proved our assertions about the cases $f = m = 2$ in Theorem 1 and we may assume $f \geqslant 3$ if $m = 2$ now. We distinguish two cases.

Case 1. $m = 2$ ($f \geqslant 3$). By Lemma 1, (5), the interval $[N, N + K]$ contains three distinct integers $a_i x_i^2$ with

$$a_i \leqslant \exp\{C(K^{1/2}(\log K)^{1/2} + \sum_{p|a_0} \log p)\} =: A.$$

By Lemma 3 we have $A^4(\log 3\ K)^2 >> \log N$. It follows that

$$K^{1/2}(\log K)^{1/2} + \sum_{p|a_0} \log p > \delta_1 \log\log N$$

for some absolute positive constant δ_1. In particular, if

$$K^{1/2}(\log K)^{1/2} \leqslant \frac{1}{2}\delta_1 \log\log N,$$

then $\Sigma_{p|a_0} \log p > \delta_1/2 \log\log N$. Since

$$\sum_{p|a_0} \log p \leqslant \log a_0 \quad \text{and} \quad \sum_{p|a_0} \log p \leqslant \sum_{p \leqslant P(a_0)} \log p \leqslant \delta_0 P(a_0)$$

for some absolute constant δ_0 we conclude that if

$$K < \delta_2 (\log\log N)^2 (\log\log\log N)^{-1}$$

for some small absolute constant δ_2 then

$$a_0 > (\log N)^{\delta_1/2} \quad \text{and} \quad P(a_0) > \delta_1/(2\delta_0) \log\log N.$$

Case 2. $m \geqslant 3$. Now $[N, N + K]$ contains, by Lemma 1, (5), two distinct integers $a_i x_i^m$ with

$$a_i \leqslant \exp\left(Cm\left(K^{1/2}(\log K)^{1/2} + \sum_{p|a_0} \log p\right)\right) =: A.$$

By Lemma 2, (2), assuming that $K \leqslant N^{2/3}$, we have

$$(*) \qquad m^4\left(K^{1/2}(\log K)^{1/2} + \sum_{p|a_0} \log p\right) >> \log\log N.$$

We have to bound the exponent m now. By Lemma 2, (1) and Lemma 1, (1), assuming $K \leqslant N^{2/3}$ again, we have

$$m \leqslant \left(\sum_{p|a_0} p \prod_{p \leqslant K} p\right)^{C_1} \qquad \text{if the } x_i \text{ are not both 1,}$$

$$\log N \leqslant \left(\prod_{p|a_0} p \prod_{p \leqslant K} p\right)^{C_1} \qquad \text{if the } x_i \text{ are both 1.}$$

In the latter case ($x_i = 1$ for both i) we get $\log\log N << K + \Sigma_{p|a_0} \log p$. In the other case we obtain $\log m << K + \Sigma_{p|a_0} \log p$. Inserting this bound for m in (*) we get

$$K + \sum_{p|a_0} \log p > \delta_3 \log\log\log N$$

for some absolute positive constant δ_3 (< 1). In particular, if

$$K \leq c_1 \log\log\log N \; (< N^{2/3})$$

for some small positive constant c_1, then $\Sigma_{p|a_0} \log p > \frac{1}{2}\, \delta_3 \log\log\log N$, which implies

$$a_0 > (\log\log N)^{\delta_3/2} \quad \text{and} \quad P(a_0) > \delta_3/(2\delta_0) \cdot \log\log\log N. \qquad \blacksquare$$

Note that it is essential that the bound for m in Lemma 2, (1) depends only on the prime divisors of a_1a_2, not just on a_1 and a_2.

COROLLARY 1. *The product of two or more distinct integers from an interval of the form* $I_N = [N, N + c_1 \log\log\log N]$, $N \geq 16$, *is never a power.*

Remark. By Theorem 1, $\Pi_{i=1}^{f} n_i^{m_i} \notin \mathbb{N}^m$ if $n_1, \cdots, n_f \in I_N$, $N \geq 16, f \geq 2$ for any $m \in \mathbb{N}$ with $m \geq 2$ and any $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m, m_i) = 1$ for $i = 1, \cdots, f$. It would be interesting to relax the conditions on the multiplicities $m_1, \cdots, m_f$, possibly to "not all $m_1, \cdots, m_f$ are multiples of m." This seems to be a difficult matter, however: observe that two distinct powers $n_1 = x_1^{m_1}$, $n_2 = x_2^{m_2}$ satisfy $n_1^{m_2} n_2^{m_1} \in \mathbb{N}^m$ with $m = m_1m_2$. It has not been established that two distinct powers cannot be neighboring integers; the only known general result in this respect is that two sufficiently large distinct powers cannot be consecutive integers (Tijdeman, 1976).

COROLLARY 2. *Let* $a \in \mathbb{N}$. *The product of three or more distinct integers* $n_1, \cdots, n_f$ *from an interval* I_N, $N \geq 16$, *is not of the form ab for any power b, except for finitely many sets* $\{n_1, \cdots, n_f\}$. *(If* $a \in \mathbb{N}^2$ *then one may replace "three" by "two" in the above assertion).*

In [2], [3] we introduced the notion of an almost power: let $\phi: \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. An integer n is called an ϕ-almost power if it can be written as $n = ab$, where b is a power and $a \in \mathbb{N}$ with $(1 <)$ $a \leq \phi(n)$. A different, perhaps more natural, notion of an almost power results if one replaces the condition $(1 <)\, a \leq \phi(n)$ by $(1 <)\, P(a) \leq \phi(n)$: n is a power iff there exists an $m \in \mathbb{N}$ with $m \geq 2$ such that $v_p(n) \in m\mathbf{Z}$ for all primes p, while in the latter definition of almost power we require $v_p(n) \in m\mathbf{Z}$ for all $p > \phi(n)$.

COROLLARY 3. *The product of three or more distinct integers from an interval* I_N, $N \geq 16$, *is never a* ϕ*-almost power (in both senses, with* $\phi(n) = (\log\log n)^{c_4}$ *and* $\phi(n) = c_5 \log\log\log n$, *respectively, where* c_4 *and* c_5 *are positive absolute constants).*

Proof. For $f \geqslant 3$, suppose $n_1, \cdots, n_f$ are f distinct integers in I_N; write $n = \Pi_{i=1}^{f} n_i = ab$, where b is some power and $a \in \mathbb{N}$. Then

$$a > (\log \log N)^{c_2} \quad \text{and} \quad P(a) > c_3 \log \log \log N.$$

Note that since $N \leqslant n_i \leqslant 2N$ we have $n = \Pi_{i=1}^{f} n_i \leqslant (2N)^f$; hence

$$\log \log N \leqslant \log \log n \leqslant C \log \log N,$$

where C is some absolute constant. Since $a \geqslant 2$, $P(a) \geqslant 2$, it follows easily that

$$a > (\log \log n)^{c_4} \quad \text{and} \quad P(a) > c_5 \log \log \log n$$

if c_4 and c_5 are sufficiently small positive absolute constants. ■

4. Products of neighboring integers and almost powers: Special cases

The intervals I_N in Section 3 are rather short. This is due to the fact that the exponent m is unspecified. When m is fixed (e.g., when one asks for neighboring integers whose product is a square) then one can allow for longer intervals.

THEOREM 2. *Let $m \in \mathbb{N}$ with $m \geqslant 2$. For $N \geqslant 16$ write*

$$K^{(m)}(N) = c_6 m^{-8} (\log \log N)^2 (\log \log \log N)^{-1}.$$

For $f \geqslant 2$, let $n_1, \cdots, n_f$ be f distinct integers in an interval of the form

$$I_N^{(m)} = [N, N + K^{(m)}(N)], \quad N \geqslant 16$$

and let $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m, m_i) = 1$ for $i = 1, \cdots, f$. Write

$$\prod_{i=1}^{f} n_i^{m_i} = ab \quad \text{where } a \in \mathbb{N},\ b \in \mathbb{N}^m.$$

Then

$$a > (\log N)^{c_7 m^{-4}} \quad \text{and} \quad P(a) > c_8 m^{-4} \log \log N,$$

except if $m = 2, f = 2$ and $a \notin \mathbb{N}^2$. Here c_6, c_7 and c_8 are fixed positive numbers.

COROLLARY 4. *The product of two or more distinct integers from an interval $[N, N + c_9 (\log \log N)^2 (\log \log \log N)^{-1}]$, $N \geqslant 16$, is never a square or a cube.*

Proof of Theorem 2. See the proof of Theorem 1; we do not have to bound m now and in Case 2 we conclude from (*) that if

$$K^{1/2} (\log K)^{1/2} < c m^{-4} \log \log N \quad \text{for some small } c > 0$$

then $\Sigma_{p|a_0} \log p >> m^{-4} \log \log N$. ■

In Theorems 1 and 2 the sets $\{n_1, \cdots, n_f\}$ are arbitrary sets contained in short intervals. One can enlarge the lengths of the intervals if the sets $\{n_1, \cdots, n_f\}$ are restricted in one of the following ways: the number of elements f is "small" or the average distance

$$\frac{n_f - n_1}{f-1} \quad (n_1 < \cdots < n_f)$$

is "small."

THEOREM 3. *Let $m \in \mathbb{N}$ with $m \geqslant 2$, let $F \geqslant 2, \Delta \geqslant 1$ and $0 \leqslant \varepsilon < 1$. For $N \geqslant 3$, let*

$$K_i(N) = \frac{1}{2}\exp(c_i(\log\log N)^{1-\varepsilon}), \quad i = 1,2,$$

where $c_1 = c_{10}m^{-4}F^{-1}$ and $c_2 = c_{10}m^{-4}\Delta^{-1}$. For $f \geqslant 2$, let $n_1 < \cdots < n_f$ be f distinct integers in an interval of the form $[N, N + K_i(N)]$, $N \geqslant 3$, with

$$f \leqslant F(\log\log N)^{\varepsilon} \quad \textit{if } i = 1,$$

$$\frac{n_f - n_1}{f-1} \leqslant \Delta(\log\log N)^{\varepsilon} \quad \textit{if } i = 2.$$

Let $m_1, \cdots, m_f \in \mathbb{N}$ with $\gcd(m, m_i) = 1$ for $i = 1, \cdots, f$, and write

$$\prod_{i=1}^{f} n^{m_i} = ab \quad \textit{where } a \in \mathbb{N},\ b \in \mathbb{N}^m.$$

Then

$$a > (\log N)^{c_{11}m^{-4}} \quad \textit{and} \quad P(a) > c_{12}m^{-4}\log\log N,$$

except if $m = 2, f = 2$ and $a \notin \mathbb{N}^2$. Here c_{10}, c_{11}, c_{12} are positive absolute constants.

COROLLARY 5. *Let $m, f \in \mathbb{N}$ with $m \geqslant 2, f \geqslant 2$. Let $n_1, \cdots, n_f$ be distinct integers in an interval of the form*

$$\left[N, N + \frac{1}{2}(\log N)^{c_{13}}\right] \quad \textit{where } c_{13} = c_{10}m^{-4}f^{-1}.$$

Then $\Pi_{i=1}^{f} n_i \notin \mathbb{N}^m$. Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let $\Delta \geqslant 1$. Let $n_1, \cdots, n_f$ be distinct integers in an interval of the form

$$\left[N, N + \frac{1}{2}(\log N)^{c_{14}}\right] \quad \textit{where } c_{14} = c_{10}m^{-4}\Delta^{-1} \textit{ with } \frac{n_f - n_1}{f-1} \leqslant \Delta.$$

Then $\Pi_{i=1}^{f} n_i \notin \mathbb{N}^m$.

Proof of Theorem 3. This is similar to the proof of Theorem 2; instead of Lemma 1, (5) use Lemma 1, (3) if $i = 1$ and Lemma 1, (4) if $i = 2$.

5. Intervals containing integers having a power as their product

It seems reasonable to guess that the assertions in Corollaries 1 and 4 are true for longer "short intervals." In this final section we prove that these assertions certainly do not hold for sufficiently long (but still "short") intervals.

THEOREM 4. *For* $N \geqslant 3$ *let* $K_3(N) = \exp(12(\log N \log\log N)^{1/2})$. *For every* $m \in \mathbb{N}$ *with* $m \geqslant 2$ *there exists an infinite set* $N_m \subset \mathbb{N}$ *such that for every* $N \in N_m$ *the interval* $[N, N + K_3(N)]$ *contains a subset* $\{n_1, \cdots, n_f\}$ *consisting of f integers,* $f = f(N) \geqslant 2$, *with the property that*

$$\prod_{i=1}^{f} n_i^{m_i} \in \mathbb{N}^m \quad \textit{for certain } m_1, \cdots, m_f \in \{1, \cdots, m-1\}.$$

COROLLARY 5. *There exist infinitely many* $N \in \mathbb{N}$ *such that* $[N, N + K_3(N)]$ *contains two or more distinct integers having a power* (*in fact, a square*) *as their product.*

Proof of Theorem 4. It is known (e.g., see [4, Theorem 5.4]) that there exist infinitely many $N \in \mathbb{N}$ such that the interval $[N, N + K_3(N)]$ contains a subset S^* of integers with $\omega(S^*) < \sqrt{|S^*|}$ and $|S^*| > K_3(N)^{1/3}$, where $|S|$ denotes the number of elements of a set S and $\omega(S)$ the number of elements $|P|$ of the set P of prime divisors of $\Pi_{s\in S}\, s$. Let $m \in \mathbb{N}$ with $m \geqslant 2$. For sufficiently large N the interval $[N, N + K_3(N)]$ contains at most one element from $\mathbb{N}^d$, for every $d > 1$. Delete from S^* those s with the property that $s \in \mathbb{N}^d$ for some $d > 1$ with $d \mid m$. For N sufficiently large (in terms of m), and denoting the number of divisors of m by $d(m)$, we have, for the resulting set S,

$$\begin{aligned} |S| &\geqslant |S^*| - d(m) \\ &\geqslant 2\,\frac{\log m}{\log 2}\sqrt{|S^*|} \\ &\geqslant \frac{\log m}{\log 2}(\omega(S^*) + 1) \\ &\geqslant \frac{\log m}{\log 2}(\omega(S) + 1). \end{aligned}$$

For every subset $T \subset S$ we define $\phi(T) = (\varepsilon_p)_{p\in P}$, where

$$\varepsilon_p \in \{0,1, \cdots, m-1\}, \quad \varepsilon_p \equiv \sum_{s\in S} v_p(s) \pmod m.$$

We have $2^{|S|}$ distinct sets T and at most $m^{|P|} = m^{\omega(S)}$ distinct tuples $\phi(T)$. By the box principle there exists a tuple $(\varepsilon_p)_{p\in P}$ such that there exist at least $2^{|S|}/m^{\omega(S)}$ $(\geqslant m)$ distinct T with $\phi(T) = (\varepsilon_p)_{p\in P}$, say $\phi(T_i) = (\varepsilon_p)_{p\in P}$ for i $= 1, \cdots, m$. Put $a = \Pi_{p\in P}\, p^{\varepsilon_p}$, then $\Pi_{t\in T_i}\, t \in a\mathbf{N}^m$ for $i = 1, \cdots, m$. Hence

$$\prod_{t\in T_1\cup\cdots\cup T_m} t^{m(t)} = \prod_{i=1}^{m}\prod_{t\in T_i} t \in \mathbf{N}^m,$$

where $m(t)$ denotes the number of T_i, $1 \leqslant i \leqslant m$, with $t \in T_i$. Let $n_1, \cdots, n_f$ be those $t \in T_1 \cup \cdots \cup T_m$ with

$$m(t) \in \{1, \cdots, m-1\}.$$

Since $T_1, \cdots, T_m$ are not all equal we have $f \geqslant 1$ and since $t \notin \mathbf{N}^d$ for any $d > 1$ with $d \mid m$ we have $f \geqslant 2$. Hence

$$\prod_{i=1}^{f} n_i^{m_i} \in \mathbf{N}^m \quad \text{where } f \geqslant 2 \text{ and } m_i = m(n_i) \in \{1, \cdots, m-1\}. \quad \blacksquare$$

THEOREM 5. *Let $K_4(N) = c_{15}N^{1/2-\varepsilon_0}$, where c_{15} and ε_0 are certain positive constants. For every $N \geqslant 1$ the interval $[N, N + K_4(N)]$ contains two or more distinct integers having a power as their product.*

Proof. It follows from the argument in the proof of Theorem 4 that if S is a set of positive integers with $\omega(S) + 1 \leqslant |S|$ which does not contain a square then there exists a subset T of S with $|T| \geqslant 2$ and $\Pi_{t\in T}\, t \in \mathbf{N}^2$. It is known (e.g., see [4, page 16], or [5]) that if $n, k \in \mathbf{N}$ with $k \geqslant c_{15}n^{1/2-\varepsilon_0}$, with ε_0 a small positive constant and c_{15} a large constant, then

$$\omega((n+1)\cdots(n+k)) \leqslant k-2.$$

Now the set $S^* = \{n+1, \cdots, n+k\}$ contains a subset T with $|T| \geqslant 2$ and $\Pi_{t\in T}\, t \in \mathbf{N}^2$: if S^* contains two squares then this is obvious; if S^* contains one or zero squares then we apply the above argument to $S = S^*$ minus the square in S^*, or to $S = S^*$, respectively. $\blacksquare$

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